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METHODS FOR SOLVING INVERSE PROBLEMS IN MATHEMATICAL PHYSICS

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Preface

The theory of inverse problems for differential equations is being extensively developed within the framework of mathematical physics. In the study of the so-called direct problems the solution of a given differential equation or system of equations is realised by means of supplementary conditions, while in inverse problems the equation itself is also unknown. The determination of both the governing equation and its solution necessitates imposing more additional conditions than in related direct problems.

The sources of the theory of inverse problems may be found late in the 19th century or early 20th century. They include the problem of equilibrium figures for the rotating fluid, the kinematic problems in seismology, the inverse Sturm–Liuville problem and more. Newton's problem of discovering forces making planets move in accordance with Kepler's laws was one of the first inverse problems in dynamics of mechanical systems solved in the past. Inverse problems in potential theory in which it is required to determine the body's position, shape and density from available values of its potential have a geophysical origin. Inverse problems of electromagnetic exploration were caused by the necessity to elaborate the theory and methodology of electromagnetic fields in investigations of the internal structure of Earth's crust.

The influence of inverse problems of recovering mathematical physics equations, in which supplementary conditions help assign either the values of solutions for fixed values of some or other arguments or the values of certain functionals of a solution, began to spread to more and more branches as they gradually took on an important place in applied problems arising in "real-life" situations. From a classical point of view, the problems under consideration are, in general, ill-posed. A unified treatment and advanced theory of ill-posed and conditionally well-posed problems are connected with applications of various regularization methods to such problems in mathematical physics. In many cases they include the subsidiary information on the structure of the governing differential equation, the type of its coefficients and other parameters. Quite often the unique solvability of an inverse problem is ensured by the surplus information of this sort. A definite structure of the differential equation coefficients leads to an inverse problem being well-posed from a common point of view. This book treats the subject of such problems containing a sufficiently complete and systematic theory of inverse problems and reflecting a rapid growth and development over recent years. It is based on the original works of the authors and involves an experience of solving inverse problems in many branches of mathematical physics: heat and mass transfer, elasticity theory, potential theory, nuclear physics, hydrodynamics, etc. Despite a great generality of the presented research, it is of a constructive nature and gives the reader an understanding of relevant special cases as well as providing one with insight into what is going on in general.

In mastering the challenges involved, the monograph incorporates the well-known classical results for direct problems of mathematical physics and the theory of differential equations in Banach spaces serving as a basis for advanced classical theory of well-posed solvability of inverse problems for the equations concerned. It is worth noting here that plenty of inverse problems are intimately connected or equivalent to nonlocal direct problems for differential equations of some combined type, the new problems arising in momentum theory and the theory of approximation, the new types of linear and nonlinear integral and integro-differential equations of the first and second kinds. In such cases the well-posed solvability of inverse problem entails the new theorems on unique solvability for nonclassical direct problems we have mentioned above. Also, the inverse problems under consideration can be treated as problems from the theory of control of systems with distributed or lumped parameters.

It may happen that the well-developed methods for solving inverse problems permit one to establish, under certain constraints on the input data, the property of having fixed sign for source functions, coefficients and solutions themselves. If so, the inverse problems from control theory are in principal difference with classical problems of this theory. These special inverse problems from control theory could be more appropriately referred to as problems of the "forecast-monitoring" type. The property of having fixed sign for a solution of "forecast-monitoring" problems will be of crucial importance in applications to practical problems of heat and mass transfer, the theory of stochastic diffusion equations, mathematical economics, various problems of ecology, automata control and computerized tomography. In many cases the well-posed solvability of inverse problems is established with the aid of the contraction mapping principle, the Birkhoff–Tarsky principle, the Newton–Kantorovich method and other effective operator methods, making it possible to solve both linear and nonlinear problems following constructive iterative procedures.

The monograph covers the basic types of equations: elliptic, parabolic and hyperbolic. Special emphasis is given to the Navier–Stokes equations as well as to the well-known kinetic equations: Bolzman equation and neutron transport equation.

Being concerned with equations of parabolic type, one of the widespread inverse problems for such equations amounts to the problem of determining an unknown function connected structurally with coefficients of the governing equation. The traditional way of covering this is to absorb some additional information on the behavior of a solution at a fixed point $u(x_0, t) = \varphi(t)$. In this regard, a reasonable interpretation of problems with the overdetermination at a fixed point is approved. The main idea behind this approach is connected with the control over physical processes for a proper choice of parameters, making it possible to provide at this point a required temperature regime. On the other hand, the integral overdetermination

$$\int_{\Omega} u(x,t)\, w(x)\, dx = \varphi(t)\,,$$

where w and φ are the known functions and u is a solution of a given parabolic equation, may also be of help in achieving the final aim and comes first in the body of the book. We have established the new results on uniqueness and solvability. The overwhelming majority of the Russian and foreign researchers dealt with such problems merely for linear and semilinear equations. In this book the solvability of the preceding problem is revealed for a more general class of quasilinear equations. The approximate methods for constructing solutions of inverse problems find a wide range of applications and are gaining increasing popularity.

One more important inverse problem for parabolic equations is the problem with the final overdetermination in which the subsidiary information is the value of a solution at a fixed moment of time: $u(x,T) = \varphi(x)$. Recent years have seen the publication of many works devoted to this canonical problem. Plenty of interesting and profound results from the explicit formulae for solutions in the simplest cases to various sufficient conditions of the unique solvability have been derived for this inverse problem and gradually enriched the theory parallel with these achievements. We offer and develop a new approach in this area based on properties of Fredholm's solvability of inverse problems, whose use permits us to establish the well-known conditions for unique solvability as well.

It is worth noting here that for the first time problems with the integral overdetermination for both parabolic and hyperbolic equations have been completely posed and analysed within the Russian scientific school headed by Prof. Aleksey Prilepko from the Moscow State University. Later the relevant problems were extensively investigated by other researchers including foreign ones. Additional information in such problems is provided in the integral form and admits a physical interpretation as a result of measuring a physical parameter by a perfect sensor. The essense of the matter is that any sensor, due to its finite size, always performs some averaging of a measured parameter over the domain of action.

Similar problems for equations of hyperbolic type emerged in theory and practice. They include symmetric hyperbolic systems of the first order, the wave equation with variable coefficients and the system of equations in elasticity theory. Some conditions for the existence and uniqueness of a solution of problems with the overdetermination at a fixed point and the integral overdetermination have been established.

Let us stress that under the conditions imposed above, problems with the final overdetermination are of rather complicated forms than those in the parabolic case. Simple examples help motivate in the general case the absence of even Fredholm's solvability of inverse problems of hyperbolic type. Nevertheless, the authors have proved Fredholm's solvability and established various sufficient conditions for the existence and uniqueness of a solution for a sufficiently broad class of equations.

Among inverse problems for elliptic equations we are much interested in inverse problems of potential theory relating to the shape and density of an attracting body either from available values of the body's external or internal potentials or from available values of certain functionals of these potentials. In this direction we have proved the theorems on global uniqueness and stability for solutions of the aforementioned problems. Moreover, inverse problems of the simple layer potential and the total potential which do arise in geophysics, cardiology and other areas are discussed. Inverse problems for the Helmholz equation in acoustics and dispersion theory are completely posed and investigated. For more general elliptic equations, problems of finding their sources and coefficients are analysed in the situation when, in addition, some or other accompanying functionals of solutions are specified as compared with related direct problems.

In spite of the fact that the time-dependent system of the Navier–Stokes equations of the dynamics of viscous fluid falls within the category of equations similar to parabolic ones, separate investigations are caused by some specificity of its character. The well-founded choice of the inverse problem statement owes a debt to the surplus information about a solution as supplementary to the initial and boundary conditions. Additional information of this sort is capable of describing, as a rule, the indirect manifestation of the liquid motion characteristics in question and admits plenty of representations. The first careful analysis of an inverse problem for the Navier–Stokes equations was carried out by the authors and provides proper guidelines for deeper study of inverse problems with the overdetermination at a fixed point and the same of the final observation conditions. This book covers fully the problem with a perfect sensor involved, in which the subsidiary information is prescribed in the integral form. Common settings of inverse problems for the Navier–Stokes system are similar to parabolic and hyperbolic equations we have considered so far and may also be treated as control problems relating to viscous liquid motion.

The linearized Bolzman equation and neutron transport equation are viewed in the book as particular cases of kinetic equations. The linearized Bolzman equation describes the evolution of a deviation of the distribution function of a one-particle-rarefied gas from an equilibrium. The statements of inverse problems remain unchanged including the Cauchy problem and the boundary value problem in a bounded domain. The solution existence and solvability are proved. The constraints imposed at the very beginning are satisfied for solid sphere models and power potentials of the particle interaction with angular cut off.

For a boundary value problem the conditions for the boundary data reflect the following situations: the first is connected with the boundary absorption, the second with the thermodynamic equilibrium of the boundary with dissipative particles dispersion on the border. It is worth noting that the characteristics of the boundary being an equilibria in thermodynamics lead to supplementary problems for investigating inverse problems with the final overdetermination, since in this case the linearized collision operator has a nontrivial kernel. Because of this, we restrict ourselves to the stiff interactions only.

Observe that in studying inverse problems for the Bolzman equation we employ the method of differential equations in a Banach space. The same method is adopted for similar problems relating to the neutron transport. Inverse problems for the transport equation are described by inverse problems for a first order abstract differential equation in a Banach space. For this equation the theorems on existence and uniqueness of the inverse problem solution are proved. Conditions for applications of these theorems are easily formulated in terms of the input data of the initial transport equation. The book provides a common setting of inverse problems which will be effectively used in the nuclear reactor theory.

Differential equations in a Banach space with unbounded operator coefficients are given as one possible way of treating partial differential equations. Inverse problems for equations in a Banach space correspond to abstract forms of inverse problems for partial differential equations. The method of differential equations in a Banach space for investigating various inverse problems is quite applicable. Abstract inverse problems are considered for equations of first and second orders, capable of describing inverse problems for partial differential equations.

It should be noted that we restrict ourselves here to abstract inverse problems of two classes: inverse problems in which, in order to solve the differential equation for $u(t)$, it is necessary to know the value of some operator or functional $B\,u(t) = \varphi(t)$ as a function of the argument t, and problems with pointwise overdetermination: $u(T) = u_T$.

For the inverse problems from the first class (problems with evolution overdetermination) we raise the questions of existence and uniqueness of a solution and receive definite answers. Special attention is being paid to the problems in which the operator B possesses some smoothness properties. In context of partial differential equations, abstract inverse problems are suitable to problems with the integral overdetermination, that is, for the problems in which the physical value measurement is carried out by a perfect sensor of finite size. For these problems the questions of existence and uniqueness of strong and weak solutions are examined, and the conditions of differentiability of solutions are established. Under such an approach the emerging equations with constant and variable coefficients are studied.

It is worth emphasizing here that the type of equation plays a key role in the case of equations with variable coefficients and, therefore, its description is carried out separately for parabolic and hyperbolic cases. Linear and semilinear equations arise in the hyperbolic case, while parabolic equations include quasilinear ones as well. Semigroup theory is the basic tool adopted in this book for the first order equations. Since the second order equations may be reduced to the first order equations, we need the relevant elements of the theory of cosine functions.

A systematic study of these problems is a new original trend initiated and well-developed by the authors.

The inverse problems from the second class, from the point of possible applications, lead to problems with the final overdetermination. So far they have been studied mainly for the simplest cases. The authors began their research in a young and growing field and continue with their pupils and colleagues. The equations of first and second orders will be of great interest, but we restrict ourselves here to the linear case only. For second order equations the elliptic and hyperbolic cases are extensively investigated. Among the results obtained we point out sufficient conditions of existence and uniqueness of a solution, necessary and sufficient conditions for the existence of a solution and its uniqueness for equations with a self-adjoint main part and Fredholm's-type solvability conditions. For differential equations in a Hilbert structure inverse problems are studied and conditions of their solvability are established. All the results apply equally well to inverse problems for mathematical physics equations, in particular, for parabolic equations, second order elliptic and hyperbolic equations, the systems of Navier–Stokes and Maxwell equations, symmetric hyperbolic systems, the system of equations from elasticity theory, the Bolzman equation and the neutron transport equation.

The overview of the results obtained and their relative comparison are given in concluding remarks. The book reviews the latest discoveries of the new theory and opens the way to the wealth of applications that it is likely to embrace.

In order to make the book accessible not only to specialists, but also to students and engineers, we give a complete account of definitions and notions and present a number of relevant topics from other branches of mathematics.

It is to be hoped that the publication of this monograph will stimulate further research in many countries as we face the challenge of the next decade.

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Contents

METHODS FOR SOLVING INVERSE PROBLEMS IN MATHEMATICAL PHYSICS

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Chapter 1

Inverse Problems for Equations of Parabolic Type

1.1 Preliminaries

In this section we give the basic notations and notions and present also a number of relevant topics from functional analysis and the general theory of partial differential equations of parabolic type. For more detail we recommend the well-known monograph by Ladyzhenskaya et al. (1968).

The symbol Ω is used for a bounded domain in the Euclidean space $\mathbf{R}^n$, $x = (x_1, \dots, x_n)$ denotes an arbitrary point in it. Let us denote by Q_T a cylinder $\Omega \times (0, T)$ consisting of all points $(x, t) \in \mathbf{R}^{n+1}$ with $x \in \Omega$ and $t \in (0, T)$.

Let us agree to assume that the symbol $\partial\Omega$ is used for the boundary of the domain Ω and S_T denotes the lateral area of Q_T. More specifically, S_T is the set $\partial\Omega \times [0, T] \in \mathbf{R}^{n+1}$ consisting of all points (x, t) with $x \in \partial\Omega$ and $t \in [0, T]$.

In a limited number of cases the boundary of the domain Ω is supposed to have certain smoothness properties. As a rule, we confine our attention to domains Ω possessing piecewise-smooth boundaries with nonzero interior angles whose closure $\bar{\Omega}$ can be represented in the form $\bar{\Omega} = \cup_{k=1}^{m} \bar{\Omega}_k$ for

$\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, and every $\bar{\Omega}_k$ can homeomorphically be mapped onto a unit ball (a unit cube) with the aid of functions $\psi_i^k(x)$, $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$, with the Lipschitz property and the Jacobians of the transformations

$$\left| \frac{\partial \psi^k}{\partial x} \right|$$

are bounded from below by a positive constant.

We say that the boundary $\partial\Omega$ is of class C^l, $l \geq 1$, if there exists a number $\rho > 0$ such that the intersection of $\partial\Omega$ and the ball B_ρ of radius ρ with center at an arbitrary point $x^0 \in \partial\Omega$ is a connected surface area which can be expressed in a local frame of reference $(\xi_1, \xi_2, \ldots, \xi_n)$ with origin at the point x^0 by the equation $\xi_n = \omega(\xi_1, \ldots, \xi_{n-1})$, where $\omega(\xi_1, \ldots, \xi_{n-1})$ is a function of class C^l in the region $\bar{D}$ constituting the projection of D onto the plane $\xi_n = 0$. We will speak below about the class $C^l(\bar{D})$.

We expound certain exploratory devices for investigating inverse problems by using several well-known inequalities. In this branch of mathematics common practice involves, for example, the **Cauchy inequality**

$$\left| \sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \eta_j \right| \leq \left(\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \right)^{1/2} \left(\sum_{i,j=1}^{n} a_{ij}\, \eta_i\, \eta_j \right)^{1/2},$$

which is valid for an arbitrary nonnegative **quadratic form** $a_{ij}\, \xi_i\, \eta_j$ with $a_{ij} = a_{ji}$ and arbitrary real numbers $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$. This is especially true of **Young's inequality**

$$(1.1.1) \qquad a\, b \leq \frac{1}{p}\, \delta^p\, a^p + \frac{1}{q}\, \delta^{-q}\, b^q, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

which is more general than the preceding and is valid for any positive a, b, δ and p, $q > 1$.

In dealing with measurable functions $u_k(x)$ defined in Ω we will use also **Hölder's inequality**

$$(1.1.2) \qquad \left| \int_\Omega \prod_{k=1}^{s} u_k(x)\, dx \right| \leq \prod_{k=1}^{s} \left(\int_\Omega |\, u_k(x)\,|^{\lambda_k}\, dx \right)^{1/\lambda_k},$$

$$\lambda_k \geq 1, \qquad \sum_{k=1}^{s} \lambda_k^{-1} = 1.$$

In the particular case where $s = 2$ and $\lambda_1 = \lambda_2 = 2$ inequality (1.1.2) is known as the **Cauchy–Schwartz inequality**.

Throughout this section, we operate in certain functional spaces, the elements of which are defined in Ω and Q_T. We list below some of them. In what follows all the functions and quantities will be real unless the contrary is explicitly stated.

The spaces $L_p(\Omega)$, $1 \le p < \infty$, being the most familiar ones, come first. They are introduced as the Banach spaces consisting of all measurable functions in Ω that are p-integrable over that set. The norm of the space $L_p(\Omega)$ is defined by

$$\| u \|_{p,\Omega} = \Big(\int_\Omega | u(x) |^p \Big)^{1/p} .$$

It is worth noting here that in this chapter the notions of measurability and integrability are understood in the sense of Lebesgue. The elements of $L_p(\Omega)$ are the classes of equivalent functions on Ω.

When $p = \infty$ the space $L_\infty(\Omega)$ comprises all measurable functions in Ω that are essentially bounded having

$$\| u \|_{\infty,\Omega} = \operatorname*{ess\,sup}_\Omega | u(x) | .$$

We obtain for $p = 2$ the **Hilbert space** $L_2(\Omega)$ if the scalar product in that space is defined by

$$(u, v) = \int_\Omega u(x)\, v(x)\; dx .$$

The **Sobolev spaces** $W_p^l(\Omega)$, where l is a positive integer, $1 \le p < \infty$, consists of all functions from $L_p(\Omega)$ having all generalized derivatives of the first l orders that are p-integrable over Ω. The norm of the space $W_p^l(\Omega)$ is defined by

$$\| u \|_{p,\Omega}^{(l)} = \Bigg(\sum_{k=0}^{l} \sum_{|\alpha|=k} \| D_x^\alpha u \|_{p,\Omega}^p \Bigg)^{1/p} ,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a **multiindex**, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$,

$$D_x^\alpha u \equiv \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

and $\sum_{|\alpha|=k}$ denotes summation over all possible αth derivatives of u. **Generalized derivatives** are understood in the sense of Sobolev (see the definitions in Sobolev (1988)). For $\alpha = 1$ and $\alpha = 2$ we will write, as usual, u_x and u_{xx}, respectively, instead of $D_x u$ and $D_x^2 u$. This should not cause any confusion.

It is fairly common to define the space $\overset{\circ}{W}{}_2^1(\Omega)$ as a subspace of $W_2^1(\Omega)$ in which the set of all functions in Ω that are infinite differentiable and have compact support is dense. The function $u(x)$ has compact support in a bounded domain Ω if $u(x)$ is nonzero only in a bounded subdomain Ω' of the domain Ω lying at a positive distance from the boundary of Ω.

When working in **Hölder's spaces** $C^h(\bar{\Omega})$ and $C^{l+h}(\bar{\Omega})$, we will assume that the boundary of Ω is smooth. A function $u(x)$ is said to satisfy Hölder's condition with exponent h, $0 < h < 1$, and **Hölder's constant** $H_{\Omega}^h(u)$ in $\bar{\Omega}$ if

$$\sup_{x,\, x' \in \bar{\Omega}} \frac{|\, u(x) - u(x') \,|}{|\, x - x' \,|^h} \equiv H_{\Omega}^h(u) < \infty .$$

By definition, $C^h(\bar{\Omega})$ is a Banach space, the elements of which are continuous on $\bar{\Omega}$ functions u having bounded

$$|\, u \,|_{\Omega}^{(h)} = \sup_{\Omega} \,|\, u \,| + H_{\Omega}^h(u) .$$

In turn, $C^{l+h}(\bar{\Omega})$, where l is a positive integer, can be treated as a Banach space consisting of all differentiable functions with continuous derivatives of the first l orders and a bounded norm of the form

$$|\, u \,|_{\Omega}^{(l+h)} = \sum_{k=0}^{l} \sum_{|\,\alpha\,|=k} \sup_{\Omega} \,|\, D_x^{\alpha} u \,| + \sum_{|\,\alpha\,|=l} H_{\Omega}^h(D_x^l u) .$$

The functions depending on both the space and time variables with dissimilar differential properties on x and t are much involved in solving nonstationary problems of mathematical physics.

Furthermore, $L_{p,\,q}(Q_T)$, $1 \le p, q < \infty$, is a Banach space consisting of all measurable functions u having bounded

$$\|\, u \,\|_{p,\,q,\,Q_T} = \left[\int_0^T \left(\int_{\Omega} |\, u \,|^p \, dx \right)^{q/p} dt \right]^{1/q} .$$

The **Sobolev space** $W_p^{l_1,\, l_2}(Q_T)$, $p \ge 1$, with positive integers $l_i \ge 0$, $i = 1, 2$, is defined as a Banach space of all functions u belonging to the space $L_p(Q_T)$ along with their weak x-derivatives of the first l_1 orders and t-derivatives of the first l_2 orders. The norm on that space is defined by

$$\|\, u \,\|_{p,\, Q_T}^{(l_1,\, l_2)} = \left[\int_{Q_T} \left(\sum_{k=0}^{l_1} \sum_{|\,\alpha\,|=k} |\, D_x^{\alpha} \, u \,|^p + \sum_{k=1}^{l_2} |\, D_t^k \, u \,|^p \right) dx \; dt \right]^{1/p} .$$

The symbol $\overset{\circ}{W}{}_2^{2,1}(Q_T)$ is used for a subspace of $W_2^{2,1}(Q_T)$ in which the set of all smooth functions in Q_T that vanish on S_T is dense.

The space $C^{2+\alpha,\,1+\alpha/2}(Q_T)$, $0<\alpha<1$, is a Banach space of all functions u in Q_T that are continuous on $\bar{Q}_T$ and that possess smooth x-derivatives up to and including the second order and t-derivatives of the first order. In so doing, the functions themselves and their derivatives depend continuously on x and t with exponents α and $\alpha/2$, respectively. The norm on that space is defined by

$$|u|_{Q_T}^{2+\alpha,\,1+\alpha/2}=\sum_{k=0}^{2}\sum_{|\alpha|=k}\sup_{Q_T}|D_x^\alpha u|+\sup_{Q_T}|D_t u|$$

$$+\sum_{|\alpha|=2}\sup_{(x,t),(x',t)\in Q_T}|D_x^\alpha u(x,t)-D_x^\alpha u(x',t)|/|x-x'|^\alpha$$

$$+\sup_{(x,t),(x,t')\in Q_T}|D_t u(x,t)-D_t u(x,t')|/|t-t'|^{\alpha/2}$$

$$+\sum_{|\alpha|=2}\sup_{(x,t),(x,t')\in Q_T}|D_x^\alpha u(x,t)-D_x^\alpha u(x,t')|/|t-t'|^{\alpha/2}$$

$$+\sup_{(x,t),(x,t')\in Q_T}|D_t u(x,t)-D_t u(x',t)|/|x-x'|^\alpha .$$

In specific cases the function u depending on x and t will be treated as an element of the space $V_2^{1,0}(Q_T)$ comprising all the elements of $W_2^{1,0}(Q_T)$ that are continuous with respect to t in the $L_2(\Omega)$-norm having finite

$$\uparrow u \uparrow_{Q_T}=\sup_{[0,T]}\|u(\cdot,u)\|_{2,\Omega}+\|u_x\|_{2,Q_T},$$

where $u_x=(u_{x_1},\dots,u_{x_n})$ and $u_x^2=|u_x|^2$. The meaning of the continuity of the function $u(\cdot,t)$ with respect to t in the $L_2(\Omega)$-norm is that

$$\|u(\cdot,t+\Delta t)-u(\cdot,t)\|_{2,\Omega}\longrightarrow 0 \quad\text{as}\quad \Delta t\to 0 .$$

For later use, the symbol $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ will appear once we agree to consider only those elements of $V_2^{1,0}(Q_T)$ that vanish on S_T.

In a number of problems a function depending on x and t can be viewed as a function of the argument t with values from a Banach space over Ω. For example, $L_q(0,T;W_p^l(\Omega))$ is a set of all functions $u(\cdot,t)$ on $(0,T)$ with values in $W_p^l(\Omega)$ and norm

$$\|u\|_{L_q(0,T;W_p^l(\Omega))}=\left\{\int_0^T\left[\|u(\cdot,t)\|_{p,\Omega}^{(l)}\right]^q dt\right\}^{1/q}.$$

Obviously, the spaces $L_q(0,T;\, L_p(\Omega))$ and $L_{p,\,q}(Q_T)$ can be identified in a natural way. In a similar line, the space

$$C([0,T];\, W_p^l(\Omega))$$

comprises all continuous functions on $[0,T]$ with values in $W_p^l(\Omega)$. We obtain the Banach space $C([0,T]; W_p^l(\Omega))$ if the norm on it is defined by

$$\| u \|_{C([0,T];\, W_p^l(\Omega))} = \sup_{[0,\,T]} \| u(\,\cdot\,,t) \|_{p,\,\Omega}^{(1)} .$$

We quote below some results concerning **Sobolev's embedding theory** and relevant inequalities which will be used in the sequel.

Recall that the **Poincare–Friedrichs inequality**

$$\int_\Omega u^2(x)\, dx \le c_1(\Omega) \int_\Omega | u_x |^2(x)\, dx \tag{1.1.3}$$

holds true for all the functions u from $\overset{\circ}{W}{}_2^1(\Omega)$, where Ω is a bounded domain in the space $\mathbf{R}^n$. The constant $c_1(\Omega)$ depending only on the domain Ω is bounded by the value $4\,(\operatorname{diam}\Omega)^2$.

Theorem 1.1.1 *Let Ω be a bounded domain in the space $\mathbf{R}^n$ with the piecewise smooth boundary $\partial\Omega$ and let S_r be an intersection of Ω with any r-dimensional hypersurface, $r \le n$ (in particular, if $r = n$ then $S_r \equiv \Omega$; if $r = n-1$ we agree to consider $\partial\Omega$ as S_r). Then for any function $u \in W_p^l(\Omega)$, where $l \ge 1$ is a positive integer and $p > 1$, the following assertions are valid:*

(a) *for $n > pl$ and $r > n - pl$ there exists a trace of u on S_T belonging to the space $L_q(S_r)$ with any finite $q \le pr/(n-pl)$ and the estimate is true:*

$$\| u \|_{q,\, S_r} \le c\, \| u \|_{p,\,\Omega}^{(1)} . \tag{1.1.4}$$

For $q < pr/(n-pl)$ the operator embedding $W_p^l(\Omega)$ into $L_q(S_r)$ is completely continuous;

(b) *for $n = pl$ the assertion of item (a) holds with any $q < \infty$;*

(c) *for $n < pl$ the function u is Hölder's continuous and belongs to the class $C^{k+h}(\bar\Omega)$, where $k = l - 1 - [n/p]$ and $h = 1 + [n/p] - n/p$ if n/p is not integer and $\forall h < 1$ if n/p is integer. In that case the estimate*

$$| u |_\Omega^{(k+h)} \le c\, \| u \|_{p,\,\Omega}^{(1)} \tag{1.1.5}$$

is valid (here $[n/p]$ denotes, as usual, the integral part of n/p).

Notice that the constants c arising from (1.1.4)–(1.1.5) depend only on n, p, l, r, q, S_r and Ω and do not depend on the function u. The proof of Theorem 1.1.1 can be found in Sobolev (1988).

In establishing some subsequent results we will rely on **Rellich's theorem**, whose precise formulation is due to Courant and Hilbert (1962).

Theorem 1.1.2 *If Ω is a bounded domain, then $\overset{\circ}{W}{}_2^1(\Omega)$ is compactly embedded into the space $L_2(\Omega)$, that is, a set of elements $\{u_\alpha\}$ of the space $\overset{\circ}{W}{}_2^1(\Omega)$ with uniformly bounded norms is compact in the space $L_2(\Omega)$.*

Much progress in solving inverse boundary value problems has been achieved by serious developments in the general theory of elliptic and parabolic partial differential equations. The reader can find deep and diverse results of this theory in Ladyzhenskaya (1973), Ladyzhenskaya and Uraltseva (1968), Friedman (1964), Gilbarg and Trudinger (1983), Berezanskij (1968). Several facts are known earlier and quoted here without proofs, the others are accompanied by explanations or proofs. Some of them were discovered and proven in recent years in connection with the investigation of the series of questions that we now answer. Being of independent value although, they are used in the present book only as part of the auxiliary mathematical apparatus. The theorems concerned will be formulated here in a common setting capable of describing inverse problems of interest that make it possible to draw fairly accurate outlines of advanced theory.

Let Ω be a bounded domain in the space $\mathbf{R}^n$ with boundary $\partial\Omega$ of class C^2. In the domain Ω of such a kind we consider the **Dirichlet boundary value (direct) problem** for the elliptic equation of the second order

$$(Lu)(x) = f(x)\,, \qquad x \in \Omega\,, \tag{1.1.6}$$

$$u(x) = 0\,, \qquad x \in \partial\Omega\,, \tag{1.1.7}$$

where L is an **elliptic differential operator** of the type

$$Lu = \sum_{i,j=1}^{n} \big(a_{ij}(x) u_{x_i}\big)_{x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u\,, \quad a_{ij} = a_{ji}\,, \tag{1.1.8}$$

which is assumed to be **uniformly elliptic** for every $x \in \bar{\Omega}$ in the sense of the following conditions:

$$0 < \nu \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \le \mu \sum_{j=1}^{n} \xi_j^2 \tag{1.1.9}$$

with certain positive constants μ and ν and arbitrary real numbers $\xi_1, \ldots, \xi_n$. The left inequality (1.1.9) reflects the ellipticity property and the right one means that the coefficients a_{ij} are bounded.

In trying to solve the direct problem posed above we look for the function u by regarding the coefficients of the operator L, the source term f and the domain Ω to be known in advance.

Theorem 1.1.3 *Let the operator L satisfy* (1.1.8)–(1.1.9), $a_{ij} \in C(\bar{\Omega})$, $\frac{\partial a_{ij}}{\partial x_i} \in C(\bar{\Omega})$, $b_i \in L_\infty$ *and* $c \leq 0$ *almost everywhere (a.e.) in* Ω. *If* $f \in L_p(\Omega)$, $1 < p < \infty$, *then the Dirichlet problem* (1.1.6)–(1.1.7) *has a solution* $u \in W_p^2(\Omega)$, *this solution is unique in the indicated class of functions and obeys the estimate*

$$\| u \|_{p,\Omega}^{(2)} \leq c^* \| f \|_{p,\Omega} , \tag{1.1.10}$$

where the constant c^* is independent of u.

A similar result concerning the unique solvability can be obtained regardless of the sign of the coefficient c. However, in this case the coefficients of the operator L should satisfy some additional restrictions such as, for example, the inequality

$$\frac{\nu}{2\, c_1(\Omega)} - \left(\| b \|_{\infty,\Omega} + \frac{1}{2\,\nu} \, \| c \|_{\infty,\Omega}^2 \right) > 0 , \tag{1.1.11}$$

where $b = \Big[\sum_{i=1}^n b_i^2(x) \Big]^{1/2}$ and $c_1(\Omega)$ is the same constant as in (1.1.3).

For further motivations we cite here the weak principle of maximum for elliptic equations following the monograph of Gilbarg and Trudinger (1983), p. 170–173. To facilitate understanding, it will be convenient to introduce some terminology which will be needed in subsequent reasonings. A function $u \in W_2^1(\Omega)$ is said to satisfy the inequality $u \leq 0$ on $\partial\Omega$ if its positive part $u^+ = \max\{u, 0\}$ belongs to $\overset{\circ}{W}{}_2^1(\Omega)$. This definition permits us to involve inequalities of other types on $\partial\Omega$. Namely, $u \geq 0$ on $\partial\Omega$ if $-u \leq 0$ on $\partial\Omega$; functions u and v from $W_2^1(\Omega)$ satisfy the inequality $u \leq v$ on $\partial\Omega$ if $u - v \leq 0$ on $\partial\Omega$;

$$\sup_{\partial\Omega} u = \inf \{ k \in \mathbf{R} \colon \ u \leq k \text{ on } \partial\Omega \} .$$

We say that a function u satisfies the inequality $Lu \geq 0$ in Ω in the weak or generalized sense if

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij}(x)\, u_{x_i}\, v_{x_j} - \sum_{i=1}^n b_i(x)\, u_{x_i}\, v - c(x)\, u\, v \Big) \, dx \leq 0$$

for all nonnegative functions $v \in C^1(\Omega)$ such that $v(x) = 0$ for $x \in \partial\Omega$.

Theorem 1.1.4 (the weak principle of maximum) *Let the conditions of Theorem* 1.1.3 *hold for the operator* L *and let a function* $u \in W_2^1(\Omega)$ *satisfy the inequality* $Lu \geq 0$ *in* Ω *in a weak sense. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ .$$

Corollary 1.1.1 *Let the operator* L *be in line with the premises of Theorem* 1.1.3 *and let a function* $\varphi \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ *comply with the conditions*

$$\varphi(x) \geq 0 \quad a.e.\ in \quad \Omega \quad and \quad \varphi(x) \not\equiv \text{const} .$$

Then there exists a measurable set $\Omega' \subset \Omega$ *with*

$$\text{mes}_n \Omega' > 0$$

such that $L\varphi < 0$ *in* Ω'.

Proof On the contrary, let $L\varphi \geq 0$ in Ω. If so, the theorem yields either $\varphi \leq 0$ in Ω or $\varphi = \text{const}$ in Ω. But this contradicts the hypotheses of Corollary 1.1.1 and proves the current corollary. ■

Corollary 1.1.2 *Let the operator* L *meet the requirements of Theorem* 1.1.3 *and let a function* $\varphi \in W_2^2(Q_T) \cap \overset{\circ}{W}{}_2^1(\Omega)$ *follow the conditions*

$$\varphi(x) \geq 0 \quad a.e.\ in \quad \Omega \quad and \quad L\varphi(x) \not\equiv \text{const} \quad in \quad \Omega.$$

Then there exists a measurable set $\Omega' \subset \Omega$ *with*

$$\text{mes}_n \Omega' > 0$$

such that $L\varphi < 0$ *in* Ω'.

Proof Since $L\varphi \not\equiv 0$, we have $\varphi \not\equiv 0$, giving either $\varphi \equiv \text{const} > 0$ or $\varphi \not\equiv \text{const}$. If $\varphi \equiv \text{const} > 0$, then

$$(L\varphi)(x) \equiv c(x)\,\varphi(x)$$

and the above assertion is simple to follow. For $\varphi \not\equiv \text{const}$ applying Corollary 1.1.1 leads to the desired assertion. ■

For the purposes of the present chapter we refer to the **parabolic equation**

$$(1.1.12) \qquad u_t(x,t) - (Lu)(x,t) = F(x,t)\,, \qquad (x,t) \in Q_T = \Omega \times (0,T)\,,$$

supplied by the initial and boundary conditions

$$(1.1.13) \qquad u(x,0) = a(x)\,, \qquad x \in \Omega\,,$$

$$(1.1.14) \qquad u(x,t) = 0\,, \qquad (x,t) \in S_T \equiv \partial\Omega \times [0,T]\,,$$

where the operator L is supposed to be uniformly elliptic. The meaning of this property is that we should have

$$(1.1.15) \qquad \begin{aligned} Lu \equiv{} & \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(A_{ij}(x)\,u_{x_i}\Big) \\ & + \sum_{i=1}^{n} B_i(x)\,u_{x_i} + C(x)\,u\,, \\ & A_{ij} = A_{ji}\,, \quad 0 < \nu \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} A_{ij}(x)\,\xi_i\,\xi_j \le \mu \sum_{i=1}^{n} \xi_i^2\,, \\ & \nu,\ \mu \equiv \text{const} > 0\,. \end{aligned}$$

In what follows we impose for the coefficients of the operator L the following constraints:

$$(1.1.16) \quad A_{ij} \in C(\bar\Omega)\,, \quad \frac{\partial}{\partial x_j} A_{ij} \in C(\bar\Omega)\,, \quad B_i \in L_\infty(\Omega)\,, \quad C \in L_\infty(\Omega)\,.$$

The direct problem for equation (1.1.12) consists of finding a solution u of the governing equation subject to the initial condition (1.1.13) and the boundary condition (1.1.14) when operating with the functions F and a, the coefficients of the operator L and the domain $\Omega \times (0,T)$.

Definition 1.1.1 *A function u is said to be a solution of the direct problem* (1.1.12)–(1.1.14) *from the class $W_2^{2,1}(Q_T)$ if $u \in W_{2,0}^{2,1}(Q_T)$ and relations* (1.1.12)–(1.1.14) *are satisfied almost everywhere in the corresponding domains.*

Theorem 1.1.5 *Let the coefficients of the operator L satisfy* (1.1.15)–(1.1.16) *and let $F \in L_2(Q_T)$ and $a \in \overset{\circ}{W}{}_2^1(\Omega)$. Then the direct problem*

(1.1.12)–(1.1.14) *has a solution* $u \in W_{2,0}^{2,1}(Q_T)$, *this solution is unique in the indicated class of functions and the following estimate is valid:*

$$\| u \|_{2,Q_T}^{(2,1)} \leq c^* \left(\| F \|_{2,Q_T} + \| a \|_{2,\Omega}^{(1)} \right), \tag{1.1.17}$$

where the constant c^* *does not depend on* u.

In subsequent studies of inverse problems some propositions on solvability of the direct problem (1.1.12)–(1.1.14) in the "**energy**" **space** $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ will serve as a necessary background for important conclusions.

Definition 1.1.2 *A function* u *is said to be a* **weak solution** *of the direct problem* (1.1.12)–(1.1.14) *from the class* $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ *if* $u \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ *and the system* (1.1.12)–(1.1.14) *is satisfied in the sense of the following integral identity:*

$$\begin{aligned} &\int_0^t \int_\Omega \Big(- u\,\Phi_\tau + \sum_{i,j=1}^n A_{ij}\, u_{x_j}\, \Phi_{x_i} \\ &\qquad - \sum_{i=1}^n B_i\, u_{x_i}\, \Phi - C\, u\, \Phi \Big)\, dx\, d\tau \\ &\qquad + \int_\Omega u(x,t)\, \Phi(x,t)\, dx \\ &\qquad - \int_\Omega a(x)\, \Phi(x,0)\, dx \\ &\qquad = \int_0^t \int_\Omega F\, \Phi\, dx\, d\tau, \qquad 0 \leq t \leq T, \end{aligned} \tag{1.1.18}$$

where Φ *is an arbitrary element of* $W_2^{1,1}(Q_T)$ *such that* $\Phi(x,t) = 0$ *for* $(x,t) \in S_T$.

The following result is an excellent start in this direction.

Theorem 1.1.6 *Let the coefficients of the operator* L *satisfy* (1.1.15)–(1.1.16) *and let* $F \in L_{2,1}(Q_T)$ *and* $a \in L_2(\Omega)$. *Then the direct problem* (1.1.11)–(1.1.14) *has a weak solution* $u \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$, *this solution is unique in the indicated class of functions and the energy balance equation is valid:*

$$\frac{1}{2}\, \| u(\,\cdot\,,t) \|_{2,\Omega}^2 + \int_0^t \int_\Omega \Big(\sum_{i,j=1}^n A_{ij}\, u_{x_j}\, u_{x_i} \tag{1.1.19}$$

$$- \sum_{i=1}^{n} B_i \, u_{x_i} \, u - C u^2 \Big) \, dx \, d\tau$$

$$= \frac{1}{2} \, \| \, a \, \|_{2,\Omega}^2 + \int_0^t \int_\Omega F \, u \, dx \, d\tau \,,$$

$$0 \le t \le T \,.$$

Differential properties of a solution u ensured by Theorem 1.1.6 are revealed in the following proposition.

Lemma 1.1.1 *If all the conditions of Theorem* 1.1.6 *are put together with* $F \in L_2(Q_T)$, *then*

$$u \in W_{2,0}^{2,1}(\Omega \times (\varepsilon, T)) \bigcap C([\varepsilon, T], \overset{\circ}{W}{}_2^1(\Omega))$$

for any $\varepsilon \in (0, T)$.

For the further development we initiate the derivation of some estimates. If you wish to explore this more deeply, you might find it helpful first to establish the estimates for solutions $u \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ of the system (1.1.12). These are aimed to carry out careful analysis in the sequel.

Suppose that the conditions of Theorem 1.1.6 and Lemma 1.1.1 are satisfied. With this in mind, we are going to show that any solution of (1.1.12)–(1.1.14) from $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ admits for $0 \le t \le T$ the estimate

$$\| \, u(\,\cdot\,, t) \, \|_{2,\Omega} \le \exp\{-\alpha \, t\} \, \| \, a \, \|_{2,\Omega} + \int_0^t \exp\{-\alpha \, (t - \tau)\} \, \| \, F(\,\cdot\,, \tau) \, \|_{2,\Omega} \, d\tau \,, \tag{1.1.20}$$

where
$$\alpha = \left[\, \frac{\nu}{2 \, c_1(\Omega)} - \left(\mu_1 + \frac{\mu_1^2}{2 \, \nu} \right) \right] ,$$

$$\mu_1 = \max \left\{ \operatorname*{ess\,sup}_{\Omega} \, | \, C(x) \, | \,, \; \operatorname*{ess\,sup}_{\Omega} \, \Big[\, \sum_{i=1}^{n} B_i^2(x) \Big]^{1/2} \right\}$$

and $c_1(\Omega)$ is the constant from the Poincare–Friedrichs inequality (1.1.3). Observe that we imposed no restriction on the sign of the constant α.

At the next stage, holding a number ε from the interval $(0,T)$ fixed and taking $t=\varepsilon$, we appeal to identity (1.1.18). After subtracting the resulting expression from (1.1.18) we get

$$
\begin{aligned}
(1.1.21)\qquad & \int_{\varepsilon}^{t}\int_{\Omega}\Big(-u\,\Phi_\tau+\sum_{i,j=1}^{n}A_{ij}\,u_{x_j}\,\Phi_{x_i} \\
&\qquad -\sum_{i=1}^{n}B_i\,u_{x_i}\,\Phi-C(x)\,u\,\Phi\Big)\,dx\,d\tau \\
&\qquad +\int_{\Omega}u(x,t)\,\Phi(x,t)\,dx \\
&\qquad -\int_{\Omega}u(x,\varepsilon)\,\Phi(x,\varepsilon)\,dx \\
&\qquad =\int_{\varepsilon}^{t}\int_{\Omega}F\,\Phi\,dx\,d\tau\,,\qquad 0<\varepsilon\le t\le T\,,
\end{aligned}
$$

where Φ is an arbitrary element of $W_2^{1,1}(Q_T)$ that vanish on S_T. Due to the differential properties of the function u established in Lemma 1.1.1 we can rewrite (1.1.21) for $0<\varepsilon\le t\le T$ as

$$
(1.1.22)\qquad \int_{\varepsilon}^{t}\int_{\Omega}\big(u_\tau-Lu\big)\,\Phi\,dx\,d\tau=\int_{\varepsilon}^{t}\int_{\Omega}F\,\Phi\,dx\,d\tau\,.
$$

It is important for us that the preceding relation occurs for any $\Phi\in W_2^{1,1}(\Omega\times(\varepsilon,T))$ vanishing on $\partial\Omega\times[\varepsilon,T]$.

Let $\eta(t)$ be an arbitrary function from the space $\overset{\circ}{C}{}^{\infty}([\varepsilon,T])$. Obviously, the function $\Phi=u(x,t)\,\eta(t)$ belongs to the class of all admissible functions subject to relation (1.1.22). Upon substituting $\Phi=u(x,t)\,\eta(t)$ into (1.1.22) we arrive at

$$
\begin{aligned}
(1.1.23)\qquad & \int_{\varepsilon}^{t}\bigg[\int_{\Omega}\big(u_\tau-Lu\big)\,u\,dx\bigg]\,\eta(\tau)\,d\tau \\
&\qquad =\int_{\varepsilon}^{t}\bigg[\int_{\Omega}F\,u\,dx\bigg]\,\eta(\tau)\,d\tau\,,\qquad 0<\varepsilon\le t\le T\,.
\end{aligned}
$$

It is worth noting here that $\overset{\circ}{C}{}^{\infty}([\varepsilon, T])$ is dense in the space $L_2([\varepsilon, T])$. By minor manipulations with relation (1.1.23) we are led to

$$
(1.1.24)\quad \frac{1}{2}\,\frac{d}{dt}\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega}^2 + \int\limits_{\Omega} \sum_{i,j=1}^{n} A_{ij}\, u_{x_j}\, u_{x_i}\, dx
$$

$$
= \int\limits_{\Omega} \Big(\sum_{i=1}^{n} B_i(x)\, u_{x_i}\, u + C(x)\, u^2 \Big)\, dx
$$

$$
+ \int\limits_{\Omega} F(x,t)\, u\, dx\,, \quad 0 < \varepsilon \le t \le T\,.
$$

By successively applying (1.1.2) and (1.1.15) to (1.1.24) we are led to

$$
(1.1.25)\quad \frac{1}{2}\,\frac{d}{dt}\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega}^2 + \nu\,\|\,u_x(\,\cdot\,,t)\,\|_{2,\Omega}^2
$$

$$
\le \mu_1\,\|\,u_x(\,\cdot\,,t)\,\|_{2,\Omega}\,\cdot\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega}
$$

$$
+\,\mu_1\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega}^2 + \|\,F(\,\cdot\,,t)\,\|_{2,\Omega}\,\cdot\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega}\,,
$$

$$
0 < \varepsilon \le t \le T\,,
$$

where

$$
\mu_1 = \max\Big\{ \operatorname*{ess\,sup}_{\Omega}\,|\,C(x)|\,,\ \operatorname*{ess\,sup}_{\Omega}\Big[\sum_{i=1}^{n} B_i^2(x)\Big]^{1/2} \Big\}\,.
$$

The estimation of the first term on the right-hand side of (1.1.25) can be done relying on Young's inequality with $p = q = 2$ and $\delta^2 = \nu/\mu_1$, whose use permits us to establish the relation

$$
(1.1.26)\quad \frac{1}{2}\,\frac{d}{dt}\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega}^2 + \frac{\nu}{2}\,\|\,u_x(\,\cdot\,,t)\,\|_{2,\Omega}^2
$$

$$
\le \left(\mu_1 + \frac{\mu_1^2}{2\,\nu} \right) \|\,u(\,\cdot\,,t)\,\|_{2,\Omega}^2
$$

$$
+\,\|\,F(\,\cdot\,,t)\,\|_{2,\Omega}\,\cdot\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega}\,.
$$

Applying the Poincare–Friedrichs inequality to the second term on the right-hand side of (1.1.26) yields

$$
(1.1.27)\quad \frac{d}{dt}\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega} + \alpha\,\|\,u(\,\cdot\,,t)\,\|_{2,\Omega} \le \|\,F(\,\cdot\,,t)\,\|_{2,\Omega}\,,
$$

$$
0 < \varepsilon \le t \le T\,,
$$

where

$$\alpha = \Big[\frac{\nu}{2\,c_1(\Omega)} - \Big(\mu_1 + \frac{\mu_1^2}{2\,\nu}\Big)\Big]$$

and $c_1(\Omega)$ is the same constant as in (1.1.3).

Let us multiply both sides of (1.1.27) by $\exp\{\alpha t\}$ and integrate then the resulting expression from ε to t. Further passage to the limit as $\varepsilon \to 0+$ leads to the desired estimate (1.1.20).

The second estimate for $u \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ in question follows directly from (1.1.20):

$$(1.1.28) \quad \sup_{[0,\,t]} \| u(\,\cdot\,,\tau) \|_{2,\Omega} \le c_2(t) \left(\| a \|_{2,\Omega} + \int_0^t \| F(\,\cdot\,,\tau) \|_{2,\Omega} \, d\tau \right), \quad 0 \le t \le T,$$

where

$$c_2(t) = \exp\{|\,\alpha\,|\,t\}.$$

In the derivation of an alternative estimate we have to integrate relation (1.1.26) from ε to t with respect to t and afterwards pass to the limit as $\varepsilon \to 0+$. The outcome of this is

$$(1.1.29) \quad \frac{\nu}{2} \int_0^t \| u_x(\,\cdot\,,\tau) \|_{2,\Omega}^2 \, d\tau \le \frac{1}{2} \| u(\,\cdot\,,0) \|_{2,\Omega} \times \sup_{[0,\,t]} \| u(\,\cdot\,,\tau) \|_{2,\Omega} + \Big(\mu_1 + \frac{\mu_1^2}{2\,\nu}\Big)\, t \sup_{[0,\,t]} \| u(\,\cdot\,,\tau) \|_{2,\Omega}^2 + \sup_{[0,\,t]} \| u(\,\cdot\,,\tau) \|_{2,\Omega} \times \int_0^t \| F(\,\cdot\,,t) \|_{2,\Omega} \, d\tau, \quad 0 \le t \le T.$$

Substituting estimate (1.1.28) into (1.1.29) yields that any weak solution

$u \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ of the direct problem (1.1.12)–(1.1.14) satisfies the estimate

$$(1.1.30)\qquad \int_0^t \| u_x(\cdot,\tau)\|_{2,\Omega}^2 \, d\tau \le c_3(t) \left[\| a\|_{2,\Omega} + 2\int_0^t \| F(\cdot,t)\|_{2,\Omega} \, d\tau \right],$$

$$0 \le t \le T,$$

where

$$c_3(t) = \frac{1}{\nu}\, c_2(t) \left[1 + 2t\, c_2(t) \Big(\mu_1 + \frac{\mu_1^2}{2\nu} \Big) \right].$$

The next goal of our studies is to obtain the estimate of $\| u_x(\cdot,t)\|_{2,\Omega}$ for the solutions asserted by Theorem 1.1.6 in the case when $t \in (0, T]$. Before giving further motivations, one thing is worth noting. As stated in Lemma 1.1.1, under some additional restrictions on the input data any solution u of the direct problem (1.1.12)–(1.1.14) from $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ belongs to the space $W_{2,0}^{2,1}(\Omega \times (\varepsilon, T))$ for any $\varepsilon \in (0, T)$. This, in particular, means that the derivative $u_{x_i}(\cdot,t)$ belongs to the space $L_2(\Omega)$ for any $t \in (\varepsilon, T)$ and is really continuous with respect to t in the $L_2(\Omega)$-norm on the segment $[\varepsilon, T]$.

Let t be an arbitrary number from the half-open interval $(0, T]$. Holding a number ε from the interval $(0, t)$ fixed we deduce that there exists a moment $\tau^* \in [\varepsilon, t]$, at which the following relation occurs:

$$(1.1.31)\qquad \int_\varepsilon^t \| u_x(\cdot,\xi)\|_{2,\Omega}^2 \, d\xi = (t-\varepsilon)\, \| u_x(\cdot,\tau^*)\|_{2,\Omega}^2,$$

$$\tau^* \in [\varepsilon, t], \qquad 0 < \varepsilon < t \le T.$$

In this line, it is necessary to recall identity (1.1.22). Since the set of admissible functions Φ is dense in the space $L_2(Q_T)$, this identity should be valid for any $\Phi \in L_2(Q_T)$. Because of this fact, the equation

$$(1.1.32)\qquad u_t(x,t) - (Lu)(x,t) = F(x,t)$$

is certainly true almost everywhere in $Q \equiv \Omega \times (\varepsilon, t)$ and implies that

$$(1.1.33)\qquad \int_{\tau^*}^t \int_\Omega (u_\tau - Lu)\, dx\, d\tau = \int_{\tau^*}^t \int_\Omega F^2 \, dx\, d\tau, \quad \tau^* \in [\varepsilon, t],$$

$$0 < \varepsilon < t \le T,$$

if τ^* and t were suitably chosen in conformity with (1.1.31). One can readily see that (1.1.33) yields the inequality

$$
(1.1.34) \qquad \int\limits_{\Omega} \sum_{i,j=1}^{n} A_{ij}(x)\, u_{x_j}(x,t)\, u_{x_i}(x,t)\; dx
$$

$$
+ \int\limits_{\tau^*}^{t} \int\limits_{\Omega} \left[u_\tau^2 + (Lu)^2 \right] \, dx \; d\tau
$$

$$
\leq \int\limits_{\Omega} \sum_{i,j=1}^{n} A_{ij}(x)\, u_{x_j}(x,\tau^*)\, u_{x_i}(x,\tau^*)\; dx
$$

$$
+ 2 \left| \int\limits_{\tau^*}^{t} \int\limits_{\Omega} u_\tau \left[\sum_{i=1}^{n} B_i(x)\, u_{x_i} \right.\right.
$$

$$
\left.\left. + \, C(x)\, u \right] \, dx \; d\tau \right| + \int\limits_{\tau^*}^{t} \int\limits_{\Omega} F^2 \, dx \; d\tau \, .
$$

With the aid of Young's inequality (1.1.1) the second term on the right-hand side of (1.1.34) can be estimated as follows:

$$
(1.1.35) \quad 2 \left| \int\limits_{\tau^*}^{t} \int\limits_{\Omega} u_\tau \left[\sum_{i=1}^{n} B_i(x)\, u_{x_i} + C(x)\, u \right] \, dx \; d\tau \right|
$$

$$
\leq \mu_1 \int\limits_{\tau^*}^{t} \int\limits_{\Omega} \left[\, 2\,\delta\, |\, u_\tau\,|^2 + \frac{1}{\delta} \left(|\, u_x\,|^2 + |\, u\,|^2 \right) \right] \, dx \; d\tau \, ,
$$

where

$$
\mu_1 = \max \left\{ \operatorname*{ess\,sup}_{\Omega} |\, C(x)\,| , \quad \operatorname*{ess\,sup}_{\Omega} \left[\sum_{i=1}^{n} B_i^2(x) \right]^{1/2} \right\}
$$

and δ is an arbitrary positive number.

By merely setting $\delta = 1/(4\,\mu_1)$ we derive from (1.1.34)–(1.1.35) one useful inequality

$$\| u_x(\,\cdot\,,t)\|^2_{2,\Omega} \le \frac{\mu}{\nu}\, \| u_x(\,\cdot\,,\tau^*)\|^2_{2,\Omega} + \frac{4\,\mu_1^2\,(1+c_1(\Omega))}{\nu} \int\limits_{\tau^*}^{t} \int\limits_{\Omega} |\,u_x\,|^2 \,dx\,d\tau + \frac{1}{\nu} \int\limits_{\tau^*}^{t} \int\limits_{\Omega} F^2 \,dx\,d\tau\,, \tag{1.1.36}$$

whose development is based on the Poincare-Friedrichs inequality (1.1.3) and conditions (1.1.15). Having substituted (1.1.31) into (1.1.36) we find that

$$\| u_x(\,\cdot\,,t)\|^2_{2,\Omega} \le c_4(t) \int\limits_{0}^{t} \int\limits_{\Omega} |\,u_x\,|^2 \,dx\,d\tau + \frac{1}{\nu} \int\limits_{0}^{t} \int\limits_{\Omega} F^2 \,dx\,d\tau\,,$$

$$0 < \varepsilon < t \le T\,,$$

where

$$c_4(t) = \frac{\mu}{\nu\,(t-\varepsilon)} + \frac{4\,\mu_1^2\,(1+c_1(\Omega))}{\nu}$$

and ε is an arbitrary positive number from the interval $(0,\,t)$.

The first term on the right-hand side of the preceding inequality can be estimated on the basis of (1.1.30) as follows:

$$\| u_x(\,\cdot\,,t)\|^2_{2,\Omega} \le c_5(t)\,\|\,a\,\|^2_{2,\Omega} + c_6(t) \int\limits_{0}^{t} \|\,F(\,\cdot\,,\tau)\,\|^2_{2,\Omega}\,d\tau\,, \tag{1.1.37}$$

$$t \in (0,\,T]\,,$$

where

$$c_5(t) = 2\,c_3(t)\,c_4(t)$$

and

$$c_6(t) = \nu^{-1} + 8\,t^2\,c_3(t)\,c_4(t)\,.$$

In this context, it is necessary to say that estimates (1.1.20), (1.1.28), (1.1.30) and (1.1.30) hold for any solution $u \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ of the direct problem (1.1.12)–(1.1.14) provided that the conditions of Theorem 1.1.6 and Lemma 1.1.1 hold.

Differential properties of a solution u ensured by Theorem 1.1.5 are established in the following assertion.

Lemma 1.1.2 *If, in addition to the premises of Theorem* 1.1.5, $F_t \in L_2(Q_T)$ *and* $a \in W_2^2(\Omega)$, *then the solution* $u(x,t)$ *belongs to* $C([0,T]; W_2^2(\Omega))$, *its derivative* $u_t(x,t)$ *belongs to*

$$C([0,T], L_2(\Omega)) \bigcap C([\varepsilon, T], \overset{\circ}{W}{}_2^1(\Omega)), \qquad 0 < \varepsilon < T.$$

Moreover, u_t *gives in the space* $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ *a solution of the direct problem*

$$\begin{aligned} &w_t(x,t) - (Lw)(x,t) = F_t(x,t), && (x,t) \in Q_T, \\ &w(x,0) = (La)(x) + F(x,0), && x \in \Omega, \\ &w(x,t) = 0, && x \in S_T. \end{aligned} \tag{1.1.38}$$

Roughly speaking, Lemma 1.1.2 describes the conditions under which one can "differentiate" the system (1.1.12)–(1.1.14) with respect to t.

Let us consider the system (1.1.38) arguing as in the derivation of (1.1.20), (1.1.28), (1.1.30) and (1.1.37). All this enables us to deduce that in the context of Lemma 1.1.2 a solution u of the system (1.1.12)–(1.1.14) has the estimates

$$\begin{aligned} \| u_t(\cdot, t) \|_{2,\Omega} \leq \exp\{-\alpha t\} \| La + F(\cdot, 0) \|_{2,\Omega} \\ + \int_0^t \exp\{-\alpha(t-\tau)\} \| F_\tau(\cdot, \tau) \|_{2,\Omega} \, d\tau, \\ 0 \leq t \leq T, \end{aligned} \tag{1.1.39}$$

$$\begin{aligned} \sup_{[0,t]} \| u_\tau(\cdot, \tau) \|_{2,\Omega} \leq c_2(t) \Bigg[\| La + F(\cdot, 0) \|_{2,\Omega} \\ + \int_0^t \| F_\tau(\cdot, \tau) \|_{2,\Omega} \, d\tau \Bigg], \quad 0 \leq t \leq T, \end{aligned} \tag{1.1.40}$$

$$\begin{aligned} \int_0^t \| u_{\tau x}(\cdot, \tau) \|_{2,\Omega}^2 \, d\tau \leq c_3(t) \Bigg[\| La + F(\cdot, 0) \|_{2,\Omega} \\ + 2 \int_0^t \| F_\tau(\cdot, \tau) \|_{2,\Omega} \, d\tau \Bigg]^2, \\ 0 \leq t \leq T, \end{aligned} \tag{1.1.41}$$

$$\| u_{tx}(\cdot, \tau) \|_{2,\Omega}^2 \le c_5(t) \Bigg[\| La + F(\cdot, 0) \|_{2,\Omega}^2 + c_6(t) \int_0^t \| F_\tau(\cdot, \tau) \|_{2,\Omega}^2 \, d\tau \Bigg]^2, \quad 0 < t \le T, \tag{1.1.42}$$

where α, $c_2(t)$, $c_3(t)$, $c_5(t)$ and $c_6(t)$ are involved in estimates (1.1.20), (1.1.28), (1.1.30) and (1.1.37), respectively.

In subsequent chapters we shall need, among other things, some special properties of the parabolic equation solutions with nonhomogeneous boundary conditions. Let a function $u \in V_2^{1,0}(Q_T)$ be a **generalized solution** of the direct problem

$$u_t(x,t) - (Lu)(x,t) = F(x,t), \qquad (x,t) \in Q_T, \tag{1.1.43}$$

$$u(x,0) = a(x), \qquad x \in \Omega, \tag{1.1.44}$$

$$u(x,t) = b(x,t), \qquad (x,t) \in S_T, \tag{1.1.45}$$

where the operator L is specified by (1.1.15)–(1.1.16), it being understood that the function u satisfies the integral identity

$$\begin{aligned} \int_0^t \int_\Omega \Big(- u\, \Phi_\tau + \sum_{i,j=1}^n A_{ij}(x)\, u_{x_j}\, \Phi_{x_i} & \\ - \sum_{i=1}^n B_i(x)\, u_{x_i}\, \Phi - C(x)\, u\, \Phi \Big) \, dx\, d\tau & \\ + \int_\Omega u(x,t)\, \Phi(x,t)\, dx - \int_\Omega a(x)\, \Phi(x,0)\, dx & \\ = \int_0^t \int_\Omega F\, \Phi \, dx\, d\tau, \qquad 0 \le t \le T, & \end{aligned} \tag{1.1.46}$$

where Φ is an arbitrary element of $W_2^{1,1}(Q_T)$ such that

$$\Phi(x,t) = 0$$

for all $(x,t) \in S_T$.

Note that we preassumed here that the function $b(x,t)$ can be extended and defined almost everywhere in the cylinder $\bar{Q}_T$. The boundary condition (1.1.45) means that the boundary value of the function

$$u(x,t) - b(x,t)$$

is equal to zero on S_T. Some differential properties of the boundary traces of functions from the space $W_q^{2m,\,m}(Q_T)$ are revealed in Ladyzhenskaya et al. (1968).

Lemma 1.1.3 *If $u \in W_q^{2m,\,m}(Q_T)$ and $2r + s < 2m - 2/q$, then*

$$D_t^r\, D_t^s\, u(\,\cdot\,, 0) \in W_q^{2m\,-\,2r\,-\,s\,-\,2/q}(\Omega)\,.$$

Moreover, if $2r + s < 2m - 1/q$, then

$$D_t^r\, D_t^s\, u|_{\text{on } S_T} \in W_q^{2m\,-\,2r\,-\,s\,-\,1/q,\; m\,-\,r\,-\,s/2\,-\,1/(2q)}(S_T)\,.$$

In what follows we will show that certain conditions provide the solvability of the direct problem (1.1.43)–(1.1.45) in the space $V_2^{1,\,0}(Q_T)$ (for more detail see Ladyzhenskaya et al. (1968)).

Theorem 1.1.7 *There exists a solution $u \in V_2^{1,\,0}(Q_T)$ of problem* (1.1.43)–(1.1.45) *for any $a \in L_2(\Omega)$, $b \in W_2^{2,\,1}(Q_T)$ and $F \in L_{2,1}(Q_T)$, this solution is unique in the indicated class of functions and the stability estimate is true:*

$$\sup_{[0,\,T]}\; \|\, u(\,\cdot\,, t)\,\|_{2,\,\Omega} + \|\, u_x\,\|_{2,\,Q_T} \;\le\; c^*\,\big(\,\|\, F\,\|_{2,\,1,\,Q_T} + \|\, a\,\|_{2,\,\Omega} + \|\, b\,\|_{2,\,Q_T}^{(2,\,1)}\big)\,.$$

To decide for yourself whether solutions to parabolic equations are positive, a first step is to check the following statement.

Theorem 1.1.8 (Ladyzhenskaya et al. (1968) or Duvant and Lions (1972)) *Let $F \in L_2(Q_T)$, $a \in L_2(\Omega)$, $b \in W_2^{2,\,1}(Q_T)$ and let*

$$\begin{aligned}
&\textit{the coefficient } C(x) \le 0 && \textit{for} \quad x \in \Omega\,;\\
&a(x) \ge 0 && \textit{for} \quad x \in \Omega\,;\\
&b(x,t) \ge 0 && \textit{for} \quad (x,t) \in S_T\,;\\
&F(x,t) \ge 0 && \textit{for} \quad (x,t) \in Q_T\,.
\end{aligned}$$

Then any solution $u \in V_2^{1,\,0}(Q_T)$ of problem (1.1.43)–(1.1.45) *satisfies the inequality $u(x,t) \ge 0$ almost everywhere in Q_T.*

Our next step is to formulate two assertions revealing this property in more detail. In preparation for this, we introduce additional notations. The symbol $K(x_0, \rho)$ is used for a cube of the space $\mathbf{R}^n$ centered at a point x_0 with ρ on edge. By subrectangles of a rectangle

$$R = K(x_0, \rho) \times (t_0 - \tau\rho^2, t_0)$$

we shall mean the following sets:

$$R^- = K(x_0, \rho') \times (t_0 - \tau_1 \rho^2, t_0 - \tau_0 \rho^2),$$
$$R^* = K(x_0, \rho'') \times (t_0 - \tau \rho^2, t_0 - \tau_2 \rho^2),$$

where $0 < \rho' < \rho''$ and $0 \le \tau_0 < \tau_1 < \tau_2 < \tau_3 < \tau$.

Recall that a function $u \in V_2^{1,0}(Q_T)$ is called a **weak supersolution** to the equation $u_t - Lu = 0$ in Q_T if this function satisfies the inequality

$$\int_0^t \int_\Omega \Big(-u\,\Phi_\tau + \sum_{i,j=1}^n A_{ij}(x)\, u_{x_i}\, \Phi_{x_j}$$
$$- \sum_{i=1}^n B_i(x)\, u_{x_i}\, \Phi - C(x)\, u\, \Phi \Big)\, dx\, d\tau$$
$$+ \int_\Omega u(x,t)\, \Phi(x,t)\, dx$$
$$- \int_\Omega u(x,0)\, \Phi(x,0)\, dx \ge 0, \qquad 0 \le t \le T,$$

for all bounded functions Φ from the space $\in W_2^{1,1}(Q_T)$ such that

$$\Phi(x,t) \ge 0 \qquad \text{and} \qquad \Phi(x,t) = 0$$

for all $(x,t) \in S_T$.

Lemma 1.1.4 (Trudinger (1968)) *Let $u(x,t)$ be a weak supersolution to the equation*

$$u_t - Lu = 0$$

in $R \subset Q$ and $0 \le u \le M$ in R. Then

$$\rho^{-(n-2)} \| u \|_{1,R^*} \le \gamma \min_{(x,t)\in R^-} u(x,t),$$

where the constant γ is independent of the function u.

Lemma 1.1.5 *Let all the conditions of Theorem* 1.1.8 *hold. Then the function* $u \in \overset{\circ}{V}{}_2^{1,0}(Q_T)$ *possesses the following properties:*

(1) *if* $a(x) \not\equiv 0$ *in* Ω, *then* $u(x,t) > 0$ *in* Ω *for any* $t \in (0,T]$;

(2) *if* $a(x) \equiv 0$ *in* Ω *and* $F(x,t) \not\equiv 0$ *in* Q_T, *then* $u(x,t) \geq 0$ *in* Q_T *and* $u(x,T) > 0$ *in* Ω.

Proof Let a function $v \in V_2^{1,0}(Q_T)$ be a generalized solution of the direct problem

$$\begin{aligned} &(1.1.47) \qquad && v_t(x,t) - (Lv)(x,t) = F(x,t)\,, && (x,t) \in Q_T\,,\\ &(1.1.48) \qquad && v(x,0) = a(x)\,, && x \in \Omega\,,\\ &(1.1.49) \qquad && v(x,t) = 0\,, && (x,t) \in S_T\,. \end{aligned}$$

It is clear that the difference

$$u_1(x,t) = u(x,t) - v(x,t)$$

belongs to the space $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ and by Theorem 1.1.8 we obtain for all $(x,t) \in Q_T$ the governing inequality

$$u_1(x,t) \geq 0$$

or, what amounts to the same in Q_T,

$$u(x,t) \geq v(x,t)\,.$$

The lemma will be proved if we succeed in justifying assertions (1)–(2) for the function $v(x,t)$ and the system (1.1.47)–(1.1.49) only.

We first choose monotonically nondecreasing sequences of nonnegative functions

$$\left\{F^{(k)}\right\}_{k=1}^{\infty}, \qquad F^{(k)} \in C^2(\bar{Q}_T)\,,$$

and

$$\left\{a^{(k)}\right\}_{k=1}^{\infty}, \qquad a^{(k)} \in \overset{\circ}{C}{}^2(\bar{\Omega})\,,$$

such that $F^{(k)} \to F$ as $k \to \infty$ almost everywhere in Q_T and $a^{(k)} \to a$ as $k \to \infty$ almost everywhere in Ω. They are associated with a sequence of direct problems

$$\begin{aligned} &(1.1.50) \qquad && v_t^{(k)}(x,t) - (Lv^{(k)})(x,t) = F^{(k)}(x,t)\,, && (x,t) \in Q_T\,,\\ &(1.1.51) \qquad && v^{(k)}(x,0) = a^{(k)}(x)\,, && x \in \Omega\,,\\ &(1.1.52) \qquad && v^{(k)}(x,t) = 0\,, && (x,t) \in S_T\,. \end{aligned}$$

Using the results obtained in Ladyzhenskaya et al. (1968), Chapter 4, we conclude that there exists a unique solution $v^{(k)} \in W_q^{2,1}(Q_T)$, $q > n+2$, of problem (1.1.50)–(1.1.52). Therefore, the function $v^{(k)}$ and its derivatives $v_{x_i}^{(k)}$ satisfy Hölder's condition with respect to x and t in Q_T. This provides support for the view, in particular, that $v^{(k)}$ is continuous and bounded in Q_T and so the initial and boundary conditions can be understood in a classical sense. The stability estimate (1.1.20) implies that

$$(1.1.53) \quad \| (v - v^{(k)})(\,\cdot\,,t) \|_{2,\Omega} \leq c^* \left(\| a - a^{(k)} \|_{2,\Omega} + \| F - F^{(k)} \|_{2,Q_T} \right), \quad 0 \leq t \leq T.$$

We proceed to prove item (1). When $a(x) \not\equiv 0$ in Ω, we may assume without loss of generality that in Ω

$$a^{(k)} \not\equiv 0$$

for any $k \in \mathbf{N}$. We have mentioned above that the function $v^{(k)}$ is continuous and bounded in Q_T. Under these conditions Theorem 1.1.8 yields in Q_T

$$v^{(k)}(x,t) \geq 0 .$$

From **Harnack's inequality** it follows that

$$v^{(k)}(x,t) > 0$$

for any $t \in (0,T]$, $x \in \Omega$ and $k \in \mathbf{N}$. We begin by placing problem (1.1.50)–(1.1.52) with regard to

$$w(x,t) = v^{(k+1)}(x,t) - v^{(k)}(x,t) .$$

It is interesting to learn whether $w(x,t) \geq 0$ in Q_T and, therefore, the sequence $\left\{ v^{(k)} \right\}_{k=1}^{\infty}$ is monotonically nondecreasing. It is straightforward to verify this as before. On the other hand, estimate (1.1.53) implies that for any $t \in (0,T]$ there exists a subsequence $\left\{ v^{(k_p)} \right\}_{p=1}^{\infty}$ such that

$$v^{(k_p)}(x,t) \to v(x,t)$$

as $p \to \infty$ for almost all $x \in \Omega$. Since $\left\{ v^{(k_p)} \right\}_{p=1}^{\infty}$ is monotonically nondecreasing, $v(x,t) > 0$ for almost all $x \in \Omega$ and any $t \in (0,T]$.

We proceed to prove item (2). When $a(x) \not\equiv 0$ in Ω and $F(x,t) \not\equiv 0$ in Q_T, we may assume that $F^{(k)} \not\equiv 0$ in Q_T for any $k \in \mathbf{N}$. Arguing as in item (1) we find that $v^{(k)} > 0$ in Q_T and by Harnack's inequality deduce that $v^{(k)}(x,T) > 0$ for all $x \in \Omega$. What is more, we establish with the aid of (1.1.53) that $v(x,T) > 0$ almost everywhere in Ω and thereby complete the proof of Lemma 1.1.5. ■

Other ideas in solving nonlinear operator equations of the second kind are connected with the **Birkhoff–Tarsky fixed point principle**. This principle applies equally well to any operator equation in a partially ordered space. Moreover, in what follows we will disregard metric and topological characteristics of such spaces.

Let E be a partially ordered space in which any bounded from above (below) subset $D \subset E$ has a least upper bound $\sup D$ (greatest lower bound $\inf D$). Every such set D falls in the category of **conditionally complete lattices**.

The set of all elements $f \in E$ such that $a \leq f \leq b$, where a and b are certain fixed points of E, is called an **order segment** and is denoted by $[a, b]$. An operator A: $E \mapsto E$ is said to be **isotonic** if $f_1 \leq f_2$ with $f_1, f_2 \in E$ implies that

$$A f_1 \leq A f_2 \,.$$

The reader may refer to Birkhoff (1967), Lyusternik and Sobolev (1982).

Theorem 1.1.9 (Birkhoff–Tarsky) *Let E be a conditionally complete lattice. One assumes, in addition, that A is an isotonic operator carrying an order segment $[a, b] \subset E$ into itself. Then the operator A can have at least one fixed point on the segment $[a, b]$.*

1.2 The linear inverse problem: recovering a source term

In this section we consider inverse problems of finding a source function of the parabolic equation (1.1.12). We may attempt the function F in the form

$$F = f(x)\, h(x,t) + g(x,t)\,, \tag{1.2.1}$$

where the functions h and g are given, while the unknown function f is sought.

Being concerned with the operators L, $\mathcal{B}$, l, the functions h, g, a, b and χ, and the domain Q_T we now study in the cylinder $Q_T \equiv \Omega \times (0, T)$ the inverse problem of finding a pair of the functions $\{u, f\}$, satisfying the equation

$$\begin{aligned} u_t(x,t) - (Lu)(x,t) & \\ = f(x)\, h(x,t) + g(x,t)\,, &\quad (x,t) \in Q_T\,, \end{aligned} \tag{1.2.2}$$

the initial condition

$$u(x,0) = a(x)\,, \qquad x \in \Omega\,, \tag{1.2.3}$$

the boundary condition

$$(\mathcal{B}u)(x,t) = b(x,t)\,, \qquad (x,t) \in S_T = \partial\Omega \times [0,T]\,, \tag{1.2.4}$$

and the overdetermination condition

$$(lu)(x) = \chi(x)\,, \qquad x \in \Omega\,. \tag{1.2.5}$$

Here the symbol L is used for a linear uniformly elliptic operator, whose coefficients are independent of t for any $x \in \bar{\Omega}$:

$$\begin{aligned} (Lu)(x,t) \equiv & \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \Big[A_{ij}(x)\, u_{x_j}(x,t) \Big] \\ & + \sum_{i=1}^{n} B_i(x)\, u_{x_i}(x,t) \\ & + C(x)\, u(x,t)\,, \end{aligned} \tag{1.2.6}$$

$$A_{ij} = A_{ji}, \quad 0 < \nu \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} A_{ij}\, \xi_i\, \xi_j \le \mu \sum_{i=1}^{n} \xi_i^2, \qquad \nu,\, \mu \equiv \text{const} > 0.$$

The meaning of an operator $\mathcal{B}$ built into the boundary condition (1.2.4) is that

$$\begin{aligned} &\text{either} \qquad (\mathcal{B}u)(x,t) \equiv u(x,t) \\ &\text{or} \qquad (\mathcal{B}u)(x,t) \equiv \frac{\partial u(x,t)}{\partial N} + \sigma(x)\, u(x,t)\,, \end{aligned} \tag{1.2.7}$$

where

$$\frac{\partial u}{\partial N} \equiv \sum_{i,j=1}^{n} A_{ij}\, u_{x_j}(x,t) \cos\,(\widehat{\mathbf{n}, Ox_i})$$

and $\mathbf{n}$ is the external normal to $\partial\Omega$. Throughout the entire subsection, we will assume that the function σ is continuous on the boundary $\partial\Omega$ and $\sigma \ge 0$.

The expression for lu from (1.2.5) reduces to

(1.2.8)
$$\text{either} \quad (lu)(x) \equiv u(x,t_1)\,, \quad 0 < t_1 \le T\,, \quad x \in \Omega\,,$$
$$\text{or} \quad (lu)(x) \equiv \int_0^T u(x,\tau)\,\omega(\tau)\,d\tau\,, \quad x \in \Omega\,,$$

if t_1 is held fixed and ω is known in advance.

Although the complete theory could be recast in this case, we confine ourselves to the homogeneous conditions (1.2.3)–(1.2.4) and the function g equal to zero in (1.2.2). Indeed, consider the direct problem of recovering a function v from the relations

(1.2.9)
$$\begin{aligned} &v_t\,(x,t) - (Lv)\,(x,t) = g(x,t)\,, && (x,t) \in Q_T\,,\\ &v(x,0) = a(x)\,, && x \in \Omega\,,\\ &(\mathcal{B}v)\,(x,t) = b(x,t)\,, && (x,t) \in S_T\,, \end{aligned}$$

if the subsidiary information is available on the operators L and $\mathcal{B}$ and the functions g, a and b. While solving problem (1.2.9) one can find a unique solution v in the corresponding class of functions. Therefore, (1.2.2)–(1.2.5) and (1.2.9) imply that a pair of the functions $\{u{-}v,\ f\}$ satisfies the equation

(1.2.10)
$$(u-v)_t - L(u-v) = f(x)\,h(x,t)\,, \qquad (x,t) \in Q_T\,,$$

the initial condition

(1.2.11)
$$(u-v)\,(x,0) = 0\,, \qquad x \in \Omega\,,$$

the boundary condition

(1.2.12)
$$\big[\mathcal{B}(u-v)\big]\,(x,t) = 0\,, \qquad (x,t) \in S_T\,,$$

and the overdetermination condition

(1.2.13)
$$\big[l(u-v)\big]\,(x) = \chi_1(x)\,, \qquad x \in \Omega\,,$$

where $\chi_1(x) = \chi(x) - (lv)(x)$ and v is the unknown function to be determined as a solution of the direct problem (1.2.9). This approach leads to the inverse problem of a suitable type.

More a detailed exposition is based on the inverse problem with Dirichlet boundary data corresponding to the first relation (1.2.7). The overdetermination here will be taken in the integral form associated with the second relation (1.2.8).

To get an improvement of such an analysis, we set up the inverse problem of finding a pair of the functions $\{u,\ f\}$ satisfying the set of relations

$$(1.2.14) \qquad u_t(x,t) - (Lu)(x,t) = f(x)\, h(x,t)\,, \qquad (x,t) \in Q_T\,,$$

$$(1.2.15) \qquad u(x,0) = 0\,, \qquad x \in \Omega\,,$$

$$(1.2.16) \qquad u(x,t) = 0\,, \qquad (x,t) \in S_T\,,$$

$$(1.2.17) \qquad \int_0^T u(x,\tau)\, \omega(\tau)\, d\tau = \varphi(x)\,, \qquad x \in \Omega\,,$$

where the operator L, the functions h, ω, φ and the domain Ω are given.

A rigorous definition for a solution of the above inverse problem is presented for later use in

Definition 1.2.1 *A pair of the functions* $\{u,\ f\}$ *is said to be a* **generalized solution** *of the inverse problem* (1.2.14)–(1.2.17) *if*

$$u \in \overset{\circ}{W}{}^{2,\,1}_{2,\,0}(Q_T)\,, \qquad f \in L_2(\Omega)$$

and all of the relations (1.2.14)–(1.2.17) *occur.*

Let us briefly outline our further reasoning. We first derive an operator equation for the function f in the space $L_2(\Omega)$. Second, we will show that the equation thus obtained is equivalent, in a certain sense, to the inverse problem at hand. Just for this reason the main attention will be paid to the resulting equation. Under such an approach the unique solvability of this equation under certain restrictions on the input data will be proved and special investigations will justify the validity of Fredholm's alternative for it. Because of this, we can be pretty sure that the inverse problem concerned is of Fredholm's character, that is, the uniqueness of the solution implies its existence.

Following the above scheme we are able to derive an operator equation of the second kind for the function f assuming that the coefficients of the operator L satisfy conditions (1.1.15)–(1.1.16) and

$$h,\ h_t \in L_\infty(Q_T)\,,$$

$$(1.2.18) \qquad \left| \int_0^T h(x,t)\, \omega(t)\, dt \right| \geq \delta > 0$$

$$\text{for} \quad x \in \bar{\Omega} \quad (\delta \equiv \text{const}\,)\,, \quad \omega \in L_2(0,\,T)\,.$$

By regarding an arbitrary function f from the space $L_2(\Omega)$ to be fixed and substituting it into equation (1.2.14) we are now in a position on account of Theorem 1.1.5 to find $u \in W_{2,0}^{2,1}(Q_T)$ as a unique solution of the direct problem (1.2.14)–(1.2.16). If this happens, Lemma 1.1.2 guarantees that the function u in question possesses the extra smoothness:

$$u(\,\cdot\,,t) \in C\big([0,T];\, W_2^2(\Omega)\big)$$

and

$$u_t(\,\cdot\,,t) \in C\big([0,T];\, L_2(\Omega)\big)\,.$$

In the light of these properties the intention is to use the linear operator

$$A_1\colon\ L_2(\Omega) \mapsto L_2(\Omega)$$

acting in accordance with the rule

$$(A_1\, f)(x) = \frac{1}{h_1(x)} \int\limits_0^T u_t(x,t)\,\omega(t)\; dt\,, \qquad x \in \Omega\,, \tag{1.2.19}$$

where

$$h_1(x) = \int\limits_0^T h(x,t)\,\omega(t)\; dt\,.$$

Of special interest is a linear operator equation of the second kind for the function f over the space $L_2(\Omega)$:

$$f = A_1\, f + \psi, \tag{1.2.20}$$

where a known function ψ belongs to the space $L_2(\Omega)$.

In the sequel we will assume that the Dirichlet (direct) problem for the elliptic operator

$$(Lv)(x) = 0\,, \quad x \in \Omega\,, \qquad v(x) = 0\,, \quad x \in \partial\Omega\,, \tag{1.2.21}$$

has only a trivial solution unless the contrary is explicitly stated. Possible examples of the results of this sort were cited in Theorem 1.1.3.

The following proposition provides proper guidelines for establishing interconnections between the solvability of the inverse problem (1.2.14)–(1.2.17) and the existence of a solution to equation (1.2.20) and vice versa.

Theorem 1.2.1 *One assumes that the operator L satisfies conditions (1.1.15)-(1.1.16), $h,\ h_t \in L_\infty(Q_T)$ and*

$$\left| \int_0^T h(x,\tau)\,\omega(\tau)\,d\tau \right| \geq \delta > 0$$

for $x \in \bar{\Omega}$ ($\delta \equiv \text{const}$), $\omega \in L_2(0,T)$ and

$$\varphi \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega)\,.$$

Let the Dirichlet problem (1.2.21) have a trivial solution only. If we agree to consider

$$\text{(1.2.22)}\qquad \psi(x) = -\,\frac{1}{h_1(x)}\,(L\varphi)(x)\,,\qquad h_1(x) = \int_0^T h(x,\tau)\,\omega(\tau)\,d\tau\,,$$

then the following assertions are valid:

(a) *if the linear equation (1.2.20) is solvable, then so is the inverse problem (1.2.14)-(1.2.17);*

(b) *if there exists a solution $\{u,\ f\}$ of the inverse problem (1.2.14)-(1.2.17), then the function f involved gives a solution to equation (1.2.20)*

Proof We proceed to prove item (a) accepting (1.2.22) to be true and equation (1.2.20) to have a solution, say f. If we substitute the function f into (1.2.14), then (1.2.14)-(1.2.16) can be solved as a direct problem. On account of Theorem 1.1.5 there exists a unique solution $u \in W_{2,0}^{2,1}(Q_T)$ and Lemma 1.1.1 gives

$$u(\,\cdot\,,t) \in C\big([0,T];\, W_2^2(\Omega)\big)$$

and

$$u_t(\,\cdot\,,t) \in C\big([0,T];\, L_2(\Omega)\big)\,.$$

The assertion will be proved if we succeed in showing that the function u so constructed satisfies the supplementary overdetermination condition (1.2.17). By merely setting

$$\text{(1.2.23)}\qquad \int_0^T u(x,\tau)\,\omega(\tau)\,d\tau = \varphi_1(x)\,,\qquad x \in \Omega\,,$$

it makes sense to bear in mind the above properties of the function u, by means of which we find out that

$$\varphi_1 \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega) .$$

Let us multiply both sides of (1.2.14) by the function $\omega(t)$ and integrate then the resulting expression with respect to t from 0 to T. After obvious rearranging we are able to write down

$$\int_0^T u_t(x,t)\,\omega(t)\; dt - (L\varphi_1)(x) \;=\; f(x)\, h_1(x) , \qquad x \in \Omega . \tag{1.2.24}$$

On the other hand, we must take into account that f is a solution of (1.2.20), meaning

$$h_1(x)\,(A_1\, f)(x) - (L\varphi)(x) = f(x)\, h_1(x) , \qquad x \in \Omega . \tag{1.2.25}$$

From (1.2.24)–(1.2.25) it follows that the function $\varphi - \varphi_1$ is just a solution of the direct stationary boundary value problem for the Laplace operator

$$\big[L(\varphi - \varphi_1)\big](x) = 0 , \quad x \in \Omega , \qquad (\varphi - \varphi_1)(x) = 0 , \quad x \in \partial\Omega , \tag{1.2.26}$$

having only a trivial solution by the assumption imposed at the very beginning. Therefore, $\varphi_1 = \varphi$ almost everywhere in Ω and the inverse problem (1.2.14)–(1.2.17) is solvable. Thus, item (a) is completely proved.

Let us examine item (b) assuming that there exists a pair of the functions $\{u,\, f\}$ solving the inverse problem (1.2.14)–(1.2.17). Relation (1.2.14) implies that

$$\int_0^T u_t(x,t)\,\omega(t)\; dt - \int_0^T (Lu)(x,t)\,\omega(t)\; dt = f(x)\, h_1(x) , \tag{1.2.27}$$

where $h_1(x) = \int_0^T h(x,\tau)\,\omega(\tau)\; d\tau$.

With the aid of the overdetermination condition (1.2.17) and relation (1.2.22) one can rewrite (1.2.27) as

$$\int_0^T u_t(x,t)\,\omega(t)\; dt + \varphi(x)\, h_1(x) = f(x)\, h_1(x) . \tag{1.2.28}$$

Recalling the definition of the operator A_1 (see (1.2.19)) we conclude that (1.2.28) implies that the function f is a solution to equation (1.2.20), thereby completing the proof of the theorem. ■

The following result states under what sufficient conditions one can find a unique solution of the inverse problem at hand.

Theorem 1.2.2 *Let the operator L comply with* (1.1.15)-(1.1.16), *$h,\ h_t \in L_\infty(Q_T)$ and let*

$$\left| \int_0^T h(x,t)\,\omega(t)\,dt \right| \geq \delta > 0 \qquad (\delta \equiv \text{const}\,),$$

$\omega \in L_2([0,T])$, $\varphi \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. One assumes, in addition, that the Dirichlet problem (1.2.21) *has a trivial solution only and the inequality holds:*

$$m_1 < 1\,, \tag{1.2.29}$$

where

$$m_1 = \delta^{-1}\,\|\,\omega\,\|_{2,(0,T)} \cdot \|\,m_2\,\|_{2,(0,T)}\,,$$

$$m_2(t) = \exp\{-\alpha\,t\}\ \operatorname*{ess\,sup}_{\Omega}\,|\,h(x,0)\,| + \int_0^t \exp\{-\alpha(t-\tau)\}\ \operatorname*{ess\,sup}_{\Omega}\,|\,h_\tau(\,\cdot\,,\tau)\,|\ d\tau\,,$$

$$\alpha = \left[\ \frac{\nu}{2\,c_1(\Omega)} - \left(\mu_1 + \frac{1}{2\,\nu}\ \mu_1^2\right)\right],$$

$$\mu_1 = \max\left\{\operatorname*{ess\,sup}_{\Omega}\ |\,C(x)\,|\,,\ \operatorname*{ess\,sup}_{\Omega}\ \left[\ \sum_{i=1}^n\ B_i^2(x)\right]^{1/2}\right\}$$

and $c_1(\Omega)$ is the constant from the Poincare-Friedrichs inequality (1.1.3). *Then there exists a solution $u \in W_{2,0}^{2,1}(Q_T)$, $f \in L_2(\Omega)$ of the inverse problem* (1.2.14)-(1.2.17), *this solution is unique in the indicated class of functions and the following estimates are valid with constant c^* from* (1.1.17)*:*

$$\|\,f\,\|_{2,\Omega} \leq \frac{\delta^{-1}}{1-m_1}\ \|\,L\varphi\,\|_{2,\Omega}\,, \tag{1.2.30}$$

$$\|\,u\,\|_{2,Q_T}^{(2,1)} \leq \frac{c^*\,\delta^{-1}}{1-m_1}\,\|\,L\varphi\,\|_{2,\Omega} \times \left(\ \int_0^T \operatorname*{ess\,sup}_{\Omega}\ |\,h(x,t)\,|^2\,dt\right)^{1/2}. \tag{1.2.31}$$

Proof We begin our proof by considering equation (1.2.20). A case in point is one useful remark that if (1.2.20) has a solution, then Theorem 1.2.1 will ensure the solvability of the inverse problem concerned.

We are going to show that for the linear operator A_1 the estimate

$$\| A_1 f \|_{2,\Omega} \leq m_1 \| f \|_{2,\Omega}, \qquad f \in L_2(\Omega), \tag{1.2.32}$$

is valid with constant m_1 of the form (1.2.29). Really, (1.2.19) is followed by

$$\| A_1 f \|_{2,\Omega} \leq \delta^{-1} \| \omega \|_{2,(0,T)} \left(\int_0^T \| u_t(\cdot, t) \|_{2,\Omega}^2 \, dt \right)^{1/2} . \tag{1.2.33}$$

On the other hand, as $f \in L_2(\Omega)$, the system (1.2.14)–(1.2.16) turns out to be of the same type as the system (1.1.12)–(1.1.14), making it possible to apply Lemma 1.1.2 and estimate (1.1.39) in the form

$$\| u_t(\cdot, t) \|_{2,\Omega} \leq m_2(t) \| f \|_{2,\Omega}, \qquad t \in [0, T],$$

where

$$m_2(t) = \exp\{-\alpha t\} \operatorname*{ess\,sup}_{\Omega} | h(x, 0) | + \int_0^t \exp\{-\alpha (t - \tau)\} \operatorname*{ess\,sup}_{\Omega} | h_\tau(\cdot, \tau) | \, d\tau,$$

$$\alpha = \left[\frac{\nu}{2 c_1(\Omega)} - \left(\mu_1 + \frac{1}{2\nu} \mu_1^2 \right) \right],$$

$$\mu_1 = \max \left\{ \operatorname*{ess\,sup}_{\Omega} | C(x) |, \operatorname*{ess\,sup}_{\Omega} \left[\sum_{i=1}^n B_i^2(x) \right]^{1/2} \right\}$$

and the constant $c_1(\Omega)$ is involved in the Poincare–Friedrichs inequality (1.1.3). Now the desired estimate (1.2.32) follows directly from the combination of (1.2.33) and the last inequality.

As $m_1 < 1$, the linear equation (1.2.20) has a unique solution with any function ψ from the space $L_2(\Omega)$ and, in particular, we might agree with $\psi = -L\varphi / h_1$, what means that (1.2.22) holds. If so, estimate (1.2.30) is certainly true. Therefore, Theorem 1.2.1 (see item (a)) implies the existence of a solution of the inverse problem (1.2.14)–(1.2.17).

In conclusion it remains to prove the uniqueness for the inverse problem (1.2.14)–(1.2.17) solution found above. Assume to the contrary that there were two distinct sets $\{u_1, f_1\}$ and $\{u_2, f_2\}$, each of them being a solution of the inverse problem at hand. When this is the case, the function f_1 cannot coincide with f_2, since their equality would immediately imply (due to the uniqueness theorem for the direct problem (1.1.12)–(1.1.14)) the equality between u_1 and u_2.

Item (b) of Theorem 1.2.1 yields that either of the functions f_1 and f_2 gives a solution to equation (1.2.20). However, this disagrees with the uniqueness of the equation (1.2.20) solution stated before. Just for this reason the above assumption concerning the nonuniqueness of the inverse problem (1.2.14)–(1.2.17) solution fails to be true. Now estimate (1.2.31) is a direct implication of (1.1.17) and (1.2.30), so that we finish the proof of the theorem. ■

We now turn our attention to the inverse problem with the **final overdetermination** for the parabolic equation

$$u_t(x,t) - (Lu)(x,t) = f(x)\,h(x,t)\,, \qquad (x,t) \in Q_T\,, \tag{1.2.34}$$

$$u(x,0) = 0\,, \qquad x \in \Omega\,, \tag{1.2.35}$$

$$u(x,t) = 0\,, \qquad (x,t) \in S_T\,, \tag{1.2.36}$$

$$u(x,T) = \varphi(x)\,, \qquad x \in \Omega\,. \tag{1.2.37}$$

In such a setting we have at our disposal the operator L, the functions h and φ and the domain $Q_T = \Omega \times (0,T)$.

To facilitate understanding, we give a rigorous definition for a solution of the inverse problem (1.2.34)–(1.2.37).

Definition 1.2.2 *A pair of the functions $\{u,\, f\}$ is said to be a* **generalized solution** *of the inverse problem* (1.2.34)–(1.2.37) *if* $u \in W_{2,0}^{2,1}(Q_T)$, $f \in L_2(\Omega)$ *and all of the relations* (1.2.34)–(1.2.37) *occur.*

We outline briefly further treatment of the inverse problem under consideration. More a detailed exposition of final overdetermination will appear in Chapter 4 for the system of Navier–Stokes equations.

Assume that the coefficients of the operator L meet conditions (1.1.15)–(1.1.16) and

$$h,\, h_t \in L_\infty(Q_T)\,, \quad |\,h(x,T)\,| \geq \delta > 0 \quad \text{for } x \in \bar{\Omega} \quad (\delta \equiv \text{const})\,. \tag{1.2.38}$$

Under this agreement, the collection of relations (1.2.34)–(1.2.36) can be treated as a direct problem by taking an arbitrary function f from the space

$L_2(\Omega)$ and substituting it into equation (1.2.34). According to Theorem 1.1.5 there exists a unique solution $u \in W_{2,0}^{2,1}(Q_T)$ of the direct problem (1.2.34)–(1.2.36) with the extra smoothness property:

$$u(\,\cdot\,,t) \in C\big([0,T];\, W_2^2(\Omega)\big)$$

and

$$u_t(\,\cdot\,,t) \in C\big([0,T];\, L_2(\Omega)\big)\,.$$

For further analysis we refer to the linear operator

$$A_2\colon\ L_2(\Omega) \mapsto L_2(\Omega)$$

acting in accordance with the rule

$$(A_2\, f)(x) = \frac{1}{h(x,T)}\; u_t(x,T)\,, \qquad x \in \Omega\,, \tag{1.2.39}$$

and the linear operator equation of the second kind for the function f over the space $L_2(\Omega)$:

$$f = A_2\, f + \psi\,, \tag{1.2.40}$$

where a known function ψ belongs to the space $L_2(\Omega)$.

Theorem 1.2.3 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16) *and let $h,\ h_t \in L_\infty(Q_T)$, $|\,h(x,T)\,| \geq \delta > 0$ for $x \in \bar{\Omega}$ ($\delta \equiv$ const), $\varphi \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. Assuming that the Dirichlet problem* (1.2.21) *can have a trivial solution only, set*

$$\psi(x) = -\,\frac{1}{h(x,T)}\; (L\varphi)(x)\,. \tag{1.2.41}$$

Then the following assertions are valid:

(a) *if the linear equation* (1.2.40) *is solvable, then so is the inverse problem* (1.2.34)–(1.2.37);

(b) *if there exists a solution $\{u,\, f\}$ of the inverse problem* (1.2.34)–(1.2.37), *then the function f involved gives a solution to the linear equation* (1.2.40).

Theorem 1.2.3 can be proved in the same manner as we carry out the proof of Theorem 1.2.1 of the present chapter or that of Theorem 4.2.1 from Chapter 4.

So, the question of the inverse problem solvability is closely connected with careful analysis of equation (1.2.40) of the second kind. By exactly the same reasoning as in the case of inequality (1.2.32) we deduce that the operator A_2 admits the estimate

$$\| A_2 f \|_{2,\Omega} \leq m_3 \| f \|_{2,\Omega}, \qquad f \in L_2(\Omega), \tag{1.2.42}$$

where

$$m_3 = \frac{1}{\delta} \left\{ \exp\{-\alpha T\} \operatorname*{ess\,sup}_{\Omega} | h(x,0)| + \int_0^T \exp\{-\alpha (T-\tau)\} \operatorname*{ess\,sup}_{\Omega} | h_t(\cdot, t) | \, dt \right\},$$

$$\alpha = \left[\frac{\nu}{2\, c_1(\Omega)} - \left(\mu_1 + \frac{1}{2\nu} \mu_1^2 \right) \right],$$

$$\mu_1 = \max \left\{ \operatorname*{ess\,sup}_{\Omega} | C(x) |, \ \operatorname*{ess\,sup}_{\Omega} \left[\sum_{i=1}^n B_i^2(x) \right]^{1/2} \right\}$$

and $c_1(\Omega)$ is the constant from the Poincare–Friedrichs inequality (1.1.3).

After that, applying estimate (1.2.42) and the fixed point principle to the linear operator A_2 with the subsequent reference to Theorem 1.2.3 we obtain an important result.

Theorem 1.2.4 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16) *and let h, $h_t \in L_\infty(Q_T)$, $| h(x,T) | \geq \delta > 0$ for $x \in \bar{\Omega}$* ($\delta \equiv$ const) *and*

$$\varphi \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega).$$

One assumes, in addition, that the Dirichlet problem (1.2.21) *has a trivial solution only. If the inequality*

$$m_3 < 1 \tag{1.2.43}$$

is valid with constant m_3 arising from (1.2.42), *then there exists a solution $u \in W_{2,0}^{2,1}(Q_T)$, $f \in L_2(\Omega)$ of the inverse problem* (1.2.34)–(1.2.37), *this solution is unique in the indicated class of functions and the following estimates*

$$\| f \|_{2,\Omega} \le \frac{\delta^{-1}}{1 - m_3} \| L\varphi \|_{2,\Omega} , \tag{1.2.44}$$

$$\| u \|_{2,Q_T}^{(2,1)} \le \frac{c^* \delta^{-1}}{1 - m_3} \| L\varphi \|_{2,\Omega} \times \left(\int_0^T \operatorname*{ess\,sup}_{\Omega} | h(x,t) |^2 \, dt \right)^{1/2} \tag{1.2.45}$$

are valid with constant c^* *from* (1.1.17).

Theorem 1.2.4 can be proved in a similar way as we did in the proof of Theorem 1.2.2.

We now present some remarks and examples illustrating the results obtained.

Remark 1.2.1 In dealing with the **Laplace operator**

$$(Lu)(x,t) \equiv \Delta u(x,t) \equiv \sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial x_i^2}$$

we assume that the function h depending only on t satisfies the conditions

$$h\,, \ h' \in C[0,T]\,, \qquad h(t)\,, \quad h'(t) \ge 0\,, \quad h(T) \ne 0\,.$$

Plain calculations give

$$\alpha = \frac{1}{2\, c_1(\Omega)} > 0 \qquad \text{and} \qquad m_3 = 1 - \tilde{m}_3\,,$$

where

$$\tilde{m}_3 = \frac{1}{\alpha\, h(T)} \int_0^T h(t) \exp\{-\alpha\,(T-t)\}\, dt\,.$$

Since $m_3 > 0$, the inequality $\tilde{m}_3 < 1$ holds true. On the other hand, $\tilde{m}_3 > 0$ for an arbitrary function $h(t) \ge 0$ with $h(T) \ne 0$. Therefore, $0 < m_3 < 1$ for any $T > 0$ and, in that case, Theorem 1.2.4 turns out to be of global character and asserts the unique solvability for any T, $0 < T < \infty$.

Example 1.2.1 Let us show how one can adapt the **Fourier method of separation of variables** in solving inverse problems with the final overdetermination. With this aim, we now turn to the inverse problem of recovering the functions $u(x)$ and $f(x)$ from the set of relations

(1.2.46) $u_t(x,t) = u_{xx}(x,t) + f(x)\,, \qquad 0 < x < \pi\,, \quad 0 < t < T\,,$

(1.2.47) $u(x,0) = 0\,, \qquad 0 < x < \pi\,,$

(1.2.48) $u(0,t) = u(\pi,t) = 0\,, \qquad 0 \le t \le T\,,$

(1.2.49) $u(x,T) = \varphi(x)\,, \qquad 0 < x < \pi\,,$

keeping $\varphi \in W_2^2(0,1)$ with the boundary values

$$\varphi(0) = \varphi(\pi) = 0\,.$$

It is worth noting here that in complete agreement with Remark 1.2.1 the inverse problem at hand has a unique solution for any T, $0 < T < \infty$. Following the standard scheme of separating variables with respect to the system (1.2.46)–(1.2.48) we arrive at

$$u(x,t) = \sum_{k=1}^{\infty} \int_0^t f_k \exp\left\{-k^2(t-\tau)\right\} d\tau \, \sin kx \tag{1.2.50}$$

$$= \sum_{k=1}^{\infty} f_k\, k^{-2} \left[1 - \exp\left\{-k^2 t\right\}\right] \sin kx\,,$$

where

$$f_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx \, dx\,.$$

The system $\{X_k(x) = \sin kx\}_{k=1}^{\infty}$ and the sequence $\{\lambda_k = k^2\}_{k=1}^{\infty}$ are found as the eigenfunctions and the eigenvalues of the **Sturm–Liouville operator** associated with the spectral problem

(1.2.51) $X_k''(x) + \lambda_k X(x) = 0\,, \quad 0 < x < \pi\,, \qquad X_k(0) = X_k(\pi) = 0\,.$

Being a basis for the space $L_2(0,\pi)$, the system $\{\sin kx\}_{k=1}^{\infty}$ is orthogonal and complete in it. In this view, it is reasonable to look for the **Fourier coefficients** f_k of the unknown function f with respect to the system

$\{\sin kx\}_{k=1}^{\infty}$. Subsequent calculations will be done formally by reasonings substantiated. Substituting (1.2.50) into (1.2.49) yields

$$\varphi(x) = \sum_{k=1}^{\infty} f_k\, k^{-2} \left(1 - \exp\{-k^2 T\}\right) \sin kx\,. \tag{1.2.52}$$

The expansion in the **Fourier series** of the function φ with respect to the basis $\{\sin kx\}_{k=1}^{\infty}$ is as follows:

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k \sin kx\,, \qquad \varphi_k = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin kx\; dx\,.$$

Equating the corresponding coefficients we thus have

$$f_k = k^2 \left(1 - \exp\{-k^2 T\}\right)^{-1} \varphi_k\,, \tag{1.2.53}$$

thereby justifying that the function f in question can be expressed explicitly:

$$f(x) = \sum_{k=1}^{\infty} k^2 \left(1 - \exp\{-k^2 T\}\right)^{-1} \varphi_k \sin kx\,. \tag{1.2.54}$$

On account of (1.2.50) and (1.2.53) we conclude that this procedure works with another unknown function u. The outcome of this is

$$u(x,t) = \sum_{k=1}^{\infty} \left(1 - \exp\{-k^2 T\}\right)^{-1} \left(1 - \exp\{-k^2 t\}\right) \varphi_k \sin kx\,. \tag{1.2.55}$$

The expansion (1.2.54) needs investigation. As far as the underlying orthogonal system is complete, **Parseval's equation** takes the form

$$\| f \|_{2,\Omega}^2 = \sum_{k=1}^{\infty} \frac{2}{\pi}\, k^4 \left(1 - \exp\{-k^2 T\}\right)^{-2} \varphi_k^2\,.$$

Therefore, for the existence of a solution f in the space $L_2(0,\pi)$ it is necessary to have at your disposal a function φ such that the series on the right-hand side of the preceding equality would be convergent. With the relation

$$\sum_{k=1}^{\infty} \frac{2}{\pi}\, k^4 \left(1 - \exp\{-k^2 T\}\right)^{-2} \varphi_k^2 \le \frac{2}{\pi} \left(1 - \exp\{-T\}\right)^{-2} \sum_{k=1}^{\infty} k^4 \varphi_k^2$$

in view, the convergence of the series in (1.2.54) in the space $L_2(0, \pi)$ depends on how well the Fourier coefficients of the function φ behave when $k \to +\infty$. A similar remark concerns the character of convergence in (1.2.55). Special investigations of (1.2.54) and (1.2.55) can be conducted in the framework of the general theory of Fourier series. If, in particular, $\varphi(x) = \sin x$, then (1.2.54)–(1.2.55) obviously imply that

$$u(x,t) = \big(1 - \exp\{-T\}\big)^{-1} \big(1 - \exp\{-t\}\big)^{-1} \sin x\,,$$

$$f(x) = \big(1 - \exp\{-T\}\big)^{-1} \sin x\,.$$

We would like to give a simple, from a mathematical point of view, example illustrating one interesting property of the inverse problem with the final overdetermination. Theorem 1.2.4, generally speaking, does not guarantee any global result for inverse problems of the type (1.2.34)–(1.2.37). Because the original assumptions include inequality (1.2.43) this theorem allows us to establish a local result only. However, the forthcoming example will demonstrate that the locality here happens to be of the so-called "inverse" character in comparison with that of direct problems. That is to say, the final observation moment T cannot be made as small (close to zero) as we like and vice versa the moment T can be taken at any level exceeding a certain fixed value T^* expressed in terms of input data.

Example 1.2.2 Being concerned with the functions h and φ, we are interested in the **one-dimensional inverse problem** for the heat conduction equation

$$(1.2.56)\qquad \begin{aligned} & u_t(x,t) = 2u_{xx}(x,t) + f(x)\, h(x,t)\,, && x \in (0,\,1)\,, \quad t \in (0,\,T)\,, \\ & u(x,0) = 0\,, && x \in (0,\,1)\,, \\ & u(0,t) = u(1,t) = 0\,, && t \in [0,\,T]\,, \\ & u(x,T) = \varphi(x)\,, && x \in (0,\,1)\,. \end{aligned}$$

Let $h(x,t) = t + x$ and φ be an arbitrary function with the necessary smoothness and compatibility. It is required to indicate special cases in which Theorem 1.2.4 will quarantee the unique solvability of (1.2.56). Direct calculations of m_3 from (1.2.42) show that

$$m_3 = T^{-1}\,.$$

The inverse problem (1.2.56) will meet condition (1.2.43) if

$$(1.2.57)\qquad T > 1\,.$$

So, according to Theorem 1.2.4 the inverse problem (1.2.56) has a unique solution if the final moment of observation T satisfies (1.2.57). Just this inequality is aimed to substantiate why the final measurement of the function u should be taken at any moment T exceeding the fixed value $T^* = 1$. In the physical language, this is a way of saying that for the unique recovery of the coefficient $f(x)$ in solving the inverse problem (1.2.56) in the framework of Theorem 1.2.4 it should be recommended to avoid the final measuring "immediately after" the initial (starting) moment of observation (monitoring).

In the above example we obtain a natural, from our standpoint, result concerning the character of locality in the inverse problem with the final overdetermination. Indeed, assume that there exists a certain moment $\widetilde{T}$, at which we are able to solve the inverse problem (1.2.56) and thereby recover uniquely the coefficient $f(x)$ with the use of the final value $u(x, \widetilde{T})$. Because the source term $f(x)\, h(x,t)$ is known, other ideas are connected with the transition to the related direct problem with the value $u(x, \widetilde{T})$ as an initial condition and the determination of the function $u(x,t)$ at any subsequent moment $t > \widetilde{T}$. Summarizing, in context of the theory of inverse problems the principal interest here lies in a possibility to take the moment $\widetilde{T}$ as small as we like rather than as large as we like.

In subsequent investigations we will establish other sufficient conditions for the unique solvability of the inverse problem with the final overdetermination. In contrast to Theorem 1.2.4 the results stated below will guarantee the global existence and uniqueness of the solution.

Among other things, we will be sure that the inverse problem (1.2.56) has, in fact, a unique solution $\{u, f\}$ for any $T \in (0, +\infty)$ (see Theorem 1.3.5 below). However, the success of obtaining this result will depend on how well we motivate specific properties of the parabolic equation solutions established in Theorem 1.1.8 and Lemmas 1.1.4–1.1.5.

1.3 The linear inverse problem: the Fredholm solvability

This section places special emphasis on one interesting property of the inverse problem (1.2.2)–(1.2.5) that is related to its Fredholm character. A case in point is that the events may happen in which the uniqueness theorem implies the theorem of the solution existence. We outline further general scheme by considering the inverse problem (1.2.34)–(1.2.37) with the final overdetermination. In Theorem 1.2.3 we have proved that the solvability of this inverse problem follows from that of the operator equation (1.2.40) of the second kind and vice versa, so there is some reason to be concerned about this. In subsequent studies the linear operator specified by (1.2.39) comes first. This type of situation is covered by the following assertion.

Theorem 1.3.1 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16) *and let h, $h_t \in L_\infty(Q_T)$, $|h(x,T)| \geq \delta > 0$ for $x \in \bar{\Omega}$ ($\delta \equiv$ const). Then the operator A_2 is completely continuous on $L_2(\Omega)$.*

Proof First of all we describe one feature of the operator A_2 emerging from Lemmas 1.1.1 and 1.1.2. As usual, this amounts to considering an arbitrary function f from the space $L_2(\Omega)$ to be fixed and substituting it into (1.2.34). Such a trick permits us to demonstrate that the system (1.2.34)–(1.2.36) is of the same type as the system (1.1.12)–(1.1.14). When solving problem (1.2.34)–(1.2.36) in the framework of Theorem 1.1.5 one finds in passing a unique function $u \in W_{2,0}^{2,1}(Q_T)$ corresponding to the function having been fixed above. Lemma 1.1.2 implies that

$$u_t \in C([0,T]; L_2(\Omega)) \bigcap C([\varepsilon,T]; \overset{\circ}{W}{}_2^1(\Omega)), \qquad 0 < \varepsilon < T.$$

Therefore, the operator A_2 specified by (1.2.39) acts, in fact, from $L_2(\Omega)$ into $\overset{\circ}{W}{}_2^1(\Omega)$.

In the estimation of $A_2(f)$ in the $\overset{\circ}{W}{}_2^1(\Omega)$-norm we make use of inequality (1.1.42) taking in terms of the system (1.2.34)–(1.2.36) the form

$$(1.3.1) \qquad \| u_{tx}(\,\cdot\,,T)\|_{2,\Omega}^2 \leq c_5(T) \Bigg[\| f\, h(\,\cdot\,,0)\|_{2,\Omega}^2 + c_6(T) \int_0^T \| f\, h_t(\,\cdot\,,t)\|_{2,\Omega}^2 \, dt \Bigg], \quad \forall f \in L_2(\Omega),$$

where c_5 and c_6 are the same as in (1.1.42) and do not depend on f. Combination of relations (1.2.39) and (1.3.1) gives the estimate

$$(1.3.2) \qquad \| (A_2 f)_x \|_{2,\Omega} \leq c_7 \| f \|_{2,\Omega}, \qquad \forall f \in L_2(\Omega),$$

where

$$c_7 = \Bigg\{ c_5(T) \Big[\operatorname*{ess\,sup}_{\Omega} |h(x,0)|^2 + c_6(T) \int_0^T \operatorname*{ess\,sup}_{\Omega} |h_t(x,t)|^2 \, dt \Big] \Bigg\}^{1/2}.$$

Note that estimate (1.3.2) is valid for any function f from the space $L_2(\Omega)$ and the constant c_7 is independent of f.

As can readily be observed, estimate (1.3.2) may be of help in establishing that the linear operator A_2 is completely continuous on $L_2(\Omega)$. Indeed, let $\mathcal{D}$ be a bounded set of the space $L_2(\Omega)$. By virtue of the properties of the operator A_2 and estimate (1.3.2) the set $A_2(\mathcal{D})$ belongs to $\overset{\circ}{W}{}_2^1(\Omega)$ and is bounded in $\overset{\circ}{W}{}_2^1(\Omega)$. In that case Rellich's theorem (Theorem 1.1.2) implies that the set $A_2(\mathcal{D})$ is compact in the space $L_2(\Omega)$. In so doing any bounded set of the space $L_2(\Omega)$ is mapped onto a set which is compact in $L_2(\Omega)$. By definition, the operator A_2 is completely continuous on $L_2(\Omega)$ and the theorem is proved. ■

Corollary 1.3.1 *Under the conditions of Theorem* 1.3.1 *the following Fredholm alternative is valid for equation* (1.2.40): *either a solution to equation* (1.2.40) *exists and is unique for any function* ψ *from the space* $L_2(\Omega)$ *or the homogeneous equation*

$$f = A_2 f \tag{1.3.3}$$

can have a nontrivial solution.

The result cited above states, in particular, that if the homogeneous equation (1.3.3) has a trivial solution only, then equation (1.2.40) is uniquely solvable for any $\psi \in L_2(\Omega)$. In other words, Corollary 1.3.1 asserts that for (1.2.40) the uniqueness theorem implies the existence one. With regard to the inverse problem (1.2.34)–(1.2.37) we establish the following theorem.

Theorem 1.3.2 *Let the operator* L *satisfy conditions* (1.1.15)–(1.1.16) *and let* $h,\ h_t \in L_\infty(Q_T)$, $|\, h(x,T) \,| \geq \delta > 0$ *for* $x \in \bar{\Omega}$ ($\delta \equiv \text{const}$),

$$\varphi \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega)\,.$$

If the Dirichlet problem (1.2.21) *has a trivial solution only, then the following assertions are valid:*

(a) *if the linear homogeneous equation* (1.3.3) *has a trivial solution only, then there exists a solution of the inverse problem* (1.2.34)–(1.2.37) *and this solution is unique in the indicated class of functions;*

(b) *if the uniqueness theorem holds for the inverse problem* (1.2.34)–(1.2.37), *then there exists a solution of the inverse problem* (1.2.34)–(1.2.37) *and this solution is unique in the indicated class of functions.*

Proof We proceed to prove item (a). Let (1.3.3) have a trivial solution only. By Corollary 1.3.1 there exists a solution to the nonhomogeneous equation (1.2.40) for any $\psi \in L_2(\Omega)$ (and, in particular, for ψ of the form (1.2.41)) and this solution is unique. The existence of the inverse problem (1.2.34)–(1.2.37) solution follows now from Theorem 1.2.3 and it remains to show only its uniqueness. Assume to the contrary that there were two distinct solutions $\{u_1, f_1\}$ and $\{u_2, f_2\}$ of the inverse problem (1.2.34)–(1.2.37). It is clear that f_1 cannot be equal to f_2, since their coincidence would immediately imply the equality between u_1 and u_2 by the uniqueness theorem for the direct problem of the type (1.2.34)–(1.2.36). According to item (b) of Theorem 1.2.3 the function $f_1 - f_2$ is just a nontrivial solution to the homogeneous equation (1.3.3). But this disagrees with the initial assumption. Thus, item (a) is completely proved.

We proceed to examine item (b). Let the uniqueness theorem hold for the the inverse problem (1.2.34)–(1.2.37). This means that the homogeneous inverse problem

$$
\begin{aligned}
&(1.3.4) \qquad && u_t(x,t) - (Lu)(x,t) = f(x)\, h(x,t)\,, \qquad && (x,t) \in Q_T\,,\\
&(1.3.5) && u(x,0) = 0\,, && x \in \Omega\,,\\
&(1.3.6) && u(x,t) = 0\,, && (x,t) \in S_T\,,\\
&(1.3.7) && u(x,T) = 0\,, && x \in \Omega\,,
\end{aligned}
$$

might have a trivial solution only. Obviously, the homogeneous equation (1.3.3) is associated with the inverse problem (1.3.4)–(1.3.7) in the framework of Theorem 1.2.3.

Let us show that (1.3.3) can have a trivial solution only. On the contrary, let $f \in L_2(\Omega)$ be a nontrivial solution to (1.3.3). Substituting f into (1.3.4) and solving the direct problem (1.3.4)–(1.3.6) by appeal to Theorem 1.1.5, we can recover a function $u \in \overset{\circ}{W}{}^{2,1}_{2,0}(Q_T)$ with the extra smoothness property indicated in Lemma 1.1.2. It is straightforward to verify that the function u satisfies also the overdetermination condition (1.3.7) by a simple observation that equation (1.3.4) implies that

$$
u_t(x,T) - (Lu)(x,T) = f(x)\, h(x,T)\,, \qquad (x,t) \in \Omega\,.
$$

On the other hand, the function f is subject to relation (1.3.3), that is,

$$
h(x,T)(A_2 f)(x) = f(x)\, h(x,T)\,, \qquad (x,t) \in \Omega\,.
$$

From definition (1.2.39) of the operator A_2, two preceding relations in combination with the boundary condition (1.3.6) it follows that the function $u(x,T)$ solves the Dirichlet problem

$$
L\big[u(x,T)\big] = 0\,, \quad x \in \Omega\,; \qquad u(x,T) = 0\,, \quad x \in \partial\Omega\,;
$$

which possesses only a trivial solution under the conditions of the theorem. Therefore, $u(x,T) = 0$ for $x \in \Omega$ and the pair $\{u, f\}$ thus obtained is just a nontrivial solution of the inverse problem (1.3.4)–(1.3.7). But this contradicts one of the conditions of item (b) concerning the uniqueness theorem. Thus, the very assumption about the existence of a nontrivial solution to the homogeneous equation (1.3.3) fails to be true. Finally, equation (1.3.3) can have a trivial solution only and the assertion of item (b) follows from the assertion of item (a). Thus, we arrive at the statement of the theorem. ■

The result thus obtained gives a hint that the inverse problem with the final overdetermination is of the Fredholm character. Before placing the corresponding alternative, we are going to show that within the framework of proving the **Fredholm solvability** it is possible to get rid of the triviality of the Dirichlet problem (1.2.21) solution. In preparation for this, one should "shift" the spectrum of the operator L. This is acceptable if we assume that conditions (1.1.15)–(1.1.16) are still valid for the operator L. Under this agreement there always exists a real number λ such that the stationary problem

$$L\chi(x) + \lambda\chi(x) = 0\,, \quad x \in \Omega\,; \qquad \chi(x) = 0\,, \quad x \in \partial\Omega\,; \tag{1.3.8}$$

has a trivial solution only. Via the transform

$$u(x,t) = \exp\{-\lambda\, t\}\, v(x,t)$$

we establish that the inverse problem (1.2.34)–(1.2.37) is equivalent to the following one:

$$\begin{aligned} & v_t(x,t) - (Lv)(x,t) - \lambda v(x,t) \\ & \qquad = f(x)\, \exp\{\lambda t\}\, h(x,t)\,, && (x,t) \in Q_T\,, && (1.3.9)\\ & v(x,0) = 0\,, && x \in \Omega\,, && (1.3.10)\\ & v(x,t) = 0\,, && (x,t) \in S_T\,, && (1.3.11)\\ & v(x,T) = \exp\{\lambda T\}\, \varphi(x)\,, && x \in \Omega\,. && (1.3.12) \end{aligned}$$

Arguing as in specifying the operator A_2 we refer to a linear operator with the values

$$(\tilde{A}_2\, f)(x) = \frac{\exp\{-\lambda T\}}{h(x,T)}\, v_t(x,T)\,, \qquad x \in \Omega\,, \tag{1.3.13}$$

and consider the related linear equation of the second kind

$$(1.3.14) \quad f = \widetilde{A}_2 f + \widetilde{\psi}\,, \quad \text{where} \quad \widetilde{\psi}(x) = -\frac{1}{h(x,T)}\left[(L\varphi)(x) + \lambda\varphi(x)\right].$$

For the inverse problem (1.3.9)-(1.3.12) and equation (1.3.14) it is possible to obtain a similar result as in Theorem 1.2.3 without the need for the triviality of the unique solution of the corresponding stationary direct problem (1.3.8). The well-founded choice of the parameter λ assures us of the validity of this property. By analogy with Theorem 1.3.1 the linear operator $\widetilde{A}_2$ turns out to be compact. By exactly the same reasoning as in Theorem 1.3.2 we introduce a preliminary lemma.

Lemma 1.3.1 *Let the operator L satisfy conditions* (1.1.15)-(1.1.16) *and let $h,\ h_t \in L_\infty(Q_T)$, $|\,h(x,T)\,| \geq \delta > 0$ for $x \in \bar{\Omega}$ ($\delta \equiv \text{const}$) and*

$$\varphi \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega)\,.$$

Then the following assertions are valid:

(a) *if the linear equation $f = \widetilde{A}_2 f$ has a trivial solution only, there exists a solution of the inverse problem* (1.3.9)-(1.3.12) *and this solution is unique in the indicated class of functions;*

(b) *if the uniqueness theorem holds for the inverse problem* (1.3.9)-(1.3.12), *there exists a solution of the inverse problem* (1.3.9)-(1.3.12) *and this solution is unique in the indicated class of functions.*

It is worth emphasizing once again that the inverse problems (1.2.34)-(1.2.37) and (1.3.9)-(1.3.12) are equivalent to each other from the standpoint of existence and uniqueness. With this equivalence in view, Lemma 1.3.1 permits us to prove the assertion of Theorem 1.3.2 once we get rid of the triviality of the inverse problem (1.2.21) solution after the appropriate "shift" of the spectrum of the operator L. This profound result is formulated below as an alternative.

Theorem 1.3.3 *Let the operator L satisfy conditions* (1.1.15)-(1.1.16) *and let $h,\ h_t \in L_\infty(Q_T)$, $|\,h(x,T)\,| \geq \delta > 0$ for $x \in \bar{\Omega}$ ($\delta \equiv \text{const}$) and*

$$\varphi \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega)\,.$$

Then the following alternative is true: either a solution of the inverse problem (1.2.34)-(1.2.37) *exists and is unique or the homogeneous inverse problem* (1.3.4)-(1.3.7) *has a nontrivial solution.*

In other words, this assertion says that under a certain smoothness of input data (see Theorem 1.3.3) of the inverse problem with the final overdetermination the uniqueness theorem implies the existence one.

Remark 1.3.1 A similar alternative remains valid for the inverse problem of recovering the source function coefficient in the general setting (1.2.2)–(1.2.5) (see Prilepko and Kostin (1992a)).

We now raise the question of the solvability of the inverse problem (1.2.34)–(1.2.37) with the final overdetermination in Hölder's classes.

Definition 1.3.1 *A pair of the functions* $\{u,\ f\}$ *is said to be a classical solution of the inverse problem* (1.2.34)–(1.2.37) *if* $u \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$, $f \in C^{\alpha}(\bar{\Omega})$ *and all of the relations* (1.2.34)–(1.2.37) *occur.*

In subsequent arguments the boundary $\partial\Omega$ happens to be of class $C^{2+\alpha}$, $0 < \alpha < 1$, and the coefficients of the uniformly elliptic operator L from (1.1.15) meet the smoothness requirements

$$A_{ij}\,,\ \frac{\partial}{\partial x_j} A_{ij}\,,\ B_i\,,\ C \in C^{\alpha}(\bar{\Omega})\,. \tag{1.3.15}$$

Also, the compatibility conditions

$$\varphi(x) = 0\,, \qquad h(x,0)\cdot(L\varphi)(x) = 0\,, \quad x \in \partial\Omega\,, \tag{1.3.16}$$

are imposed. By the way, the second relation is fulfilled if, for example, $\varphi \in C^{2+\alpha}(\bar{\Omega})$ and $h(x,0) = 0$ for all $x \in \partial\Omega$.

The study of the inverse problem (1.2.34)–(1.2.37) in Hölder's classes can be carried out in just the same way as we did in the consideration of Sobolev's spaces. In this line, we obtain the following result.

Theorem 1.3.4 *Let the operator* L *satisfy conditions* (1.1.15) *and* (1.3.8) *and let* $h,\ h_t \in C^{\alpha,\alpha/2}(\bar{Q}_T)$, $|\,h(x,T)\,| \geq \delta > 0$ *for* $x \in \bar{\Omega}$ ($\delta \equiv$ const), $\varphi \in C^{2+\alpha}(\bar{\Omega})$. *Under the compatibility conditions* (1.3.16) *the following alternative is true: either a solution in the sense of Definition* 1.3.1 *of the inverse problem* (1.2.34)–(1.2.37) *exists and is unique or the homogeneous inverse problem* (1.3.4)–(1.3.7) *has a nontrivial solution.*

We should focus the reader's attention on one principal case where one can prove the global theorem of uniqueness (and hence of existence) for the inverse problem with the final overdetermination. Along with the assumptions of Theorem 1.3.4 we require that

$$h(x,t),\ h_t(x,t) \geq 0 \quad \text{for} \quad (x,t) \in Q_T\,, \qquad C(x) \leq 0, \quad x \in \Omega\,. \tag{1.3.17}$$

Returning to the homogeneous inverse problem (1.3.4)–(1.3.7) we assume that there exists a nontrivial solution

$$u \in C^{2+\alpha,\,1+\alpha/2}(\bar{Q}_T)\,, \qquad f \in C^{\alpha}(\bar{\Omega})\,,$$

in which the function f involved cannot be identically equal to zero, since otherwise the uniqueness theorem related to the direct problem for parabolic equations would immediately imply the same property for the function u.

It is customary to introduce the well-established representation for the function f

$$f(x) = f^+(x) - f^-(x) , \tag{1.3.18}$$

where

$$f^+(x) = \max\{0, f(x)\}$$

and

$$f^-(x) = \max\{0, -f(x)\} .$$

Let us substitute into the system (1.3.4) in place of f, first, the function f^+ and, second, the function f^-. When solving the direct problem (1.3.4)–(1.3.6) in either of these cases, we denote by u_1 and u_2 the solutions of (1.3.4)–(1.3.6) with the right-hand sides $f^+(x)\, h(x,t)$ and $f^-(x)\, h(x,t)$, respectively. Via the transform $f = f^+ - f^-$ we deduce by the linearity of the operator L and the uniqueness of the solution of (1.3.4)–(1.3.6) that

$$u(x,t) = u_1(x,t) - u_2(x,t) , \qquad (x,t) \in Q_T . \tag{1.3.19}$$

In addition to (1.3.4)–(1.3.6), the function u also satisfies (1.3.7). With this relation in view, we deduce from (1.3.7) and (1.3.19) that

$$u_1(x,t) = u_2(x,t) \equiv \mu(x) , \qquad x \in \Omega , \tag{1.3.20}$$

where the function μ is introduced as a common notation for the final values of the functions u_1 and u_2. This provides reason enough for decision-making that several relations take place for $i = 1,\ 2$:

$$(u_i)_t(x,t) - (Lu_i)(x,t) = f^{\pm}(x)\, h(x,t) , \qquad (x,t) \in Q_T , \tag{1.3.21}$$

$$u_i(x,0) = 0 , \qquad x \in \Omega , \tag{1.3.22}$$

$$u_i(x,t) = 0 , \qquad (x,t) \in S_T . \tag{1.3.23}$$

In addition to the conditions imposed above, the solutions of (1.3.21)–(1.3.23) must satisfy (1.3.20).

It is clear that the functions f^+ and f^- cannot be identically equal to zero simultaneously. In this context, it is of interest three possible cases. Special investigations will be done separately.

Case 1. Let $f^+(x) \not\equiv 0$ and $f^-(x) \equiv 0$. In that case $u_2(x,t) \equiv 0$ in Q_T and (1.3.20) implies that

$$u_1(x,T) = 0\,, \qquad x \in \Omega\,. \tag{1.3.24}$$

On the other hand, since $f^+(x)\,h(x,t) \geq 0$ and $f^+h \not\equiv 0$ in Q_T, on account of Lemma 1.1.5 for $x \in \Omega$ we would have $u_1(x,T) > 0$, violating (1.3.24). For this reason case 1 must be excluded from further consideration.

Case 2. Let $f^-(x) \not\equiv 0$ and $f^+(x) \equiv 0$. By the linearity of the homogeneous inverse problem (1.3.4)–(1.3.7) that case reduces to the preceding.

Case 3. Suppose that f^+ and f^- both are not identically equal to zero. Observe that the functions $v_i = (u_i)_t$, $i = 1, 2$, give solutions of the direct problems

$$(v_i)_t(x,t) - (Lv_i)(x,t) = f^{\pm}(x)\,h_t(x,t)\,, \quad (x,t) \in Q_T\,, \tag{1.3.25}$$

$$v_i(x,0) = f^{\pm}(x)\,h(x,0)\,, \quad x \in \Omega\,, \tag{1.3.26}$$

$$v_i(x,t) = 0\,, \quad (x,t) \in S_T\,, \tag{1.3.27}$$

and in so doing the relations

$$v_i(x,T) = (L\mu)(x) + f^{\pm}(x)\,h(x,T)\,, \qquad x \in \Omega\,,\ i = 1, 2\,, \tag{1.3.28}$$

occur. Let h and h_t belong to the space $C^{\alpha,\,\alpha/2}(\bar{Q}_T)$. Then by the **Newton–Leibniz formula** we can write

$$f^{\pm}(x)\,h(x,T) = f^{\pm}(x)\,h(x,0) + \int\limits_0^T f^{\pm}(x)\,h_t(x,t)\;dt\,.$$

With this relation established, one can verify that the right-hand side of equation (1.3.25) and the right-hand side of the initial condition (1.3.26) are not identically equal to zero simultaneously. Indeed, let

$$f^{\pm}(x)\,h(x,0) \,\equiv\, 0$$

and

$$f^{\pm}(x)\,h_t(x,0) \,\equiv\, 0$$

simultaneously. Then the Newton–Leibniz formula implies that

$$f^{\pm}(x)\,h(x,T) \,\equiv\, 0$$

in Ω. However, neither f^+ nor f^- is identically equal to zero. Then $h(x,T) \equiv 0$ in Ω, violating one of the conditions of the theorem $|\, h(x,T)\, | \geq \delta > 0$.

After this remark we can apply Lemma 1.1.5 to the system (1.3.25)–(1.3.27) to derive the inequality

$$v_i(x,T) > 0\,, \qquad x \in \Omega\,, \qquad i = 1,2\,.$$

Let $x_0 \in \Omega$ be a maximum point of the function μ. Then $(L\mu)(x_0) \leq 0$ and, in view of (1.3.28), the inequality $f^{\pm}(x_0) > 0$ should be valid. Therefore,

$$f^+(x_0)\, f^-(x_0) \neq 0\,,$$

which is not consistent with the fact that $f^+(x)\, f^-(x) = 0$ for $x \in \Omega$. The contradiction obtained shows that the assumption about the existence of a nontrivial solution of the inverse problem (1.3.4)–(1.3.7) fails to be true and, therefore, the homogeneous inverse problem (1.3.4)–(1.3.7) has a trivial solution only. By virtue of the alternative from Theorem 1.3.4 another conclusion can be drawn concerning the solvability of the inverse problem under consideration.

Theorem 1.3.5 *Let the operator L satisfy conditions* (1.1.15) *and* (1.3.15) *and let the coefficient $C(x) \leq 0$ for $x \in \Omega$. One assumes, in addition, that $h,\ h_t \in C^{\alpha,\,\alpha/2}(\bar{Q}_T)$, $h(x,t) \geq 0$ and $h_t(x,t) \geq 0$ for $(x,t) \in Q_T$, $h(x,T) \geq \delta > 0$ for $x \in \bar{\Omega}$ ($\delta \equiv \mathrm{const}$), $\varphi \in C^{2+\alpha}(\bar{\Omega})$. Under the compatibility conditions* (1.3.16) *the inverse problem* (1.2.34)–(1.2.37) *possesses a solution*

$$u \in C^{2+\alpha,\,1+\alpha/2}(\bar{Q}_T)\,, \qquad\qquad f \in C^{\alpha}(\bar{\Omega})$$

and this solution is unique in the indicated class of functions.

Recall how we pursued a detailed exploration of the initial inverse problem (1.2.2)–(1.2.5) in Sobolev's spaces (see Definition 1.2.2) by relating the values r and p as

$$(1.3.29) \qquad \begin{aligned} r &= 2 && \text{for} \quad p = \infty\,, \\ r &= \frac{2p}{p-2} && \text{for} \quad p \in (2,\, \infty)\,, \\ r &= \infty && \text{for} \quad p = 2\,. \end{aligned}$$

As we have mentioned above, the question of the uniqueness of the inverse problem (1.2.2)–(1.2.4) solution is equivalent to decision-making whether the corresponding homogeneous inverse problem with $g = b = 0$ in Q_T and $a = \varphi = 0$ in Ω possesses a trivial solution only. In the case of the Dirichlet boundary data and the final overdetermination the answer to the latter is provided by the following proposition.

Theorem 1.3.6 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16) *and let the coefficient $C(x) \le 0$ for $x \in \Omega$. One assumes, in addition, that h, $h_t \in L_{r,2}(Q_T)$, $h(x,T) > 0$ for $x \in \Omega$; $h(x,t) \ge 0$ and $h_t(x,t) \ge 0$ for $(x,t) \in Q_T$. Suppose that a pair of the functions $u \in W_{2,0}^{2,1}(Q_T)$ and $f \in L_p(\Omega)$ with r and p related by* (1.2.39) *gives a solution of the homogeneous inverse problem* (1.3.4)–(1.3.7). *Then $u = 0$ and $f = 0$ almost everywhere in Q_T and Ω, respectively.*

Proof On the contrary, let $f \not\equiv 0$ and $u \not\equiv 0$. As before, it is customary to introduce the new functions

$$f^+(x) = \max\{0, f(x)\}$$

and

$$f^-(x) = \max\{0, -f(x)\}.$$

Then, obviously, $f^\pm \in L_p(\Omega)$ and $hf^\pm$, $h_t f^\pm \in L_2(Q_T)$. When solving the direct problems (1.3.21)–(1.3.23) with the right-hand sides $h f^+$ and $h f^-$, respectively, as in Theorem 1.3.5, the functions u_1 and u_2 emerge as their unique solutions. Via the transform $f = f^+ - f^-$ we deduce by the linearity of the operator L and the uniqueness of the direct problem solution (compare (1.3.4)–(1.3.6) and (1.3.21)–(1.3.23)) that $u = u_1 - u_2$. Since the function u satisfies the condition of the final overdetermination (1.3.7), the values of the functions u_1 and u_2 coincide at the final moment $t = T$:

$$u_1(x,T) = u_2(x,T) \equiv \mu(x),$$

where μ is a known function. Due to the differential properties of the direct problem solution (see Lemma 1.1.2) the function μ belongs to the class $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. From such reasoning it seems clear that the functions f^+ and f^- cannot be identically equal to zero in Ω simultaneously. In this context, it is necessary to analyze three possible cases separately.

Case 1. Let $f^+(x) \not\equiv 0$ and $f^-(x) \equiv 0$ in Ω. Then $u_2(x,t) = 0$ almost everywhere in Q_T and $f(x) = f^+(x)$, $u(x,t) = u_1(x,t)$. Consequently, $u_1(x,T) = 0$ almost everywhere in Ω. On the other hand, it follows from Lemma 1.1.5 that $u_1(x,T) > 0$ in Ω if we deal in Q_T with $f^+h \ge 0$ and $f^-h \not\equiv 0$. The obtained contradiction shows that case 1 must be excluded from further consideration.

Case 2. Let $f^+(x) \equiv 0$ and $f^-(x) \not\equiv 0$ in Ω. By the linearity and homogeneity of the inverse problem (1.3.4)–(1.3.7) that case reduces to the preceding.

Case 3. Assume that $f^+(x) \not\equiv 0$ and $f^-(x) \not\equiv 0$ in Q_T. Applying Lemma 1.1.5 to the system (1.3.25)–(1.3.27) yields

$$u_i(x,T) \equiv \mu(x) > 0, \qquad x \in \Omega, \qquad i = 1,2. \tag{1.3.30}$$

In complete agreement with Lemma 1.1.2 the functions

$$v_i(x,t) \equiv (u_i)_t(x,t)\,, \qquad i = 1,\,2\,,$$

belong to $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ and the following relations

$$(1.3.31) \qquad (v_i)_t(x,t) - (Lv_i)(x,t) = f^{\pm}(x)\, h_t(x,t)\,, \qquad (x,t) \in Q_T\,,$$

$$(1.3.32) \qquad v_i(x,0) = f^{\pm}(x)\, h(x,0)\,, \qquad x \in \Omega\,,$$

$$(1.3.33) \qquad v_i(x,t) = 0\,, \qquad (x,t) \in S_T\,,$$

are valid in the sense of the corresponding integral identity.

Before we undertake the proof with the aid of Lemma 1.1.5, let us observe that the functions $f^{\pm} h(x,0)$ and $f^{\pm} h_t(x,t)$ can never be identically equal to zero once at a time. Indeed, assume to the contrary that $f^{\pm} h(x,0) \equiv 0$ and $f^{\pm} h_t(x,t) \equiv 0$ simultaneously. Then the Newton–Leibniz formula gives

$$f^{\pm}(x)\, h(x,T) = f^{\pm}(x)\, h(x,0) + \int_0^T f^{\pm}(x)\, h_t(x,t)\; dt\,,$$

yielding

$$f^{\pm}(x)\, h(x,T) \equiv 0$$

over Ω. However, by requirement,

$$f^{+} \not\equiv 0$$

and

$$f^{-} \not\equiv 0\,.$$

This provides support for the view that $h(x,T) \equiv 0$, $x \in \Omega$, which disagrees with $h(x,T) > 0$.

In accordance with what has been said, Lemma 1.1.5 is needed to derive the inequality

$$(1.3.34) \qquad (u_i)_t(x,T) \equiv v_i(x,T) > 0\,, \qquad x \in \Omega\,, \qquad i = 1,\,2\,.$$

On the other hand, Lemma 1.1.2 implies that

$$u_i \in C\big([0,T];\, W_2^2(\Omega)\big)$$

along with

$$(u_i)_t \in C\big([0,T];\, L_2(\Omega)\big)\,.$$

Having stipulated these conditions, equation (1.3.21) leads to the relations

$$(u_i)_t(x,T) - (L\mu)(x) = f^{\pm}(x)\,h(x,T)\,, \qquad x \in \Omega\,, \qquad i = 1,2\,. \tag{1.3.35}$$

Here we used also that the values of u_1 and u_2 coincide at the moment $t = T$ and were denoted by μ.

When $L\mu \not\equiv 0$ in Ω, it follows from (1.3.30) and Corollary 1.1.2 that there exists a measurable set $\Omega' \subset \Omega$ such that

$$\operatorname{mes}_n \Omega' > 0$$

and

$$(L\mu)(x) < 0\,, \qquad x \in \Omega'. \tag{1.3.36}$$

By assumption, $h(x,T) > 0$. In view of this, relations (1.3.35)–(1.3.36) imply that $f^+(x) > 0$ and $f^-(x) > 0$ in Ω'. But this contradicts in Ω the identity $f^+ \cdot f^- \equiv 0$, valid for the functions f^+ and f^- of such constructions.

For the case $L\mu \equiv 0$ relation (1.3.35) can be rewritten as

$$(u_i)_t(x,T) = f^{\pm}(x)\,h(x,T)\,, \qquad x \in \Omega\,, \qquad i = 1,2\,. \tag{1.3.37}$$

Therefore, relations (1.3.34) and (1.3.37) imply that again $f^+ > 0$ and $f^- > 0$ in Ω. As stated above, this disagrees with constructions of the functions f^+ and f^-. Thus, all possible cases have been exhausted and the theorem is completely proved. ■

Remark 1.3.2 A similar uniqueness theorem is still valid for the inverse problem of recovering the source term coefficient in the general statement (1.2.2)–(1.2.5) (see Prilepko and Kostin (1992a)).

Theorems 1.3.3 and 1.3.6 imply immediately the unique solvability of the inverse problem (1.2.34)–(1.2.37). We quote this result for the inverse problem (1.2.2)–(1.2.5) in a common setting.

Theorem 1.3.7 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16) *and the coefficient $C(x) \geq 0$ for $x \in \Omega$. One assumes, in addition, that h, $h_t \in L_\infty(Q_T)$, $\sigma \in C(\partial\Omega)$, $g \equiv 0$, $a \equiv 0$, $b \equiv 0$, $\omega \in L_2([0,T])$, $\chi \in W_2^2(\Omega)$ and $(\mathcal{B}\chi)(x) = 0$ for $x \in \partial\Omega$. Also, let $h(x,t) \geq 0$ and $h_t(x,t) \geq 0$ almost everywhere in Q_T; $\sigma(x) \geq 0$ for $x \in \partial\Omega$; $\omega(t) \geq 0$ almost everywhere on $(0,T)$ and $[l(h)](x) \geq \delta > 0$ for $x \in \bar{\Omega}$ ($\delta \equiv$ const). Then the inverse problem (1.2.2)–(1.2.5) has a solution $u \in W_2^{2,1}(Q_T)$, $f \in L_2(\Omega)$, this solution is unique in the indicated class of functions and the estimate*

$$\|\, u \,\|_{2,Q_T}^{(2,1)} + \|\, f \,\|_{2,\Omega} \leq c\, \|\, \chi \,\|_{2,\Omega}^{(2)}$$

is true, where the constant c is expressed only in terms of the input data and does not depend on u and f.

1.4 The nonlinear coefficient inverse problem: recovering a coefficient depending on x

In this section we study the inverse problem of finding a coefficient at the function u involved in the equations of parabolic type.

Let us consider the inverse problem of recovering a pair of the functions $\{u, f\}$ from the equation

$$(1.4.1)\qquad \begin{aligned} &u_t(x,t) - (Lu)(x,t) \\ &\quad = f(x)\, u(x,t) + g(x,t)\,, \quad (x,t) \in Q_T \equiv \Omega \times (0,T)\,, \end{aligned}$$

the initial condition

$$(1.4.2)\qquad u(x,0) = 0\,, \qquad\qquad x \in \Omega\,,$$

the boundary condition

$$(1.4.3)\qquad u(x,t) = b(x,t)\,, \qquad\qquad (x,t) \in S_T\,,$$

and the condition of **final overdetermination**

$$(1.4.4)\qquad u(x,T) = \varphi(x)\,, \qquad\qquad x \in \Omega\,.$$

Here we have at our disposal the operator L, the functions g, b and φ and the domain Ω.

The linear operator L is supposed to be uniformly elliptic subject to conditions (1.1.15)–(1.1.16). In what follows we will use the notation

$$E \;=\; L_\infty(\Omega)$$

and

$$E_- = \{\, f(x) \in E\colon\; f(x) \le 0,\; x \in \Omega\}\,.$$

Definition 1.4.1 *A pair of the functions $\{u,\, f\}$ is said to be a solution of the inverse problem* (1.4.1)–(1.4.4) *if $u \in W_2^{2,1}(Q_T)$, $f \in E_-$ and all of the relations* (1.4.1)–(1.4.4) *take place.*

Recall that, in general, the boundary condition function b is defined on the entire cylinder Q_T. The boundary condition (1.4.3) is to be understood in the sense that the function $u - b$ vanishes on S_T.

One can readily see that the present inverse problem of recovering the coefficient is nonlinear. This is due to the fact that the linear differential equation contains a product of two unknown functions. As a first step towards the solution of this problem, it is necessary to raise the question of the solution uniqueness. Assume that there were two distinct pairs of functions u_1, f_1 and u_2, f_2, solving the inverse problem (1.4.1)–(1.4.4) simultaneously. Note that f_1 cannot coincide with f_2, since otherwise the same would be valid for the functions u_1 and u_2 due to the direct problem uniqueness theorem with regard to (1.4.1)–(1.4.3). Since the functions $v = u_1 - u_2$ and $f = f_1 - f_2$ give the solutions of the inverse problem

$$\begin{aligned}
&(1.4.5) \qquad && v_t(x,t) - (Lv)(x,t) - f_1(x)\, v(x,t) && \\
& && \qquad = f(x)\, u_2(x,t)\,, && (x,t) \in Q_T\,, \\
&(1.4.6) \qquad && v(x,0) = 0\,, && x \in \Omega\,, \\
&(1.4.7) \qquad && v(x,t) = 0\,, && (x,t) \in S_T\,, \\
&(1.4.8) \qquad && v(x,T) = 0\,, && x \in \Omega\,,
\end{aligned}$$

the way of proving the uniqueness for the nonlinear coefficient inverse problem (1.4.1)–(1.4.4) amounts to making a decision whether the linear inverse problem (1.4.5)–(1.4.6) has a trivial solution only.

In what follows we will assume that the coefficient $C(x)$ of the operator L satisfies the inequality $C(x) \le 0$ in Ω and

$$(1.4.9) \qquad \begin{aligned}
&g,\ g_t \in L_2(Q_T)\,; \quad g \ge 0 \quad g_t \ge 0 \quad \text{in} \quad Q_T\,; \\
&b,\ b_t \in W_2^{2,1}(Q_T)\,; \quad b \ge 0 \quad b_t \ge 0 \quad \text{on} \quad S_T\,.
\end{aligned}$$

With these assumptions, joint use of Theorem 1.1.8 and Lemma 1.1.5 permits us to deduce that $u_2(x,t) \ge 0$ and $(u_2)_t(x,t) \ge 0$ in Q_T, and $u_2(x,T) > 0$ in Ω. This provides support for the view that the linear inverse problem (1.4.5)–(1.4.8) of finding a pair of the functions $\{v,\ f\}$ is of the same type as (1.3.4)–(1.3.7). Applying Theorem 1.3.6 yields that $v = 0$ and $f = 0$ almost everywhere in Q_T and Ω, respectively. Summarizing, we obtain the following result.

Theorem 1.4.1 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16) *and let the coefficient $C(x) \le 0$ in Ω. One assumes, in addition, that condition* (1.4.9) *holds for the functions g and b. Then the inverse problem* (1.4.1)–(1.4.4) *can have at most one solution.*

In this regard, the question of the solution existence for the nonlinear inverse problem (1.4.1)–(1.4.4) arises naturally. We begin by deriving an operator equation of the second kind for the coefficient f keeping

$$(1.4.10)\qquad \begin{gathered} g,\, b,\, g_t \in W_p^{2,1}(Q_T)\,, \quad p \geq n+1\,; \quad L\varphi \in E\,; \\ \varphi(x) \geq \delta > 0 \quad \text{in} \quad \Omega\,; \quad b(x,0) = 0 \quad \text{for} \quad x \in \partial\Omega\,. \end{gathered}$$

By relating a function f from E to be fixed we substitute it into equation (1.4.1). The well-known results of Ladyzhenskaya and Uraltseva (1968) guide the proper choice of the function u as a unique solution of the direct problem (1.4.1)–(1.4.3). It will be convenient to refer to the nonlinear operator

$$A\colon\ E \mapsto E$$

with the values

$$(1.4.11)\qquad (Af)(x) = \frac{1}{\varphi(x)}\Big[u_t(x,T) - (L\varphi)(x) - g(x,T)\Big]\,, \qquad x \in \Omega\,,$$

and consider over E the operator equation of the second kind associated with the function f:

$$(1.4.12)\qquad f = Af.$$

We will prove that the solvability of equation (1.4.12) implies that of the inverse problem (1.4.1)–(1.4.4).

Lemma 1.4.1 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16), *the coefficient $C(x) \leq 0$ for $x \in \Omega$ and condition* (1.4.10) *hold. Also, let the compatibility condition*

$$b(x,T) = \varphi(x)$$

be fulfilled for $x \in \partial\Omega$. One assumes, in addition, that equation (1.4.12) *admits a solution lying within E_-. Then there exists a solution of the inverse problem* (1.4.1)–(1.4.4).

Proof By assumption, equation (1.4.12) has a solution lying within E_-, say f. Substitution f into (1.4.1) helps find u as a solution of the direct problem (1.4.1)–(1.4.3) for which there is no difficulty to establish that $u_t \in W_p^{2,1}(Q_T)$, $u_t(\,\cdot\,,T) \in E$ and $(Lu)(\,\cdot\,,T) \in E$ (see Ladyzhenskaya and Uraltseva (1968)). We will show that the function u so defined satisfies also the overdetermination condition (1.4.4). In preparation for this, set

$$u(x,T) = \varphi_1(x)\,, \qquad x \in \Omega\,.$$

By the construction of the function u,

$$u_t(x,T) - (L\varphi_1)(x) - g(x,T) = f(x)\,\varphi_1(x)\,. \tag{1.4.13}$$

On the other hand, the function f being a solution to equation (1.4.12) provides

$$\varphi(x)\,(Af)(x) = f(x)\,\varphi(x)\,. \tag{1.4.14}$$

After subtracting (1.4.13) from (1.4.14) we conclude that the function $\varphi-\varphi_1$ satisfies the equation

$$\big[L(\varphi-\varphi_1)\big](x) + f(x)(\varphi-\varphi_1) = 0\,, \qquad x\in\Omega\,. \tag{1.4.15}$$

Combination of the boundary condition (1.4.3) and the compatibility condition gives

$$(\varphi-\varphi_1)(x) = 0\,, \qquad x\in\partial\Omega\,. \tag{1.4.16}$$

As $C(x) \le 0$ in Ω and $f \in E_-$, the stationary direct problem (1.4.15)–(1.4.16) has only a trivial solution due to Theorem 1.1.3. Therefore, the function u satisfies the final overdetermination condition (1.4.4) and the inverse problem (1.4.1)–(1.4.4) is solvable, thereby completing the proof of the lemma. ■

As we have already mentioned, the Birkhoff–Tarsky theorem is much applicable in solving nonlinear operator equations. A key role in developing this approach is to check whether the operator A is **isotonic**.

Lemma 1.4.2 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16), *the coefficient $C(x)\le 0$ in Ω, conditions* (1.4.10) *hold, $g(x,t)\ge 0$ and $g_t(x,t)\ge 0$ in Q_T. If, in addition, the compatibility conditions*

$$b(x,T) = \varphi(x) \quad \textit{and} \quad b_t(x,0) = g(x,0)$$

are fulfilled for any $x\in\partial\Omega$, then A is an isotonic operator on E_-.

Proof First of all we stress that E_- is a conditionally complete lattice. Let f_1 and f_2 be arbitrary elements of E_- with $f_1 \le f_2$. One trick we have encountered is to substitute f_1 and f_2 both into equation (1.4.1) with further passage to the corresponding direct problems for $i = 1,\,2$:

$$\begin{aligned}
&(1.4.17) \qquad u_t^i(x,t) - (Lu^i)(x,t) \\
&\qquad\qquad = f_i(x)\,u^i(x,t) + g(x,t)\,, \quad (x,t)\in Q_T\,,\\
&(1.4.18) \qquad u^i(x,0) = 0\,, \qquad\qquad x\in\Omega\,,\\
&(1.4.19) \qquad u^i(x,t) = b(x,t)\,, \qquad (x,t)\in S_T\,,
\end{aligned}$$

In this regard, the conditions of the lemma assure that the function $w^i \equiv u^i_t$ presents a solution of the direct problem

$$(1.4.20)\qquad \begin{aligned} & w^i_t(x,t) - (Lw^i)(x,t) \\ & \qquad = f_i(x)\, w^i(x,t) + g_t(x,t)\,, \quad (x,t) \in Q_T\,, \end{aligned}$$

$$(1.4.21)\qquad w^i(x,0) = g(x,0)\,, \qquad x \in \Omega\,,$$

$$(1.4.22)\qquad w^i(x,t) = b_t(x,t)\,, \qquad (x,t) \in S_T\,.$$

Being solutions of (1.4.17)–(1.4.19), the functions $v = u^2 - u^1$ and $h = f_2 - f_1$ are subject to the set of relations:

$$(1.4.23)\qquad \begin{aligned} & v_t(x,t) - (Lv)(x,t) - f_2(x)\, v(x,t) \\ & \qquad = h(x)\, u^1(x,t)\,, \qquad (x,t) \in Q_T\,, \end{aligned}$$

$$(1.4.24)\qquad v(x,0) = 0\,, \qquad x \in \Omega\,,$$

$$(1.4.25)\qquad v(x,t) = 0\,, \qquad (x,t) \in S_T\,.$$

Just now it is necessary to keep in mind that $f_1 \le f_2$. On account of Theorem 1.1.8 and Lemma 1.1.5 the systems (2.4.17)–(2.4.19) and (2.4.20)–(2.4.22) provide that $h(x)\, u^1(x,t) \ge 0$ and $h(x)\, u^1_t(x,t) \ge 0$ for $(x,t) \in Q_T$. Once again, appealing to Theorem 1.1.8 and Lemma 1.1.5, we deduce that $v_t(x,T) \ge 0$ in Ω. By definition (1.4.11) of the operator A, $Af_1 \le Af_2$, what means that A is isotonic on E_-. This proves the assertion of the lemma. ■

We now turn to a common setting and proving the principal global result concerning the unique solvability of the inverse problem at hand.

Theorem 1.4.2 *Let the operator L satisfy conditions* (1.1.15)–(1.1.16) *and the coefficient $C(x) \le 0$ in Ω. Let g, b, $b_t \in W^{2,1}_p(Q_T)$, $p \ge n+1$; $g(x,t) \ge 0$ and $g_t(x,t) \ge 0$ for $(x,t) \in Q_T$; $L\varphi \in E$; $\varphi(x) \ge \delta > 0$ in Ω ($\delta \equiv$ const). Also, we take for granted the compatibility conditions*

$$b(x,0) = 0\,, \quad b(x,T) = \varphi(x) \quad \text{and} \quad b_t(x,0) = g(x,0) \quad \text{for} \quad x \in \partial\Omega$$

and require that

$$(1.4.26)\qquad L\big[\, u^0(x,T) - \varphi(x)\big] \le 0\,, \qquad x \in \Omega\,,$$

where u^0 refers to a solution of problem (1.4.1)–(1.4.3) *with $f \equiv 0$. Then the inverse problem* (1.4.1)–(1.4.4) *has a solution $u \in W^{2,1}_p(Q_T)$, $f \in E_-$ and this solution is unique in the indicated class of functions.*

Proof First we are going to show that on E_- there exists an order segment which is mapped by the operator A onto itself. Indeed, it follows from the foregoing that (1.4.26) implies the inequality

$$(1.4.27) \qquad A(0) \equiv \frac{1}{\varphi(x)} \, L\big[\, u^0(x,T) - \varphi(x)\big] \le 0\,, \qquad x \in \Omega\,,$$

and, consequently,

$$A\colon\, E_- \mapsto E_-\,.$$

Let us take a constant M from the bound

$$(1.4.28) \qquad M \ge \frac{1}{\varphi(x)} \left[(L\varphi)(x) + g(x,T)\right], \qquad x \in \Omega\,.$$

By definition (1.4.10) of the operator A,

$$(1.4.29) \quad A(-M) \equiv \frac{1}{\varphi(x)} \left[\, u_t(x,T) - (L\varphi)(x) - g(x,T)\,\right], \qquad x \in \Omega\,,$$

where u is a solution of the system (1.4.1)–(1.4.3) with constant $-M$ standing in place of the coefficient f.

Before giving further motivations, let us recall that the coefficients of the operator L do not depend on t. For the same reason as before, the function u_t gives a solution from $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ of the direct problem

$$\begin{aligned} &w_t - Lw = -Mw + g_t\,, && (x,t) \in Q_T\,,\\ &w(x,0) = g(x,0)\,, && x \in \Omega\,,\\ &w(x,t) = b_t(x,t)\,, && (x,t) \in S_T\,, \end{aligned}$$

whence by Theorem 1.1.8 and Lemma 1.1.5 it follows that in Ω

$$u_t(x,T) \equiv w(x,T) \ge 0\,.$$

Thus, (1.4.28)–(1.4.29) imply the inequality

$$(1.4.30) \qquad A(-M) \ge -M\,.$$

Because of (1.4.27) and (1.4.30), the operator A being isotonic carries the order segment

$$\left[-M,\, 0\right] = \{\, f \in E_-\colon\, -M \le f \le 0\,\}$$

of the conditionally complete and partially ordered set E_- into itself. By the Birkhoff–Tarsky theorem (Theorem 1.1.9) the operator A has at least one fixed point in $[-M,\, 0] \subset E_-$ and, therefore, equation (1.4.11) is solvable on E_-. In conformity with Lemma 1.4.1 the inverse problem (1.4.1)–(1.4.4) has a solution. The uniqueness of this solution follows immediately from Theorem 1.4.1 and thereby completes the proof of the theorem. ■

Remark 1.4.1 A similar theorem of existence and uniqueness is valid for the case where the boundary condition is prescribed in the general form (1.2.4) (see Prilepko and Kostin (1992b)).

1.5 The linear inverse problem: recovering the evolution of a source term

This section is devoted to inverse problems of recovering the coefficients depending on t. The main idea behind method here is to reduce the inverse problem to a certain integral equation of the Volterra type resulting in global theorems of existence and uniqueness. We consider the two types of overdetermination: pointwise and integral. In the case of a pointwise overdetermination the subsidiary information is the value of the function u at a point x_0 of the domain Ω at every moment within the segment $[0, T]$. In another case the function u is measured by a sensor making a certain averaging over the domain Ω. From a mathematical point of view the result of such measurements can be expressed in the form of integral overdetermination. We begin by placing the problem statement for the latter case.

Being concerned with the functions g, ω and φ, we study in the cylinder $Q_T \equiv \Omega \times (0, T)$ the inverse problem of finding a pair of the functions $\{u, f\}$ from the equation

$$(1.5.1) \quad u_t(x,t) - \Delta u(x,t) = f(t)\, g(x,t)\,, \quad (x,t) \in Q_T\,,$$

the initial condition

$$(1.5.2) \quad u(x,0) = 0\,, \qquad x \in \Omega\,,$$

the boundary condition

$$(1.5.3) \quad u(x,t) = 0\,, \qquad (x,t) \in S_T = \partial\Omega \times [0,T]\,,$$

and the condition of **integral overdetermination**

$$(1.5.4) \quad \int\limits_{\Omega} u(x,t)\,\omega(x)\, dx = \varphi(t)\,, \qquad t \in [0,T]\,.$$

A rigorous definition for a solution of this inverse problem is presented below.

Definition 1.5.1 *A pair of the functions* $\{u, f\}$ *is said to be a* **generalized solution** *of the inverse problem* (1.5.1)–(1.5.4) *if* $u \in \overset{\circ}{W}{}_{2}^{2,1}(Q_T)$, $f \in L_2(0,T)$ *and all of the relations* (1.5.1)–(1.5.4) *occur.*

In what follows we agree to consider

$$(1.5.5)\qquad \begin{gathered} g \in C\big([0,T], L_2(\Omega)\big)\,, \quad \omega \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega)\,, \quad \varphi \in W_2^1(0,T)\,, \\ \Big| \int_\Omega g(x,t)\, \omega(x)\; dx \Big| \geq g^* \equiv \text{const} > 0 \quad \text{for} \quad 0 \leq t \leq T\,. \end{gathered}$$

The first goal of our studies is to derive a linear second kind equation of the Volterra type for the coefficient f over the space $L_2(0,T)$. The well-founded choice of a function f from the space $L_2(0,T)$ may be of help in achieving this aim. Substitution into (1.5.1) motivates that the system (1.5.1)–(1.5.3) serves as a basis for finding the function $u \in W_{2,0}^{2,1}(Q_T)$ as a unique solution of the direct problem (1.5.1)–(1.5.3). The correspondence between f and u may be viewed as one possible way of specifying the linear operator

$$A\colon\; L_2(0,T) \mapsto L_2(0,T)$$

with the values

$$(1.5.6)\qquad (Af)(t) = \frac{1}{g_1(t)} \int_\Omega u(x,t)\, \Delta\omega(x)\; dx\,,$$

where

$$g_1(t) = \int_\Omega g(x,t)\omega(x)\; dx\,.$$

In this view, it is reasonable to refer to the linear equation of the second kind for the function f over the space $L_2(0,T)$:

$$(1.5.7)\qquad f = A\, f + \psi\,,$$

where $\psi(t) = \varphi'(t)\, g_1(t)$.

Theorem 1.5.1 *Let the input data of the inverse problem* (1.5.1)–(1.5.4) *satisfy* (1.5.5). *Then the following assertions are valid:*

(a) *if the inverse problem* (1.5.1)–(1.5.4) *is solvable, then so is equation* (1.5.7)*;*

(b) *if equation* (1.5.7) *possesses a solution and the compatibility condition*

$$(1.5.8)\qquad \varphi(0) = 0$$

holds, then there exists a solution of the inverse problem (1.5.1)–(1.5.4).

Proof We proceed to prove item (a) assuming that the inverse problem (1.5.1)–(1.5.4) is solvable. We denote its solution by $\{u, f\}$. Multiplying both sides of (1.5.1) by the function $\omega(x)$ scalarly in $L_2(\Omega)$ we establish the relation

$$(1.5.9)\quad \frac{d}{dt}\int_\Omega u(x,t)\,\omega(x)\,dx + \int_\Omega u(x,t)\,\Delta\omega(x,t)\,dt = f(t)\int_\Omega g(x,t)\,\omega(x)\,dx\,.$$

Because of (1.5.4) and (1.5.6), it follows from (1.5.9) that $f = Af + \varphi'/g_1$. But this means that f solves equation (1.5.7).

We proceed to prove item (b). By the original assumption equation (1.5.7) has a solution in the space $L_2(0,T)$, say f. When inserting this function in (1.5.1), the resulting relations (1.5.1)–(1.5.3) can be treated as a direct problem having a unique solution $u \in \overset{\circ}{W}{}^{2,1}_{2,0}(Q_T)$.

In this line, it remains to show that the function u satisfies also the integral overdetermination (1.5.4). Indeed, (1.5.1) yields

$$(1.5.10)\quad \frac{d}{dt}\int_\Omega u(x,t)\,\omega(x)\,dx + \int_\Omega u(x,t)\,\Delta\omega(x,t)\,dt = f(t)\,g_1(t)\,.$$

On the other hand, being a solution to equation (1.5.7), the function u is subject to the relation

$$(1.5.11)\quad \varphi'(t) + \int_\Omega u(x,t)\,\Delta\omega(x,t)\,dt = f(t)\,g_1(t)\,.$$

After subtracting (1.5.11) from (1.5.10) we finally get

$$\frac{d}{dt}\int_\Omega u(x,t)\,\omega(x)\,dx - \varphi'(t) = 0\,.$$

Integrating the preceding differential equation and taking into account the compatibility condition (1.5.8), we find out that the function u satisfies the overdetermination condition (1.5.4) and the pair of the functions $\{u, f\}$ is just a solution of the inverse problem (1.5.1)–(1.5.4). This completes the proof of the theorem. ■

Before considering details, it will be sensible to touch upon the properties of the operator A. In what follows the symbol A^s $(s = 1, 2, \dots)$ refers to the sth degree of the operator A.

Lemma 1.5.1 *Let condition* (1.5.5) *hold. Then there exists a positive integer* s_0 *for which* A^{s_0} *is a contracting operator in* $L_2(0,T)$.

Proof Obviously, (1.5.6) yields the estimate

$$\| Af \|_{2,(0,t)} \le \frac{1}{g^*} \, \| \Delta\omega \|_{2,\Omega} \cdot \| u \|_{2,Q_t} , \qquad 0 \le t \le T . \tag{1.5.12}$$

Multiplying both sides of (1.5.1) by u scalarly in $L_2(\Omega)$ and integrating the resulting expressions by parts, we arrive at the identity

$$\frac{1}{2} \frac{d}{dt} \| u(\,\cdot\,,t) \|_{2,\Omega}^2 + \| u_x(\,\cdot\,,t) \|_{2,\Omega}^2 = \int_\Omega f(t)\, g(x,t)\, u(x,t)\; dx \,, \quad 0 \le t \le T ,$$

and, subsequently, the inequality

$$\frac{d}{dt} \| u(\,\cdot\,,t) \|_{2,\Omega} \le \| f(t)\, g(\,\cdot\,,t) \|_{2,\Omega} , \qquad 0 \le t \le T .$$

In this line,

$$\| u(\,\cdot\,,t) \|_{2,\Omega} \le \| u(\,\cdot\,,0) \|_{2,\Omega} + \sup_{[0,T]} \| g(\,\cdot\,,t) \|_{2,\Omega} \int_0^t | f(\tau) | \, d\tau, \tag{1.5.13}$$

$$0 \le t \le T .$$

Since $u(x,0) = 0$, relations (1.5.12) and (1.5.13) are followed by the estimate

$$\| Af \|_{2,(0,t)} \le \mu \biggl(\int_0^t \bigl(\| f \|_{2,(0,\tau)} \bigr)^2 \; d\tau \biggr)^{1/2} , \qquad 0 \le t \le T , \tag{1.5.14}$$

where

$$\mu \; = \; \frac{\sqrt{T}}{g^*} \, \| \Delta\omega \|_{2,\Omega} \; \sup_{[0,T]} \; \| g(\,\cdot\,,\tau) \|_{2,\Omega} \, .$$

It is worth noting here that μ does not depend on t.

It is evident that for any positive integer s the sth degree of the operator A can be defined in a natural way. In view of this, estimate (1.5.14) via the mathematical induction gives

$$\| A^s f \|_{2,(0,T)} \le \left(\frac{\mu^{2s} \, T^s}{s\,!} \right)^{1/2} \| f \|_{2,(0,T)} , \qquad s = 1,\, 2,\, \ldots . \tag{1.5.15}$$

It follows from the foregoing that there exists a positive integer $s = s_0$ such that

$$\left(\frac{\mu^{2s_0} T^{s_0}}{s_0\,!} \right)^{1/2} < 1 . \tag{1.5.16}$$

This provides support for the view that the linear operator A^{s_0} is a contracting mapping on $L_2(0,T)$ and completes the proof of the lemma. ■

Regarding the unique solvability of the inverse problem concerned, the following result could be useful.

Theorem 1.5.2 *Let* (1.5.5) *and the compatibility condition* (1.5.8) *hold. Then the following assertions are valid:*

(a) *a solution* $\{u, f\}$ *of the inverse problem* (1.5.1)-(1.5.4) *exists and is unique;*

(b) *with any initial iteration* $f_0 \in L_2(0,T)$ *the successive approximations*

$$f_{n+1} = \widetilde{A}\, f_n\,, \qquad n = 0,\, 1,\, 2,\, \dots\,, \tag{1.5.17}$$

converge to f *in the* $L_2(0,T)$*-norm (for* $\widetilde{A}$ *see* (1.5.18) *below).*

Proof We have occasion to use the nonlinear operator

$$\widetilde{A}\colon\ L_2(0,T) \mapsto L_2(0,T)$$

acting in accordance with the rule

$$\widetilde{A}\, f = A\, f + \frac{\varphi'}{g_1}\,, \tag{1.5.18}$$

where the operator A and the function g_1 arise from (1.5.6). From (1.5.18) it follows that equation (1.5.7) can be recast as

$$f = \widetilde{A}\, f. \tag{1.5.19}$$

To prove the solvability of (1.5.19) it is sufficient to show that $\widetilde{A}$ has a fixed point in the space $L_2(0,T)$. With the aid of the relations

$$\widetilde{A}^s f_1 - \widetilde{A}^s f_2 \;=\; A^s f_1 - A^s f_2 \;=\; A^s\,(f_1 - f_2)$$

we deduce from estimate (1.5.15) that

$$(1.5.20)\quad \| \widetilde{A}^{s_0} f_1 - \widetilde{A}^{s_0} f_2 \|_{2,(0,T)} = \| A^{s_0} (f_1 - f_2) \|_{2,(0,T)} \le \left(\frac{\mu^{2s_0} T^{s_0}}{s_0 !} \right)^{1/2} \| f_1 - f_2 \|_{2,(0,T)} ,$$

where s_0 has been fixed in (1.5.16). By virtue of (1.5.16) and (1.5.20) it turns out that $\widetilde{A}^{s_0}$ is a contracting mapping on $L_2(0,T)$. Therefore, $\widetilde{A}^{s_0}$ has a unique fixed point f in $L_2(0,T)$ and the successive approximations (1.5.17) converge to f in the $L_2(0,T)$-norm without concern for how the initial iteration $f_0 \in L_2(0,T)$ will be chosen.

Just for this reason equation (1.5.19) and, in turn, equation (1.5.7) have a unique solution f in $L_2(0,T)$. According to Theorem 1.5.1, this confirms the existence of the inverse problem (1.5.1)–(1.5.4) solution. It remains to prove the uniqueness of this solution. Assume to the contrary that there were two distinct solutions $\{u_1, f_1\}$ and $\{u_1, f_1\}$ of the inverse problem under consideration. We claim that in that case $f_1 \neq f_2$ almost everywhere on $[0,T]$. Indeed, if $f_1 = f_2$, then applying the uniqueness theorem to the corresponding direct problem (1.5.1)–(1.5.3) we would have $u_1 = u_2$ almost everywhere in Q_T.

Since both pairs satisfy identity (1.5.9), the functions f_1 and f_2 give two distinct solutions to equation (1.5.19). But this contradicts the uniqueness of the solution to equation (1.5.19) just established and proves the theorem. ■

Corollary 1.5.1 *Under the conditions of Theorem* 1.5.2 *a solution to equation* (1.5.7) *can be expanded in a series*

$$(1.5.21)\qquad f = \psi + \sum_{s=1}^{\infty} A^s \psi$$

and the estimate

$$(1.5.22)\qquad \| f \|_{2,(0,T)} \le \rho \| \psi \|_{2,(0,T)}$$

is valid with

$$\psi = \frac{\varphi'}{g_1} , \qquad \rho = \sum_{s=0}^{\infty} \frac{\mu^s T^{s/2}}{(s!)^{1/2}} .$$

Proof The successive approximations (1.5.17) with $f_0 = \psi$ verify that

$$f_n = \widetilde{A}\, f_{n+1} = \widetilde{A}^n f_0 = \psi + \sum_{s=1}^{\infty} A^s \psi\,. \tag{1.5.23}$$

The passage to the limit as $n \to \infty$ in (1.5.23) leads to (1.5.21), since, by Theorem 1.5.2,

$$\| f - f_n \|_{2,(0,T)} \to 0$$

as $n \to \infty$.

Being concerned with A^s satisfying (1.5.12) we get the estimate

$$\| f \|_{2,(0,T)} \le \| \psi \|_{2,(0,T)} \sum_{s=0}^{\infty} \left(\frac{\mu^{2s}\, T^s}{s\,!} \right)^{1/2} .$$

By **D'Alambert ratio test** the series on the right-hand side converges, thereby completing the proof of the theorem. ■

As an illustration to the result obtained, we will consider the inverse problem for the one-dimensional heat equation and find the corresponding **solution in the explicit form**.

Example 1.5.1 We are exploring the inverse problem of finding a pair of the functions $\{u,\ f\}$ from the set of relations

$$u_t(x,t) = u_{xx}(x,t) + F(x,t)\,, \qquad (x,t) \in (0,\pi) \times (0,T)\,, \tag{1.5.24}$$

$$u(x,0) = 0\,, \qquad x \in [0,\pi]\,, \tag{1.5.25}$$

$$u(0,t) = u(\pi,t) = 0\,, \qquad t \in (0,T]\,, \tag{1.5.26}$$

$$\int_0^{\pi} u(x,t)\, \sin x\, dx = \varphi(t)\,, \qquad t \in [0,T]\,, \tag{1.5.27}$$

where

$$F(x,t) = f(t)\, \sin x\,, \qquad \varphi(t) = t\,. \tag{1.5.28}$$

In trying to solve it we employ the Fourier method of separation of variables with regard to the system (1.5.24)–(1.5.26), making it possible to derive the formula

$$u(x,t) = \sum_{k=1}^{\infty} \int_0^t F_k(\tau)\, \exp\{-k^2(t-\tau)\}\, d\tau\, \sin kx\,, \tag{1.5.29}$$

where

$$F_k(\tau) = \frac{2}{\pi} \int_0^{\pi} F(x,\tau) \sin kx \; dx \, .$$

Because of (1.5.28) and (1.5.29), the function F is representable by

$$u(x,t) = \int_0^t f(\tau) \exp\{-(t-\tau)\} \; d\tau \; \sin x \, . \tag{1.5.30}$$

Substituting (1.5.30) into (1.5.24) and taking into account the overdetermination condition (1.5.27), we derive the second integral Volterra equation of the type (1.5.7). In principle, our subsequent arguments do follow the general scheme outlined above. However, in this particular case it is possible to perform plain calculations by more simpler reasonings. Upon substituting (1.5.30) into (1.5.27) we obtain the integral Volterra equation of the first kind for the function f

$$t = \frac{\pi}{2} \int_0^t f(\tau) \exp\{-(t-\tau)\} \; d\tau \, . \tag{1.5.31}$$

Here we used also that $\varphi(t) = t$ in view of (1.5.28). In order to solve equation (1.5.31) one can apply the well-known integral transform

$$\hat{f}(p) = \int_0^{+\infty} \exp\{-pt\} \, f(t) \; dt \, , \tag{1.5.32}$$

where the function $\hat{f}(p)$ of one complex variable is termed the **Laplace transform** of the original function $f(t)$. The symbol $\div$ is used to indicate the identity between $f(t)$ and $\hat{f}(p)$ in the sense of the Laplace transform. Within this notation, (1.5.32) becomes

$$f(t) \div \hat{f}(p) \, .$$

Direct calculations by formula (1.5.32) show, for example, that

$$\exp\{-t\} \div \frac{1}{p+1} \; ; \qquad t^n \div \frac{n\,!}{p^{n+1}} \, , \qquad n = 0, 1, 2, \ldots \, . \tag{1.5.33}$$

With a **convolution** of two functions one associates

$$\int_0^t f(\xi) \, g(t-\xi) \; d\xi \div \hat{f}(p) \, \hat{g}(p) \, , \tag{1.5.34}$$

where $f(t) \div \hat{f}(p)$ and $g(t) \div \hat{g}(p)$. The outcome of taking the Laplace transform of both sides of (1.5.31) and using (1.5.34) is the algebraic equation for the function $\hat{f}(p)$:

$$\frac{1}{p^2} = \frac{\pi}{2}\, \hat{f}\, \frac{1}{p+1}\,, \tag{1.5.35}$$

giving

$$\hat{f} = \frac{2}{\pi} \left(\frac{1}{p} + \frac{1}{p^2} \right). \tag{1.5.36}$$

On the basis of (1.5.33) and (1.5.36) it is possible to recover the original function $f(t)$ as follows:

$$f(t) = \frac{2}{\pi}\,(1+t)\,. \tag{1.5.37}$$

Then formula (1.5.30) immediately gives the function

$$u(x,t) = \frac{2}{\pi}\, t\, \sin x\,. \tag{1.5.38}$$

From such reasoning it seems clear that the pair of functions (1.5.37)–(1.5.38) is just a solution of the inverse problem (1.5.24)–(1.5.27). But this solution was found by formal evaluation. However, due to the uniqueness theorems established above there are no solutions other than the pair (1.5.37)–(1.5.38).

We now turn our attention to the inverse problem of recovering a source term in the case of a pointwise overdetermination.

Assume that there exists a perfect sensor responsible for making measurements of exact values of the function u at a certain interior point $x_0 \in \Omega$ at any moment within the segment $[0,\ T]$. As a matter of fact, the pointwise overdetermination $u(x_0\,, t) = \varphi(t)$, $t \in [0, T]$, of a given function φ arises in the statement of the inverse problem of finding a pair of the functions $\{u,\ f\}$, satisfying the equation

$$u_t(x,t) - \Delta u(x,t) = f(t)\, g(x,t)\,, \qquad (x,t) \in Q_T\,, \tag{1.5.39}$$

the initial condition

$$u(x,0) = 0\,, \qquad x \in \Omega\,, \tag{1.5.40}$$

the boundary condition

$$u(x,t) = 0\,, \qquad (x,t) \in S_T\,, \tag{1.5.41}$$

and the condition of **pointwise overdetermination**

$$u(x_0,t) = \varphi(t)\,, \qquad t \in [0,T]\,, \tag{1.5.42}$$

when the functions g and φ are known in advance.

We outline here only the general approach to solving this inverse problem. Having no opportunity to touch upon this topic we address the readers to Prilepko and Soloviev (1987a).

We are still in the framework of the Fourier method of separating variables with respect to the system of relations (1.5.39)–(1.5.41), whose use permits us to establish the expansion

$$u(x,t) = \sum_{k=1}^{\infty} \int_0^t f(\tau)\, g_k(\tau)\, \exp\{-\lambda_k\,(t-\tau)\}\; d\tau\; X_k(x)\,, \tag{1.5.43}$$

where

$$g_k(\tau) = \frac{1}{\|\,X_k\,\|_{2,\Omega}^2} \int_\Omega g(x,\tau)\, X_k(x)\; d\tau$$

and $\{\lambda_k,\, X_k\}_{k=1}^{\infty}$ are the eigenvalues and the eigenfunctions of the Laplace operator emerging from the Sturm–Liouville problem (1.2.51). By inserting (1.5.43) in (1.5.42) we get a linear integral **Volterra equation of the first kind**

$$\varphi(t) = \sum_{k=1}^{\infty} \int_0^t f(\tau)\, g_k(\tau)\, \exp\{-\lambda_k(t-\tau)\}\; d\tau\; X_k(x_0)\,. \tag{1.5.44}$$

Assuming the functions φ and g to be sufficiently smooth and accepting $|\,g(x_0,t)\,| \geq g^* > 0$, $t \in [0,T]$, we can differentiate both sides of (1.5.44) with respect to t, leading to the integral **Volterra equation of the second kind**

$$f(t) = \int_0^t K(t,\tau)\, f(\tau)\; d\tau + \psi(t)\,, \tag{1.5.45}$$

where

$$\psi(t) = \frac{\varphi'(t)}{g(x_0, t)} ,$$

$$K(t, \tau) = \frac{1}{g(x_0, t)} \sum_{k=1}^{\infty} \lambda_k \, g_k(\tau) \exp\{-\lambda_k (t - \tau)\} \, X_k(x_0) .$$

As can readily be observed, the solvability of the inverse problem (1.5.39)–(1.5.42) follows from that of equation (1.5.45) if the compatibility condition $\varphi(0) = 0$ was imposed (see a similar result in Theorem 1.5.1). The existence and uniqueness of the solution to the Volterra equation (1.5.45), in turn, can be established in the usual way. The above framework may be useful in obtaining a unique global solution of the inverse problem (1.5.39)–(1.5.42).

As an illustration of our approach we consider the following problem.

Example 1.5.2 It is required to recover a pair of the functions $\{u, f\}$ from the set of relations

$$u_t(x,t) = u_{xx}(x,t) + f(t) \sin x , \quad (x,t) \in (0,\pi) \times (0,T) , \tag{1.5.46}$$

$$u(x,0) = 0 , \quad x \in [0,\pi] , \tag{1.5.47}$$

$$u(0,t) = u(\pi,t) = 0 , \quad t \in (0,T] , \tag{1.5.48}$$

$$u\left(\frac{\pi}{2} , t \right) = t , \quad t \in (0,T] . \tag{1.5.49}$$

Now equation (1.5.44) becomes

$$t = \int_0^t f(\tau) \exp\{-(t-\tau)\} \, d\tau . \tag{1.5.50}$$

Because of its form, the same procedure works as does for equation (1.5.31). Therefore, the functions $f(t) = 1 + t$ and $u(x,t) = t \sin x$ give the desired solution.

In concluding this chapter we note that the approach and results of this section carry out to the differential operators L of rather complicated and general form (1.1.8).

Chapter 2

Inverse Problems for Equations of Hyperbolic Type

2.1 Inverse problems for x-hyperbolic systems

Quite often, mathematical models for applied problems arising in natural sciences lead to hyperbolic systems of partial differential equations of the first order. This is especially true of hydrodynamics and aerodynamics. One more important case of such hyperbolic systems is connected with the system of Maxwell equations capable of describing electromagnetic fields. Until now the most profound research was devoted to systems of equations with two independent variables associated with one-dimensional models which do not cover fully the diversity of problems arising time and again in theory and practice. The situation becomes much more complicated in the case of multidimensional problems for which careful analysis requires a somewhat different technique. Moreover, the scientists were confronted with rather difficult ways of setting up and treating them on the same footing. Because of these and some other reasons choosing the most complete posing of several ones that are at the disposal of the scientists is regarded as one of the basic problems in this field. On the other hand, a one-dimensional problem can serve, as a rule, as a powerful tool for establishing the basic pattern and features of the behavior of solutions of hyperbolic systems. Some of them are of general character and remain valid for solutions of multidimensional problems. Adopting the arguments just mentioned it would be reasonable to restrict yourself to the study of linear hyperbolic systems of partial differential equations with two independent variables.

With this aim, we consider the **system of the first order** linear equations with two independent variables $x, t \in \mathbf{R}$:

$$A(x,t)\,\frac{\partial u}{\partial x} + B(x,t)\,\frac{\partial u}{\partial t} + C(x,t)\,u = F(x,t)\,, \tag{2.1.1}$$

where

$$u = u(x,t) = \big(u_1(x,t), \ldots, u_n(x,t)\big)$$

and

$$F(x,t) = \big(F_1(x,t), \ldots, F_n(x,t)\big)$$

are vectors, $A(x,t)$, $B(x,t)$ and $C(x,t)$ are $n \times n$-matrices for any fixed x and t. The matrix A is assumed to be invertible and the matrix $A^{-1}B$ can always be diagonalized. Any such system is said to be **x-hyperbolic**.

Let a matrix T reduce $A^{-1}B$ to a diagonal matrix K, that is, $K = T^{-1}(A^{-1}B)T$. Substituting $u = Tv$ and multiplying (2.1.1) by the matrix $T^{-1}A^{-1}$ from the left yields the **canonical form** of the x-hyperbolic system

$$\frac{\partial v}{\partial x} + K\,\frac{\partial v}{\partial t} + D\,v = G\,, \tag{2.1.2}$$

where

$$D = T^{-1}\,\frac{\partial T}{\partial x} + T^{-1}A^{-1}B\,\frac{\partial T}{\partial t} + T^{-1}A^{-1}C\,T\,,$$

$$G = T^{-1}A^{-1}F\,.$$

In what follows we will always assume that any x-hyperbolic system under consideration admits the canonical form (2.1.2). Under such a formalization the method of integrating along the corresponding characteristics will be adopted as a basic technique for investigating the system (2.1.2) in solving inverse problems.

In order to understand nature a little better, we introduce as preliminaries the auxiliary inverse problem in which it is required to find a pair of the functions $v_1, v_2 \in C^1([0,L] \times [0,+\infty])$ and a pair of the functions $p_1, p_2 \in C[0,L]$, satisfying the system of relations

$$
(2.1.3) \quad \begin{cases}
\dfrac{\partial v_1(x,t)}{\partial t} + \dfrac{\partial v_1(x,t)}{\partial x} = v_2(x,t) + p_1(x)\,, & 0 \le x \le L\,, \quad t \ge 0\,, \\[2ex]
\dfrac{\partial v_2(x,t)}{\partial t} - \dfrac{\partial v_2(x,t)}{\partial x} = v_1(x,t) + p_2(x)\,, & 0 \le x \le L\,, \quad t \ge 0\,, \\[2ex]
v_1(x,0) = \varphi_1(x)\,, \quad v_2(x,0) = \varphi_2(x)\,, & 0 \le x \le L\,, \\[2ex]
v_1(L,t) = \psi_1(x)\,, \quad v_2(0,t) = \psi_2(x)\,, & t \ge 0\,.
\end{cases}
$$

In the general case a solution of problem (2.1.3) is not obliged to be unique. In this connection, we should raise the question of imposing additional restrictions if we want to ensure the uniqueness of a solution of the inverse problem under consideration.

There are various ways of taking care of these restrictions. For example, the conditions for the **exponential growth** of the derivatives of the functions v_1 and v_2 with respect to t, meaning

$$
(2.1.4) \quad \left| \frac{\partial v_1(x,t)}{\partial t} \right| \le M_1 \exp\{a_1 t\}\,, \qquad \left| \frac{\partial v_2(x,t)}{\partial t} \right| \le M_1 \exp\{a_1 t\}\,,
$$

fall within the category of such restrictions. One succeeds in showing that under a sufficiently small value L conditions (2.1.4) guarantee not only the uniqueness, but also the existence of a solution of the inverse problem (2.1.3). This type of situation is covered by the following assertion.

Theorem 2.1.1 *Let φ_1, φ_2, ψ_1 and ψ_2 be continuously differentiable functions such that $\varphi_1(L) = \psi_1(0)$ and $\varphi_2(0) = \psi_2(0)$. One assumes, in addition, that there are positive constants a and M such that for any $t \ge 0$*

$$
|\,\psi_1'(t)\,| \le M \exp\{a\,t\}
$$

and

$$
|\,\psi_2'(t)\,| \le M \exp\{a\,t\}\,.
$$

Then there exists a value $L_0 = L_0(a) > 0$ such that for any $L < L_0$ the inverse problem (2.1.3) has a solution in the class of functions satisfying estimates (2.1.4). Moreover, there exists a value $L_1 = L_1(a_1) > 0$ such that for any $L_1 < L$ the inverse problem (2.1.3) can have at most one solution in the class of functions for which estimates (2.1.4) are true.

Theorem 2.1.1 follows from one more general theorem to be proved below. Here we only note that the value L_1 decreases with increasing a_1. This property is of special interest and needs investigation. Accepting $\varphi_1 = 0$, $\varphi_2 = 0$, $\psi_1 = 0$ and $\psi_2 = 0$ we say that the nontrivial solutions v_1 and v_2 of the system (2.1.3) corresponding to certain p_1 and p_2 constitute the **eigenfunctions of the inverse problem** (2.1.3). The meaning of existence of eigenfunctions of this inverse problem is that its solution is not unique there. Some of them can be found by the well-established method of separation of variables. Let

$$v_i(x,t) = p_i(x) \int_0^t \exp\{\alpha\,\tau\}\, d\tau\,, \qquad i = 1, 2\,, \qquad \alpha \in \mathbf{R}\,. \tag{2.1.5}$$

By separating variables we get the system coming from problem (2.1.3) and complementing later discussions:

$$\begin{aligned} &p_1'(x) + \alpha\, p_1(x) = p_2(x)\,, && 0 \le x \le L\,, \\ &p_2'(x) - \alpha\, p_2(x) = -p_1(x)\,, && 0 \le x \le L\,, \\ &p_1(L) = 0\,, \qquad p_2(0) = 0\,, \end{aligned}$$

followed by

$$\begin{cases} p_i'' + (1-\alpha^2)\, p_i = 0\,, & i = 1, 2\,, \quad 0 \le x \le L\,, \\ p_1(L) = 0\,, \qquad p_2(0) = 0\,. \end{cases} \tag{2.1.6}$$

For the purposes of the present chapter we have occasion to use the function

$$L^*(\alpha) = \begin{cases} \dfrac{\arccos \alpha}{\sqrt{1-\alpha^2}}\,, & |\,\alpha\,| < 1\,, \\ 1\,, & \alpha = 1\,, \\ \dfrac{\log(\alpha + \sqrt{1-\alpha^2})}{\sqrt{1-\alpha^2}}\,, & \alpha > 1\,. \end{cases} \tag{2.1.7}$$

One can readily see that problem (2.1.6) with $L = L^*(\alpha)$ possesses the nontrivial solution

$$p_1(x) = \begin{cases} \alpha \sin\left(\sqrt{1-\alpha^2}\, x\right) - \sqrt{1-\alpha^2} \cos\left(\sqrt{1-\alpha^2}\, x\right), & |\,\alpha\,| < 1\,, \\ x - 1\,, & \alpha = 1\,, \\ \alpha \operatorname{sh}\left(\sqrt{\alpha^2 - 1}\, x\right) - \sqrt{\alpha^2-1} \operatorname{ch}\left(\sqrt{\alpha^2-1}\, x\right), & \alpha > 1\,, \end{cases}$$

$$p_2(x) = \begin{cases} \sin\left(\sqrt{1-\alpha^2}\, x\right), & |\,\alpha\,| < 1\,, \\ x\,, & \alpha = 1\,, \\ \operatorname{sh}\left(\sqrt{\alpha^2-1}\, x\right), & \alpha > 1\,. \end{cases}$$

With these relations in view, we can specify by formulae (2.1.5) the eigenfunctions of the inverse problem (2.1.3).

The function $L^*(\alpha)$ is monotonically decreasing on the semi-axis $(-1, +\infty)$ and takes the following limiting values:

$$\lim_{\alpha \to -1+0} L^*(\alpha) = +\infty \,, \qquad \lim_{\alpha \to +\infty} L^*(\alpha) = 0 \,.$$

From such reasoning it seems clear that for any $L > 0$ the inverse problem (2.1.3) has the eigenfunctions of exponential type $\alpha > -1$. Due to this fact another conclusion can be drawn that if the exponential type $\alpha > -1$ is held fixed and $L > L^*(\alpha)$, then a solution of the inverse problem turns out to be nonunique in the class of functions of this exponential type.

Returning to the x-hyperbolic system in the general statement (2.1.2) we assume now that the function G is representable by

$$G(x,t) = H(x,t)\, p(x) \,, \tag{2.1.8}$$

where an $n \times n$-matrix $H(x,t)$ is known, while the unknown vector function p is sought. Under the approved form (2.1.2), we restrict ourselves to the case where $\det K \neq 0$. Assume also that the eigenvalues $k_1, \ldots, k_n$ of the matrix K are bounded and continuously differentiable in the domain $\{0 \le x \le L,\ t \ge 0\}$. In addition, let $k_i < 0$ for $1 \le i \le s$ and let $k_i > 0$ for $s < i \le n$ $(0 \le s \le n)$.

With these ingredients, we may set up the inverse problem of finding a pair of the functions

$$v(x,t) \in C^1 \,, \qquad p(x) \in C \,,$$

which must satisfy relations (2.1.2) and (2.1.8) together with the supplementary conditions

$$\begin{cases} v(x,0) = \varphi(x) \,, & 0 \le x \le L \,, \\ v_i(0,t) = \psi_i(t) \,, & t \ge 0 \,, \quad 1 \le i \le s \,, \\ v_i(L,t) = \psi_i(t) \,, & t \ge 0 \,, \quad s < i \le n \,. \end{cases} \tag{2.1.9}$$

Given a vector $v = (x_1, \ldots, v_n) \in \mathbf{R}^n$, the norm on that space is defined by

$$\| v \| = \max_{1 \le i \le n} | v_i | \,.$$

The associated operator **matrix norm** $\| A \|$ of an $n \times n$-matrix A is specified by

$$\| A \| = \sup_{\| v \| = 1} \| A v \| \,.$$

When providing the uniqueness of a solution, we should restrict ourselves to the class of functions being of great importance in the sequel and satisfying the condition of exponential growth like

$$\text{(E)} \qquad \| v_t(x,t) \| \le c \exp\{b t\}\,.$$

The following theorem is the precise formulation of one profound result.

Theorem 2.1.2 *Let D, H, $\partial D/\partial t$ and $\partial H/\partial t$ be continuous functions. One assumes, in addition, that K and ψ are continuously differentiable and*

$$\| D \| \le M, \ \left\| \partial D/\partial t \right\| \le M, \ \left\| \partial H/\partial t \right\| \le M, \ \left\| \partial K/\partial t \right\| \le M, \ \| K \| \le \beta,$$

$$\| \psi'(t) \| \le a \exp\{\alpha t\}$$

and

$$|\det H(x,0)| \ge \gamma > 0$$

with certain constants M, a, α, β and γ. Then there exist constants b, c and $L_0 > 0$ such that for $L < L_0$ and any continuously differentiable function φ satisfying the compatibility conditions

$$\varphi_i(0) = \psi_i(0)\,, \qquad 1 \le i \le s\,,$$

and

$$\varphi_i(L) = \psi_i(0)\,, \qquad s < i \le n\,,$$

the inverse problem (2.1.2), (2.1.8), (2.1.9) *has a solution in the class of functions satisfying condition (E).*

Proof A key role in the current proof is played by the **characteristic** $\tau_i(\xi; x,t)$ satisfying the system

$$\text{(2.1.10)} \qquad \begin{cases} \dfrac{d\tau_i}{d\xi} = k_i(\xi, \tau_i)\,, \\ \tau_i(x; x,t) = t\,, \end{cases}$$

and passing through a point (x,t). Because of the representation

$$\frac{d}{d\xi}\, v_i\big(\xi, \tau_i(\xi; x,t)\big) = \frac{\partial v_i}{\partial x} + k_i\, \frac{\partial v_i}{\partial t}\,,$$

we can integrate each component of (2.1.2). The outcome of this is

$$\text{(2.1.11)} \qquad v_i(x,t) = \psi_i\big(T_i(x,t)\big) + \int\limits_{\alpha_i}^{x} (-Dv + Hp)_i \, d\xi\,,$$

where

$$T_i(x,t) = \tau_i(\alpha_i; x, t), \qquad \alpha_i = \begin{cases} 0, & 1 \le i \le s, \\ L, & s < i \le n. \end{cases}$$

Putting these together with (2.1.9) we arrive at the relations

$$\varphi_i(x,t) - \psi_i\big(T_i(x,0)\big) = \int\limits_{\alpha_i}^{x} (-Dv + Hp)_i \, d\xi, \tag{2.1.12}$$

which should be rearranged for the new functions

$$w_i(x,t) = \frac{\partial}{\partial t} v_i(x,t).$$

As a final result we get

$$v_i(x,t) = \varphi_i(x) + \int\limits_0^t w_i(x,\tau)\, d\tau. \tag{2.1.13}$$

By differentiating (2.1.11) and (2.1.12) with respect to t and x, respectively, and involving (2.1.13) we derive the system of equations

$$w_i(x,t) = \Phi_i(x,t) + \int\limits_{\alpha_i}^{x} (K_1 w)_i \, d\xi + \int\limits_{\alpha_i}^{x} \int\limits_0^{\tau_i} (K_2 w)_i \, d\tau\, d\xi + \int\limits_{\alpha_i}^{x} (K_3 p)_i \, d\xi, \tag{2.1.14}$$

$$\big(H(x,0)\, p(x)\big)_i = \widetilde{\Phi}_i(x) + \int\limits_{\alpha_i}^{x} (\widetilde{K}_1 w)_i \, d\xi + \int\limits_{\alpha_i}^{x} \int\limits_0^{\tau_i} (\widetilde{K}_2 w)_i \, d\tau\, d\xi + \int\limits_{\alpha_i}^{x} (\widetilde{K}_3 p)_i \, d\xi \tag{2.1.15}$$

related to the new variables

$$\Phi_i(x,t) = \psi_i'\big(T_i(x,t)\big) \frac{\partial \tau_i}{\partial t} + \int\limits_{\alpha_i}^{x} \sum_{j=1}^{n} \frac{\partial D_{ij}}{\partial t} \frac{\partial \tau_i}{\partial t} \varphi_j(\xi)\, d\xi,$$

$$(K_1)_{ij}(x,t,\xi) = D_{ij}\big(\xi, \tau_i(\xi; x,t)\big)\,\frac{\partial \tau_i}{\partial t}(\xi; x,t)\,,$$

$$(K_2)_{ij}(x,t,\xi) = \frac{\partial D_{ij}}{\partial t}\big(\xi, \tau_i(\xi; x,t)\big)\,\frac{\partial \tau_i}{\partial t}(\xi; x,t)\,,$$

$$(K_3)_{ij}(x,t,\xi) = \frac{\partial H_{ij}}{\partial t}\big(\xi, \tau_i(\xi; x,t)\big)\,\frac{\partial \tau_i}{\partial t}(\xi; x,t)\,,$$

$$\widetilde{\Phi}_i(x) = \varphi_i'(x) - \psi_i'\big(T_i(x,0)\big)\,\frac{\partial T_i(x,0)}{\partial x}$$

$$- \sum_{j=1}^{n} D_{ij}(x,0)\,\varphi_j(x)$$

$$- \sum_{j=1}^{n} \int_{\alpha_i}^{x} \frac{\partial D_{ij}}{\partial t}\,\frac{\partial \tau_i}{\partial t}\,\varphi_j(\xi)\,d\xi\,,$$

$$(\widetilde{K}_1)_{ij}(x,\xi) = -D_{ij}\big(\xi, \tau_i(\xi; x,0)\big)\,\frac{\partial \tau_i(\xi; x,0)}{\partial x}\,,$$

$$(\widetilde{K}_2)_{ij}(x,\xi) = -\frac{\partial D_{ij}}{\partial t}\big(\xi, \tau_i(\xi; x,0)\big)\,\frac{\partial \tau_i(\xi; x,0)}{\partial x}\,,$$

$$(\widetilde{K}_3)_{ij}(x,\xi) = -\frac{\partial H_{ij}}{\partial t}\big(\xi, \tau_i(\xi; x,0)\big)\,\frac{\partial \tau_i(\xi; x,0)}{\partial x}\,.$$

It is worth noting here two useful expressions for the derivatives

$$\frac{\partial \tau_i}{\partial x}\,(\xi; x,t) = -k_i(x,t)\,\exp\left\{\int_x^{\xi} \frac{\partial k_i}{\partial t}\big(\xi, \tau_i(\xi; x,t)\big)\,d\xi\right\},$$

$$\frac{\partial \tau_i}{\partial t}\,(\xi; x,t) = \exp\left\{\int_x^{\xi} \frac{\partial k_i}{\partial t}\big(\xi, \tau_i(\xi; x,t)\big)\,d\xi\right\}.$$

Under the conditions of the theorem the functions Φ_i, $\widetilde{\Phi}_i$, $1 \le i \le n$, are really continuous and the matrices K_1, K_2, K_3, $\widetilde{K}_1$, $\widetilde{K}_2$ and $\widetilde{K}_3$ are continuous and bounded in complete agreement with a simple observation that the inequalities

$$\|K\| \le \beta\,, \qquad \|K_t\| \le M$$

assure us of the validity of the estimates

$$\left| \frac{\partial \tau_i}{\partial x} \right| \leq \beta \exp\{M\,L\}, \qquad \left| \frac{\partial \tau_i}{\partial t} \right| \leq \exp\{M\,L\}.$$

Most substantial is the fact that the system of the integral equations (2.1.14)–(2.1.15) in the class $w \in C$, $p \in C$ is equivalent (within the substitution formulae) to the inverse problem (2.1.2), (2.1.8), (2.1.9).

Let Ω be a half-strip $\{(x,t)\colon\ 0 \leq x \leq L,\ t \geq 0\}$. The symbol $C(\Omega)$ is used for the space of all pairs of functions

$$r = (w, p) = (w_1, \dots, w_n, p_1, \dots, p_n),$$

whose components w_i, $1 \leq i \leq n$, are defined and continuous on the half-strip Ω and p_i, $1 \leq i \leq n$, are defined and continuous on the segment $[0,\ L]$. It will be sensible to introduce in the space $C(\Omega)$ the system of **seminorms**

$$p_T(r) = \max_{\substack{0 \leq i \leq n \\ 0 \leq x \leq L \\ 0 \leq t \leq T}} \big\{ \max\{|\,w_i(x,t)\,|,\ |\,p_i(x)\,|\}\big\}$$

and the operator A acting in accordance with the rule

$$\bar{r} = (\bar{w},\ \bar{p}) \ = \ A\,r = A(w,p),$$

where the components of a vector $\bar{w}$ are defined by the last three terms in (2.1.14) and $\bar{p}$ is a result of applying the matrix $H(x,0)^{-1}$ to a vector, the components of which are defined by the last three terms in (2.1.15). Also, we initiate the construction of the vector

$$r_0 = (\Phi_1, \dots, \Phi_n, \Psi_1, \dots, \Psi_n), \qquad \Psi = H(x,0)^{-1}\,\widetilde{\Phi},$$

by means of which the system of the integral equations (2.1.14)–(2.1.15) can be recast as

$$r = r_0 + A\,r. \tag{2.1.16}$$

The **Neumann series** may be of help in solving the preceding equation by introducing

$$r = \sum_{k=0}^{\infty} A^k\, r_0\,. \tag{2.1.17}$$

In the current situation the derivation of some estimates, making it possible to establish the convergence of (2.1.17) and showing that the sum of the series in (2.1.17) solves equation (2.1.16), becomes extremely important. This can be done using the **operator of integration**

$$(If)(t) = \int_0^{t+\beta l} f(\tau)\, d\tau$$

and the function defined by the recurrence relation

$$I_0(t) = 1\,, \qquad I_k(t) = I\big(I_{k-1}(t)\big)\,.$$

The components of the image of the kth power of any operator A are denoted by

$$(\bar{w}_1^{(k)}, \dots, \bar{w}_n^{(k)}, \bar{p}_1^{(k)}, \dots, \bar{p}_n^{(k)} n) = A^k(w_1^{(k)}, \dots, w_n^{(k)}, p_1^{(k)}, \dots, p_n^{(k)} n)\,.$$

With the obvious relation $\| K \| \leq \beta$ in view, the characteristic of (2.1.10) satisfies the inequality

$$\tau_i(\xi; x, t) \leq t + \beta L\,, \qquad 1 \leq i \leq n\,.$$

As far as the function $I_k(t)$ is nondecreasing and nonnegative, the following estimates are true:

$$\int_0^{\tau_i(\xi;x,t)} d\tau \leq \int_0^{t+\beta L} d\tau = I_1(t)\,, \tag{2.1.18}$$

$$\begin{aligned} &\int_{\alpha_i}^{x} I_k\big(\tau_i(\xi; x, t) + (m-1)\,\beta L\big)\, d\xi \\ &\qquad \leq \int_{\alpha_i}^{x} I_k(t + m\beta L)\, d\xi \leq L I_k(t + m\beta L)\,, \end{aligned} \tag{2.1.19}$$

$$\begin{aligned} &\int_{\alpha_i}^{x} \int_0^{\tau_i} I_k\big(\tau + (m-1)\,\beta L\big)\, d\tau\, d\xi \\ &\qquad \leq L \int_0^{t+\beta L} I_k\big(\tau + (m-1)\,\beta L\big)\, d\tau \\ &\qquad = L \int_{(m-1)\beta L}^{t+m\beta L} I_k(\tau)\, d\tau \leq L \int_0^{(t+m\beta L)+\beta L} I_k(\tau)\, d\tau \\ &\qquad = L\, I_{k+1}(t + m\beta L)\,. \end{aligned} \tag{2.1.20}$$

It is straightforward to verify the estimates

$$(2.1.21) \qquad |\,\bar{w}_i^{(k)}(x,t)\,| \le B^k\, p_{T}(r) \sum_{s=0}^{k} C_k^s\; I_s(t + k\,\beta\, L)\,,$$

$$(2.1.22) \qquad |\,\bar{p}_i^{(k)}(x)\,| \le B^k\, p_{T}(r) \sum_{s=0}^{k} C_k^s\; I_s(t + k\,\beta\, L)$$

by appeal to (2.1.18)–(2.1.20) and the well-known recurrence relation for the binomial coefficients

$$C_n^{k-1} + C_n^k = C_{n+1}^k\,.$$

Observe that estimates (2.1.21) and (2.1.22) are valid with constants

$$B = 2\,\widetilde{M} \max_{0\le x\le L} \{\, 1, \|\, H(x,0)^{-1}\,\|\,\}\,,$$

$$\widetilde{M} = \max_{(x,t)\in\Omega} \{\, \max\{\,\|\, K_1\,\|, \|\, K_2\,\|, \|\, K_3\,\|, \|\, \widetilde{K}_1\,\|, \|\, \widetilde{K}_2\,\|, \|\, \widetilde{K}_3\,\|\,\}\}$$

as long as $0 \le t \le T - k\,\beta\, L$ if $T > k\,\beta\, L$. The preceding estimates can be derived by induction on k. We proceed as usual. For $k = 1$ and $0 \le t \le T - \beta\, L$ we are led by the replacement of $\|\, K_i\,\|$ and $\|\, \widetilde{K}_i\,\|$ both by their common upper bound $\widetilde{M}$ to the following relations:

$$\begin{aligned} |\,\bar{w}_i^{(1)}\,| &\le \widetilde{M}\, p_{T}(r)\, L + \widetilde{M}\, p_{T}(r)\, L\, I_1 + \widetilde{M}\, p_{T}(r)\, L \\ &\le 2\,\widetilde{M}\, L p_{T}(r)\,(1 + I_1)\,, \\ |\,\bar{p}_i^{(1)}\,| &\le \|\, H(x,0)^{-1}\,\|\, \big(\widetilde{M}\, p_{T}(r)\, L \\ &\qquad + \widetilde{M}\, p_{T}(r)\, L\, I_1 + \widetilde{M}\, p_{T}(r)\, L\,\big) \\ &\le 2\,\widetilde{M}\, L\; \|\, H(x,0)^{-1}\,\|\; p_{T}(r)\,(1 + I_1)\,, \end{aligned}$$

which confirm (2.1.21)–(2.1.22) for $k = 1$. Suppose now that (2.1.21) and

(2.1.22) hold true for $k = m - 1$. With this in mind, we obtain

$$
\begin{aligned}
\left| \bar{w}_i^{(m)}(x,t) \right| &\leq \widetilde{M}\, B^{m-1} p_{_T}(r) L \sum_{s=0}^{m-1} C_{m-1}^s\, I_s(t + m\beta L) \\
&\quad + \widetilde{M}\, B^{m-1} p_{_T}(r)\, L \sum_{s=0}^{m-1} C_{m-1}^s\, I_{s+1}(t + m\,\beta\, L) \\
&\quad + \widetilde{M}\, B^{m-1}\, p_{_T}(r)\, L \sum_{s=0}^{m-1} C_{m-1}^s\, I_s(t + m\beta L) \\
&\leq (2\, \widetilde{M}\, L)\, B^{m-1}\, p_{_T}(r) \sum_{s=0}^{m} \left(C_{m-1}^s + C_{m-1}^{s-1} \right) I_s(t + m\,\beta\, L) \\
&\leq B^m\, p_{_T}(r) \sum_{s=0}^{m} C_m^s\, I_s(t + m\,\beta\, L)\,,
\end{aligned}
$$

thereby justifying the desired result for (2.1.21). The proof of (2.1.22) is similar to follow.

Furthermore, the function $I_k(t)$ admits the estimate

$$
I_k(t) \leq \frac{(t + k\,\beta\, L)^k}{k\,!}\,, \tag{2.1.23}
$$

which can be established by induction on k as well. Letting $k = 1$ we conclude that (2.1.23) follows directly from the definition of $I_k(t)$. If (2.1.23) holds true for $k = m - 1$, then

$$
\begin{aligned}
I_m(t) &= \int_0^{t+\beta L} I_{m-1}(\tau)\; d\tau \leq \int_0^{t+\beta L} \frac{(\tau + (m-1)\,\beta\, L)^{m-1}}{(m-1)\,!}\; d\tau \\
&= \frac{(t + m\,\beta\, L)^m}{m\,!} - \frac{((m-1)\,\beta\, L)^m}{m\,!} \leq \frac{(t + m\,\beta\, L)^m}{m\,!}\,.
\end{aligned}
$$

This provides sufficient background for the conclusion that (2.1.23) is valid.

The well-known inequality $C_k^s \leq 2^k$ for binomial coefficients leads to the chain of relations

$$
\begin{aligned}
\sum_{s=0}^{k} C_k^s\, I_s(t) &\le \sum_{s=0}^{k} 2^k\, \frac{(t+s\beta L)^s}{s\,!} \\
&\le \sum_{s=0}^{k} 2^k\, \frac{(t+k\beta L)^s}{s\,!} \\
&\le 2^k \sum_{s=0}^{\infty} \frac{(t+k\beta L)^s}{s\,!} \\
&= 2^k\, \exp\{t+k\beta L\}\,.
\end{aligned}
$$

Putting $q = 2\,B\, \exp\{3\,\beta\, L\}$ and replacing T by $T + k\beta L$, we deduce from (2.1.21) and (2.1.22) that

$$p_T(A^k r) \le q^k\, e^T\, p_{T+k\beta L}(r). \tag{2.1.24}$$

We now proceed to estimate the seminorms of the element r_0. As stated above,

$$\tau_i(\xi; x, t) \le t + k\,\beta\, L$$

and, therefore,

$$
\begin{aligned}
|\, \psi_i'\big(T_i(x,t)\big)\,| &\le a\, \exp\{\alpha T_i\} \\
&\le a\, \exp\{\alpha t + \alpha\beta L\} \\
&= a\, \exp\{\alpha\beta L\}\, \exp\{\alpha t\}\,.
\end{aligned}
$$

Since the values $\partial D_{ij}/\partial t$, $\partial \tau_i/\partial t$ and $\partial T_i/\partial t$ are bounded,

$$|\, \Phi_i(x,t)\,| \le c_0\, \exp\{\alpha\, t\}\,,$$

yielding

$$p_T(r_0) \le c_1\, \exp\{\alpha\, T\}\,. \tag{2.1.25}$$

All this enables us to estimate the members of Neumann's series in (2.1.17). From (2.1.24)–(2.1.25) it follows that

$$p_T(A^k r_0) \le c_1\, \exp\{(\alpha+1)\, T\}\, \big[q\, \exp\{\alpha\,\beta\, L\}\big]^k\,.$$

With the aid of the last inequality we try to majorize the series in (2.1.17) by a convergent geometric progression keeping $q \exp\{\alpha\,\beta\,L\} < 1$. Having stipulated this condition, the series in (2.1.17) converges uniformly over $0 \le t \le T$ and

$$p_T(r) \le \frac{c_1 \exp\{(\alpha+1)\,T\}}{1 - q \exp\{\alpha\,\beta\,L\}} \,. \tag{2.1.26}$$

It follows from the foregoing that (2.1.17) is just a solution to equation (2.1.16) and this solution is of exponential growth in agreement with relation (2.1.26).

The quantity $q \exp\{\alpha\,\beta\,L\}$ of the form

$$4\,\widetilde{M}\,L \max_{0\le x\le L} \{1, \| H(x,0)^{-1} \|\} \exp\{\,3\,\beta\,L\,\} \exp\{\,\alpha\,\beta\,L\,\}$$

is continuous and monotonically increasing as a function of L and vanishes for $L = 0$. Hence there exists $L_0 > 0$ such that for $L < L_0$

$$q \exp\{\alpha\,\beta\,L\} < 1$$

and, therefore, the inverse problem at hand possesses a solution of proper form. Thus, the theorem is completely proved. ■

Theorem 2.1.3 *Let D, H, D_t and H_t be continuous functions and let K be a continuously differentiable function. If*

$$\| D \| \le M\,, \quad \| D_t \| \le M\,, \quad \| H_t \| \le M\,,$$

$$\| K \| \le \beta\,, \qquad |\det H(x,0)\,| \ge \gamma > 0$$

with certain positive constants M, β and γ, then for any $b > 0$ there exists $L_1 > 0$ such that for $L < L_1$ the inverse problem (2.1.2), (2.1.8), (2.1.9) *can have at most one solution in the class of functions satisfying the estimate*

$$\| v_t \| \le c \exp\{b\,t\}\,.$$

Proof This assertion will be proved if we succeed in showing that the homogeneous equation corresponding to (2.1.16) has a trivial solution only. Let $r = A\,r$ and

$$p_T(r) \le c_1 \exp\{b\,T\}\,.$$

Since $A^k\,r = r$, relation (2.1.24) implies that

$$p_T(r) = p_T(A^k r) \le c_1 \left[q \exp\{\,b\,\beta\,L\}\right]^k \exp\{(b+1)\,T\}\,. \tag{2.1.27}$$

Hence, if we choose L_1 so as to satisfy $q \exp\{b\,\beta\,L\} < 1$ for $L < L_1$, then the relation $p_T(r) = 0$ is attained by letting $k \to \infty$ in (2.1.27) and is valid for any T. But it is possible only if $r = 0$ and thereby the theorem is completely proved. ■

Of particular interest is the situation in which the matrix K has fixed sign. Under the assumption imposed above the existence and uniqueness of a solution of the inverse problem concerned are obtained for any L. For the sake of definiteness, the case of negative eigenvalues will appear in more a detailed exposition.

Theorem 2.1.4 *Let D, H, D_t and H_t be continuous functions and let φ, ψ and K be continuously differentiable functions. One assumes, in addition, that*

$$\| D \| \leq M \,, \quad \| D_t \| \leq M \,, \quad \| H_t \| \leq M \,, \quad \| K_t \| \leq M \,, \quad \| K \| \leq \beta \,,$$

$$\| \psi'(t) \| \leq a \, \exp \{\alpha t\} \,, \quad | \det H(x,0) | \geq \gamma > 0 \,, \quad \varphi(0) = \psi(0) \,,$$

$$-k_i(x,t) \geq \gamma \quad in \quad \Omega \qquad (1 \leq i \leq n)$$

with certain constants M, β, γ and a. Then in the domain Ω the inverse problem (2.1.2), (2.1.8), (2.1.9) *has a solution*

$$v \in C^1, \qquad p \in C \,, \qquad \| v_t \| \leq c \, \exp \{b t\}$$

and this solution is unique in the indicated class of functions.

Proof The proof of the existence of a solution under the above agreements is carried out as usual. This amounts to fixing a point $(x,t) \in \Omega$ and considering characteristic (2.1.10). Upon integrating along this characteristic we arrive at relations (2.1.11) and (2.1.12), where all the α_i's are equal to zero. After differentiating we get the system of equations (2.1.14)–(2.1.15) of the second kind which can be rewritten in the concise form (2.1.16).

The idea behind derivation of the estimates in question is to refer, in addition to the operator I, to another integration operator J with the values

$$(Jf)(x) = \int_0^x f(\xi) \, d\xi \,.$$

The symbol $J_k(x)$ stands for the functions defined by the recursion

$$J_0(x) \equiv 1 \,, \qquad J_k(x) = J\big(J_{k-1}(x)\big) \,.$$

Using the well-known expression

$$J_k(x) = \frac{x^k}{k\,!} \,,$$

we replace (2.1.21)–(2.1.22) by the following estimates:

$$|\,\bar{w}_i^{(k)}(x,t)\,| \le B^k\, p_T(r)\, J_k(x) \sum_{s=0}^{k} C_k^s\, I_s(t+k\beta L)\,,$$

$$|\,\bar{p}_i^{(k)}(x)\,| \le B^k\, p_T(r)\, J_k(x) \sum_{s=0}^{k} C_k^s\, I_s(t+k\beta L)\,.$$

In this line, one useful inequality

$$p_T(A^k r) \le \frac{q^k\, L^k}{k\,!}\ \exp T\ p_{T+k\beta L}(r)$$

will be involved in place of (2.1.24) and will be useful in the estimation of the sum of Neumann's series:

$$p_T(r) \le c_1\ \exp\{(\alpha+1)\,T\} \sum_{k=0}^{\infty} \frac{\big[q\ \exp\{\alpha\,\beta\, L\}\big]^k}{k\,!}$$

$$= c_1\ \exp\big\{q\ \exp\{\alpha\,\beta\,L\}\big\}\ \exp\{(\alpha+1)\,T\}\,.$$

Further reasoning is similar to the proof of Theorem 2.1.2.

To prove the uniqueness here one should reproduce almost word for word the corresponding arguments adopted in proving Theorem 2.1.3 by replacing the value $\big[q\ \exp\{b\,\beta\,L\}\big]^k$ by $\big[q\ \exp\{b\beta L\}\big]^k/k\,!$. As $k \to \infty$, the last sequence tends to 0 for any L, thereby justifying the assertion of the theorem. ■

In view of the solution uniqueness established for the inverse problem of finding the right-hand side function, one can easily prove the uniqueness theorem for inverse problems of recovering other coefficients of the governing equations. The methodology of the considered problem provides proper guidelines for subsequent investigations.

Let us consider the nonlinear inverse problem of finding a matrix $D = D(x)$ built into the system (2.1.2). As the total number of unknown coefficients $d_{ij}(x)$, $1 \le i \le n$, $1 \le j \le n$, of the matrix D is equal to n^2, it is reasonable to absorb more information on the boundary behavior of n solutions of the system

$$(2.1.28)\qquad \begin{cases} v^{(k)}(x,0) = \varphi^{(k)}(x)\,, & 0 \le x \le L\,, \quad 1 \le k \le n\,, \\ v_i^{(k)}(0,t) = \psi_i^{(k)}(x)\,, & t \ge 0\,, \quad 1 \le i \le s\,, \quad 1 \le k \le n\,, \\ v_i^{(k)}(L,t) = \psi_i^{(k)}(x)\,, & t \ge 0\,, \quad s < i \le n.\,, \quad 1 \le k \le n\,. \end{cases}$$

In general, the solutions sought correspond to different right-hand side functions

$$G(x,t) = G^{(k)}(x,t)\,, \qquad\qquad 1 \le k \le n\,,$$

which are assumed to be available.

Other ideas are connected with some reduction (within the aspect of uniqueness) of the nonlinear coefficient inverse problem in view to the linear inverse problem we have resolved earlier. In preparation for this, we introduce the vector function $\mathbf{v} = (v^{(1)}, v^{(2)}, \dots, v^{(n)})$ and the diagonal hypermatrix $\widehat{K}$ with n blocks on the main diagonal, each of which coincides with the matrix K. We deal also with the partitioned matrix $\widehat{D}$ composed of $n \times n$-blocks $\widehat{D}_{ij}$, $1 \le i \le n$, $1 \le j \le n$. In the block $\widehat{D}_{ij}$ the jth row coincides with the vector $v^{(i)}$ and the others are taken to be zero. By a vector $\mathbf{G}$ we mean one whose components are identical with vectors $G^{(k)}$, that is, $\mathbf{G} = (G^{(1)}, G^{(2)}, \dots, G^{(n)})$. By merely setting

$$\mathbf{p} = (d_{11}, d_{12}, \dots, d_{1n}, d_{21}, d_{22}, \dots, d_{2n}, \dots, d_{n1}, d_{n2}, \dots, d_{nn})$$

the augmented system for the vector $\mathbf{v}$ reduces to

$$\frac{\partial \mathbf{v}}{\partial x} + \widehat{K}\,\frac{\partial \mathbf{v}}{\partial t} + \widehat{D}\,\mathbf{p} = \mathbf{G}\,. \tag{2.1.29}$$

Assume that the inverse problem of recovering a matrix D has two distinct solutions $(\mathbf{v}^{(1)}, \mathbf{p}^{(1)})$ and $(\mathbf{v}^{(2)}, \mathbf{p}^{(2)})$, where the vectors $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ are put in correspondence with the matrices $D^{(1)}$ and $D^{(2)}$, respectively. By introducing $\mathbf{u} = \mathbf{v}^{(2)} - \mathbf{v}^{(1)}$ we subtract equation (2.1.29) written for $\mathbf{v} = \mathbf{v}^{(2)}$ from the same equation but written for $\mathbf{v} = \mathbf{v}^{(1)}$. As a final result we get the inverse problem to be investigated:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial x} + \widehat{K}\,\dfrac{\partial \mathbf{u}}{\partial t} + \widehat{D}^{(2)}\,\mathbf{u} = \widehat{H}\,\mathbf{\Phi}\,, & \\ \mathbf{u}(x,0) = 0\,, & 0 \le x \le L\,, \\ u_i^{(k)}(0,t) = 0\,, & t \ge 0, \quad 1 \le i \le s, \quad 1 \le k \le n\,, \\ u_i^{(k)}(L,t) = 0\,, & t \ge 0, \quad s < i \le n, \quad 1 \le k \le n\,, \end{cases} \tag{2.1.30}$$

where $\widehat{D}^{(2)}$ stands for the diagonal hypermatrix with n blocks on the main diagonal, each of which coincides with the matrix $D^{(2)}$; $\widehat{H}$ denotes the matrix $-\widehat{D}$ associated with $\mathbf{v}^{(1)}$ and

$$\mathbf{\Phi} = \mathbf{p}^{(2)} - \mathbf{p}^{(1)}.$$

It is appropriate to mention that, because of its statement, the coefficient inverse problem can be treated as a linear inverse problem of the type (2.1.2), (2.1.8), (2.1.9). Note that the determinant of the matrix $\widehat{H}(x,0)$ is of the form

$$\det \widehat{H}(x,0) = \left[\det \begin{pmatrix} \varphi_1^{(1)}(x) & \dots & \varphi_n^{(1)}(x) \\ \dots & \dots & \dots \\ \varphi_1^{(n)}(x) & \dots & \varphi_n^{(n)}(x) \end{pmatrix} \right]^n .$$

The following corollary can easily be deduced by applying Theorem 2.1.3 to $b = 0$.

Corollary 2.1.1 *Let* $K \in C^1$, $\| K \| \leq \beta$, $\| K_t \| \leq M$ *and let* $\det \big(\varphi_j^{(i)}(x)\big) \neq 0$. *Then there exists a number* $L_0 > 0$ *such that for* $L < L_0$ *the inverse problem* (2.1.2), (2.1.28) *can have at most one solution in the class of functions*

$$v^{(k)} \in C^1, \qquad \| v_t^{(k)} \| \leq M , \qquad 1 \leq k \leq n ; \qquad D(x) \in C .$$

2.2 Inverse problems for t-hyperbolic systems

In this section the system (2.1.1) is supposed to be **t-hyperbolic**. By definition, this means an alternative form of writing

$$\text{(2.2.1)} \qquad \frac{\partial v}{\partial t} + K \, \frac{\partial v}{\partial x} + D\, v = G ,$$

where K is a diagonal matrix with entries $k_{ij} = k_i \, \delta_{ij}$ and δ_{ij} is, as usual, **Kronecker's delta**. We agree to consider the right-hand side function in the form

$$G(x,t) = H(x,t)\, p(t) ,$$

where an $n \times n$-matrix H is known and the unknown vector function p is sought. When recovering a pair of the functions $\{v, p\}$ in such a setting, equation (2.2.1) has to be supplied by the boundary condition

$$\text{(2.2.2)} \qquad v|_{\partial\Omega} = \varphi ,$$

where $\partial\Omega$ designates the boundary of the half-strip

$$\Omega = \{(x,t) \colon \ 0 \leq x \leq L, \ t \geq 0\} .$$

The function $\varphi(x,t)$ is assumed to be continuous on $\partial\Omega$ and the functions $\varphi(0,t)$, $\varphi(x,0)$ and $\varphi(L,t)$ are supposed to be continuously differentiable. Let the eigenvalues of the matrix K obey the same properties as before. Recall that $k_i(x,t)$, $1 \le i \le n$, are bounded and continuously differentiable in Ω, $k_i < 0$ for $1 \le i \le s$ and $k_i > 0$ for $s < i \le n$. In addition, they are supposed to be bounded away from zero, that is, there exists a positive constant c such that $|\, k_i(x,t)\,| \ge c$, $1 \le i \le n$, for all $(x,t) \in \Omega$.

Along with the function $\tau_i(\xi; x, t)$ being a solution of problem (2.1.10) we deal with the function $\xi_i(\tau; x, t)$, which for fixed values x and t gives the inverse function of $\tau_i(\xi; x, t)$ and satisfies the system

$$
\left\{
\begin{aligned}
&\frac{d\xi_i}{d\tau} = k_i(\xi_i\,, \tau)\,, \\
&\xi_i(t; x, t) = x\,.
\end{aligned}
\right.
\tag{2.2.3}
$$

After integrating along the characteristics specified by (2.2.3) the inverse problem (2.2.1)–(2.2.2) reduces to a system of integral equations. Let $(\alpha_i(x,t), \beta_i(x,t))$, $1 \le i \le n$, indicate a point at which the characteristic (2.2.3) intersects the boundary $\partial\Omega$ and $\beta_i(x,t) \le t$. Putting

$$
\Phi_i(x,t) = \varphi_i\Big(\alpha_i(x,t), \beta_i(x,t)\Big), \qquad 1 \le i \le n\,,
$$

we integrate the equations of the system (2.2.1) along the characteristic to establish the representations

$$
v_i(x,t) - \Phi_i(x,t) = \int\limits_{\beta_i(x,t)}^{t} (-Dv + Hp)_i \; d\tau\,, \qquad 1 \le i \le n\,.
\tag{2.2.4}
$$

The next step is to define the numbers γ_i as follows: $\gamma_i = 0$ for $1 \le i \le s$ and $\gamma_i = L$ for $s < i \le n$ and then set $\varepsilon_i(t) = \varphi_i(\gamma_i\,, t)$, $1 \le i \le n$. Furthermore, substituting into (2.2.4) $x = 0$ for $1 \le i \le s$ and $x = L$ for $s < i \le n$ yields

$$
\varepsilon_i(t) - \Phi_i(\gamma_i\,, t) = \int\limits_{\beta_i(\gamma_i\,, t)}^{t} (-Dv + Hp)_i \; d\tau\,, \qquad 1 \le i \le n\,.
\tag{2.2.5}
$$

The new variables

$$
w_i = \frac{\partial v_i}{\partial x}\,, \qquad 1 \le i \le n\,,
$$

are involved in the following relationships:

$$v_i(x,t) = \int_{\gamma_i}^{x} w_i(\xi,t)\, d\xi + \varepsilon_i(t)\,, \qquad 1 \le i \le n\,. \tag{2.2.6}$$

Let us differentiate (2.2.4) with respect to x and (2.2.5) with respect to t and eliminate then the functions v_i $(1 \le i \le n)$ from the resulting expressions with the aid of (2.2.6). The tacks and tricks demonstrated permit us to derive the equations

$$\begin{aligned} w_i(x,t) = F_i(x,t) &+ \sum_{j=1}^{n} \Bigg(\int_{\beta_i(x,t)}^{t} A_{ij}\, w_j\, d\tau \\ &+ \int_{\beta_i(x,t)}^{t} \int_{\gamma_j}^{\xi_i} B_{ij}\, w_j\, d\xi\, d\tau + \int_{\beta_i(x,t)}^{t} C_{ij}\, p_j\, d\tau \Bigg) \\ &- \sum_{j=1}^{n} h_{ij}\big(\alpha_i(x,t), \beta_i(x,t)\big) \\ &\times p_j\big(\beta_i(x,t)\big)\, \frac{\partial \beta_i}{\partial x}(x,t)\,, \end{aligned} \tag{2.2.7}$$

$$\begin{aligned} \sum_{j=1}^{n} h_{ij}(\gamma_i,t)\, p_j(t) = \widetilde{F}_i(t) &+ \sum_{j=1}^{n} h_{ij}\big(\alpha_i(\gamma_i,t), \beta_i(\gamma_i,t)\big) \\ &\times p_j\big(\beta_i(\gamma_i,t)\big)\, \frac{\partial \beta_i}{\partial \tau}\,(\gamma_i,t) \\ &+ \sum_{j=1}^{n} \Bigg(\int_{\beta_i(\gamma_i,t)}^{t} \widetilde{A}_{ij}\, w_j\, d\tau \\ &+ \int_{\beta_i(\gamma_i,t)}^{t} \int_{\gamma_j}^{\xi_i} \widetilde{B}_{ij}\, w_j\, d\xi\, d\tau \\ &+ \int_{\beta_i(\gamma_i,t)}^{t} \widetilde{C}_{ij}\, p_j\, d\tau \Bigg) \end{aligned} \tag{2.2.8}$$

by introducing the set of new notations

$$F_i(x,t) = \frac{\partial \Phi_i}{\partial x}(x,t) - \sum_{j=1}^{n} \int_{\beta_i(x,t)}^{t} \frac{\partial d_{ij}}{\partial x} \frac{\partial \xi_i}{\partial x} \varepsilon_j(\tau)\, d\tau$$

$$+ \Big[\sum_{j=1}^{n} d_{ij}\big(\alpha_i(x,t), \beta_i(x,t)\big)\, \Phi_j(x,t)\Big] \frac{\partial \beta_i}{\partial x},$$

$$A_{ij} = -d_{ij}\big(\xi_i(\tau; x,t), \tau\big) \frac{\partial \xi_i}{\partial x},$$

$$B_{ij} = -\frac{\partial d_{ij}}{\partial x} \frac{\partial \xi_i}{\partial x}, \qquad C_{ij} = \frac{\partial h_{ij}}{\partial x} \frac{\partial \xi_i}{\partial x},$$

$$\widetilde{F}_i(t) = \varepsilon'(t) - \frac{\partial \Phi_i}{\partial t}(\gamma_i, t) + \sum_{j=1}^{n} d_{ij}(\gamma_i, t)\, \varphi_j(\gamma_i, t)$$

$$- \left[\sum_{j=1}^{n} d_{ij}\big(\alpha_i(\gamma_i, t), \beta_i(\gamma_i, t)\big)\, \Phi_j(\gamma_i, t)\right] \frac{\partial \beta_i}{\partial t},$$

$$\widetilde{A}_{ij} = d_{ij}\big(\xi_i(\tau; \gamma_i, t), \tau\big) \frac{\partial \xi_i}{\partial t},$$

$$\widetilde{B}_{ij} = \frac{\partial d_{ij}}{\partial x} \frac{\partial \xi_i}{\partial t}, \qquad \widetilde{C}_{ij} = -\frac{\partial h_{ij}}{\partial x} \frac{\partial \xi_i}{\partial t},$$

$$\delta_i = \begin{cases} L, & 1 \le i \le s, \\ 0, & s < i \le n. \end{cases}$$

Here d_{ij} and h_{ij} refer to the elements of the matrices D and H, respectively.

In what follows we will assume that the matrices D, H, D_x and H_x are continuous. Under this premise the coefficients of equation (2.2.7) may have discontinuities only on the characteristics $\xi_i(t; \delta_i, 0)$, $1 \le i \le n$ (it may happen only with $\partial \Phi_i / \partial x$ and $\partial \beta_i / \partial x$ because other coefficients are continuous). In equation (2.2.8) only the coefficients $\partial \Phi_i / \partial x$ and $\partial \beta_i / \partial x$ may have discontinuities at a single point $T_i = \tau_i(\gamma_i; \delta_i, 0)$. Moreover, all discontinuities appear to be of the first kind. In that case the system of equations (2.2.7)–(2.2.8) being viewed in the class $w \in C$, $p \in C$ with regard to relations (2.2.6) will be equivalent to the inverse boundary value problem (2.2.1)–(2.2.2) in the class $u \in C^1$, $p \in C$.

Let us make the substitution

$$z_i(x,t) = w_i(x,t) + \sum_{j=1}^{n} h_{ij}\big(\alpha_i(x,t), \beta_i(x,t)\big)\, p_j\big(\beta_i(x,t)\big)\, \frac{\partial}{\partial x}\, \beta_i(x,t),$$

which is inverted by the transform

$$(2.2.9)\quad w_i(x,t) = z_i(x,t) - \sum_{j=1}^{n} h_{ij}\big(\alpha_i(x,t), \beta_i(x,t)\big)\, p_j\big(\beta_i(x,t)\big)\, \frac{\partial}{\partial x}\, \beta_i(x,t).$$

This yields the following system as far as the functions z_i and p_i are concerned:

$$(2.2.10)\qquad z_i(x,t) = F_i + \sum_{j=1}^{n} \Bigg(\int_{\beta_i}^{t} A_{ij}\, z_j\, d\tau + \int_{\beta_i}^{t}\int_{\gamma_j}^{\xi_i} B_{ij}\, z_j\, d\xi\, d\tau + \int_{\beta_i}^{t} C_{ij}\, p_j\, d\tau \Bigg) - \sum_{j=1}^{n}\sum_{k=1}^{n} \Bigg(\int_{\beta_i}^{t} A_{ij}\, h_{jk}\, \frac{\partial \beta_j}{\partial x}\, p_k\, d\tau - \int_{\beta_i}^{t}\int_{\gamma_j}^{\xi_i} A_{ij}\, h_{jk}\, \frac{\partial \beta_j}{\partial x}\, p_k\, d\xi\, d\tau \Bigg),$$

$$(2.2.11)\qquad \sum_{j=1}^{n} h_{ij}(\gamma_i, t)\, p_j(t) = \widetilde{F}_i(t) + \Bigg[\sum_{j=1}^{n} h_{ij}\big(\alpha_i(\gamma_i,t), \beta_i(\gamma_i,t)\big)\, p_j\big(\beta_i(\gamma_i,t)\big) \Bigg] \frac{\partial \beta_i}{\partial t} + \sum_{j=1}^{n} \Bigg(\int_{\beta_i}^{t} \widetilde{A}_{ij}\, z_j\, d\tau + \int_{\beta_i}^{t}\int_{\gamma_j}^{\xi_i} \widetilde{B}_{ij}\, z_j\, d\xi\, d\tau + \int_{\beta_i}^{t} \widetilde{C}_{ij}\, p_j\, d\tau \Bigg) - \sum_{j=1}^{n}\sum_{k=1}^{n} \Bigg(\int_{\beta_i}^{t} \widetilde{A}_{ij}\, h_{jk}\, \frac{\partial \beta_j}{\partial x}\, p_k\, d\tau$$

$$+\int_{\beta_i}^{t}\int_{\gamma_j}^{\xi_i} \widetilde{B}_{ij}\, h_{jk}\, \frac{\partial \beta_j}{\partial x}\, p_k\, d\xi\, d\tau\Bigg) .$$

In trying to derive necessary compatibility conditions one should consider the ith equation of the governing system (2.2.1) at the points $(\gamma_i, 0)$ and $(\delta_i, 0)$ and then set $E_i(t) = \varphi_i(L,t)$ for $1 \le i \le s$, $E_i(t) = \varphi_i(0,t)$ for $s < i \le n$ and $\chi(x) = \varphi(x,0)$. Retaining only the terms containing $p_i(t)$, $1 \le i \le n$, we are led to the relations

$$\sum_{j=1}^{n} h_{ij}(\gamma_i\,,0)\, p_j(0) = \varepsilon_i'(0) + k_i(\gamma_i\,,0)\, \chi_i'(\gamma_i) + \sum_{j=1}^{n} d_{ij}(\gamma_i\,,0)\, \varepsilon_j(0)\,, \tag{2.2.12}$$

$$\sum_{j=1}^{n} h_{ij}(\delta_i\,,0)\, p_j(0) = E_i'(0) + k_i(\delta_i\,,0)\, \chi_i'(\delta_i) + \sum_{j=1}^{n} d_{ij}(\delta_i\,,0)\, E_j(0)\,. \tag{2.2.13}$$

Observe that the right-hand side of (2.2.12) coincides with the values of the function $\widetilde{F}_i(t)$ at the point $t = 0$. Assuming the matrix

$$H_0 \equiv \big(h_{ij}(\gamma_i\,,t)\big)$$

to be nonsingular and composing the vector $\widetilde{\Phi} = H_0^{-1}\, \widetilde{F}$, we conclude that (2.2.12) and (2.2.13) imply the relations

$$E_i'(0) + k_i(\delta_i\,,0)\, \chi_i'(\delta_i) + \sum_{j=1}^{n} d_{ij}(\delta_i\,,0)\, E_j(0) = \sum_{j=1}^{n} h_{ij}(\delta_i\,,0)\, \widetilde{\Phi}_j(0)\,, \qquad 1 \le i \le n\,. \tag{2.2.14}$$

Theorem 2.2.1 *Let $K \in C^1$, $\| K \| \le M$, $k_i < 0$ for $1 \le i \le s$, $k_i > 0$ for $s < i \le n$ and let $| k_i(x,t) | \ge c > 0$, $\varphi \in C$, $\varphi(0,t)$, $\varphi(L,t)$, $\varphi(x,0) \in C^1$ and D, H, D_x, $H_x \in C$. Suppose that* $\det\big(h_{ij}(\gamma_i\,,t)\big) \ne 0$ *and the compatibility conditions (2.2.14) hold. Then there exists a solution $u \in C^1$, $p \in C$ of the inverse problem (2.2.1)–(2.2.2) and this solution is unique in the indicated class of functions.*

Proof To prove the theorem, it suffices to establish the existence and uniqueness of the solution of the system of the **integro-functional equations** (2.2.10)–(2.2.11). One way of proceeding is to compose the set of all vector functions

$$r = (z, p) = (z_1, \dots, z_n, p_1, \dots, p_n),$$

where the first n components are defined in Ω and the remaining ones have the semi-axis $[0, \infty)$ as the common domain of definition. Each such set with the usual operations of addition and multiplication on numbers is a vector space. Let us define there a linear operator U acting in accordance with the following rule: the first n components of the vector function Ur are taken to be the right-hand sides of (2.2.10) with F_i omitted and the last n components make up a vector obtained by multiplying the matrix H_0^{-1} by the initial vector, whose components are identical with the right-hand sides of (2.2.11) with $\widetilde{F}_i$ omitted. By involving one more vector function

$$r_0 = (F_1, \dots, F_n, \widetilde{\Phi}_1, \dots, \widetilde{\Phi}_n)$$

the system of equations (2.2.10)–(2.2.11) can be rewritten as

$$r = r_0 + U\, r. \tag{2.2.15}$$

Let $t_0 = 0$ and $t_1 > 0$. We claim that it is sufficient to solve problem (2.2.1)–(2.2.2) in the rectangle

$$G_1 = \{(x,t)\colon\ 0 \le x \le L,\ t_0 \le t \le t_1\}.$$

Indeed, if we have at our disposal a solution of the problem in G_1, our subsequent reasonings will be connected with further transition from the domain Ω to the domain

$$\Omega_1 = \{(x,t)\colon\ 0 \le x \le L,\ t \ge t_1\}.$$

In that case the problem of the same type arises once again. However, we will be concerned with a new boundary function $\varphi_1(x,t)$, which can be constructed as follows: $\varphi_1(x,t)$ is identical with $\varphi(x,t)$ on $\partial\Omega \cap \partial\Omega_1$ and at $t = t_1$ is equal to the problem (2.2.1)–(2.2.2) solution we have found in the domain G_1. Since equations (2.2.1) are satisfied at the points $(0, t_1)$ and (L, t_1), the boundary function $\varphi_1(x,t)$ does follow the compatibility conditions as desired. Because of this fact, the inverse problem at hand can be solved in the domain

$$G_2 = \{(x,t)\colon\ 0 \le x \le L,\ t_1 \le t \le t_1\},$$

where $t_2 > t_1$, etc. By regarding the spacing

$$h = t_1 - t_0 = t_2 - t_1 = \cdots = t_k - t_{k-1} = \cdots$$

to be fixed we look for a solution of the inverse problem in every domain

$$G_k = \{(x,t)\colon\ 0 \le x \le L,\ t_{k-1} \le t \le t_k\}\,,$$

by means of which it is possible to construct a solution of (2.2.1)–(2.2.2) from the required class everywhere over Ω.

Setting

$$\mu = \sup_{\substack{x,\,t \in D\\ 1\le i\le n}} |\, k_i(x,t)\,|$$

we are exploring the inverse problem (2.2.1)–(2.2.2) in the rectangle

$$G = \{\,(x,t)\colon\ 0 \le x \le L,\ 0 \le t \le L/\mu\,\}\,,$$

bearing in mind that the boundary function is unknown at the point $t = L/\mu$. Now $\beta_i(\gamma_i\,,t) \equiv 0$ $(1 \le i \le n)$ and this considerably simplifies equations (2.2.11) responsible, in the present framework, for the development of the Volterra integral equations of the second kind.

The system of equations (2.2.10)–(2.2.11) written in the vector form (2.2.15) can be solved by means of successive approximations satisfying the recurrence relations

$$r^{(0)} = r_0\,, \qquad\qquad r^{(k)} = r_0 + U\, r^{(k-1)}\,.$$

Owing to the choice of the initial approximation and further iterations the functions $z_i^{(k)}(x,t)$ may have discontinuities of the first kind on the characteristic

$$x = \xi_i(t;\delta_i,0)\,,$$

whereas the functions $\widetilde{\Phi}(t)$ and all the approximations $p_i^{(k)}(t)$ should be continuous on the segment $[0,\ L/\mu]$.

In mastering the difficulties involved due possible discontinuities of the functions $z_i^{(k)}(x,t)$, we try to adapt the functions $w_i^{(k)}(x,t)$ specified by (2.2.9):

$$\text{(2.2.9')} \qquad w_i^{(k)}(x,t) = z_i^{(k)}(x,t) - \sum_{j=1}^{n} h_{ij}\big(\alpha_i(x,t),\beta_i(x,t)\big) \times p_j^{(k)}\big(\beta_i(x,t)\big)\,\frac{\partial \beta_i}{\partial x}\,.$$

Observe that they are really continuous on G. To prove this assertion, it is sufficient to reveal the continuity of $w_i^{(k)}(x,t)$ in t at the point $(\bar{x},\bar{t})$, lying on the characteristic passing through the point $(\delta_i, 0)$. Via some transform in which the functions $p_j(t)$, $1 \le j \le n$, on the left-hand side of (2.2.11) are taken to be $p_j^{(k)}(t)$ and those on the right-hand one are replaced by $p_j^{(k-1)}(t)$ it is not difficult to establish this property. In addition, we write $z_j^{(k-1)}$ instead of z_j and put $t = 0$. The transform just considered permits us to reduce to zero the terms containing integrals. Consequently, by appeal to the explicit formulae for $\widetilde{F}_i(t)$, $1 \le i \le n$, we arrive at the recurrence relations

$$\sum_{j=1}^{n} h_{ij}(\gamma_i\,,0)\,p_j^{(k)}(0) = \sum_{j=1}^{n} h_{ij}(\gamma_i\,,0)\,p_j^{(k-1)}(0)\,, \qquad 1 \le i \le n\,,$$

which assure us of the validity of the equality

$$p^{(k)}(0) = p^{(k-1)}(0)$$

for any positive integer k. Here we take into account that the matrices $\big(h_{ij}(\gamma_i\,,0)\big)$ are nonsingular. Since $p^{(0)}(t) = \widetilde{\Phi}(t)$, we obtain for any k

$$p^{(k)}(0) = \widetilde{\Phi}(0)\,.$$

Below the symbol Δ is used to indicate the value of the **jump** with respect to t of a function u defined in the domain G:

$$\Delta u(\bar{x},\bar{t}) = \lim_{t\to\bar{t}+0} u(\bar{x},t) - \lim_{t\to\bar{t}-0} u(\bar{x},t)\,.$$

Now by relation (2.2.9′) we derive the following expression for the jump of the function $w^{(k)}(x,t)$:

$$\Delta\, w_i^{(k)} = \Delta\, z_i^{(k)} - \sum_{j=1}^{n} h_{ij}\, p_j^{(k)} \Big(\Delta\, \frac{\partial \beta_i}{\partial x}\Big)\,, \qquad 1 \le i \le n\,.$$

From the recurrence relation obtained for the functions $z_i^{(k)}$ by attaching the superscripts k and $k-1$ to the function z_i on the left-hand and right-hand sides of (2.2.10), respectively, it follows that

$$\Delta\, z_i^{(k)} = \Delta\, F_i$$

and, therefore,

$$\text{(2.2.16)} \qquad \Delta\, w_i^{(k)} = \Delta\, F_i - \sum_{j=1}^{n} h_{ij}\, p_j^{(k)} \left(\Delta\, \frac{\partial \beta_i}{\partial x} \right), \qquad 1 \le i \le n\,.$$

If the point $(\bar{x}, \bar{t})$ lies on the characteristic $x = \xi_i(t; \delta_i, 0)$, then $\beta_i(\bar{x}, \bar{t}) = 0$ and the variation of the function $p_j^{(k)}$ in (2.2.16) is equal to zero. Consequently, the value of this function equals $\widetilde{\Phi}(0)$ and a minor manipulation in (2.2.16) yields

$$\begin{aligned} \text{(2.2.17)} \qquad \Delta w_i^{(k)}(\bar{x}, \bar{t}) = {} & E_i'(0)\, \frac{\partial \tau_i}{\partial x}(\delta_i\,; \bar{x}, \bar{t}) \\ & - \chi_i'(\delta_i)\, \frac{\partial \xi_i}{\partial x}(0; \bar{x}, \bar{t}) \\ & + \sum_{j=1}^{n} d_{ij}(\delta_i\,, 0)\, E_j(0)\, \frac{\partial \tau_i}{\partial x}(\delta_i\,; \bar{x}, \bar{t}) \\ & - \sum_{j=1}^{n} h_{ij}(\delta_i\,, 0)\, \widetilde{\Phi}_j(0)\, \frac{\partial \tau_i}{\partial x}(\delta_i\,; \bar{x}, \bar{t})\,. \end{aligned}$$

Recall one useful result from mathematical analysis: if $x = g(y, p)$ is the inverse function of a differentiable function $y = f(x, p)$, then

$$f_x'\, g_p' = -f_p'\,.$$

This formula immediately follows by letting to zero the coefficient at dp on the right-hand side of the identity

$$dy = f_x' \left[g_y'\, dy + g_p'\, dp \right] + f_p'\, dp\,,$$

which can be established by formal differentiating of the equality

$$y = f(g(y, p), p)\,.$$

The above remark implies that

$$\frac{\partial \xi_i}{\partial \tau} \cdot \frac{\partial \tau_i}{\partial x} = - \frac{\partial \xi_i}{\partial x}\,.$$

However, $\partial \xi_i / \partial \tau = k_i$ and, given the compatibility conditions (2.2.14), the right-hand side of (2.2.17) equals zero. This provides support for the view that the functions $w_i^{(k)}(x, t)$ should be continuous.

To complete the proof, we introduce a Banach space $M(G)$ consisting of all vector functions

$$r = (z, p) = (z_1, \dots, z_n, p_1, \dots, p_n),$$

the first n components of which are measurable and bounded in the domain G and the remaining n components are measurable and bounded on the segment $[0, L/\mu]$. Developing the recurrence relations for $(\bar{z}^{(k)}, \bar{p}^{(k)})$ and $(\bar{z}^{(k-1)}, \bar{p}^{(k-1)})$ from equations (2.2.10)–(2.2.11) and putting

$$U^k r = (\bar{z}^{(k)}, \bar{p}^{(k)}),$$

we get the standard Volterra estimates

$$\| \bar{z}^{(k)} \| \le \frac{(M_0 t)^k}{k!} \, | r, M(G) |,$$

$$\| \bar{p}^{(k)} \| \le \frac{(M_0 t)^k}{k!} \, | r, M(G) |,$$

where the norm of an element $r = (z, p)$ on that space is defined by

$$| r, M(G) | = \max \{ \sup_G \| z \|, \sup_{[0, L/\mu]} \| p \| \}.$$

All this enables us to estimate the norm of the kth power of the operator U in the space $M(G)$ as follows:

$$\| U^k \| \le \frac{(M_0 T)^k}{k!}, \tag{2.2.18}$$

where $T = L/\mu$. From (2.2.18) it is clear that for all sufficiently large k the operator U^k becomes a contracting mapping. In turn, this property ensures the convergence of the sequence $\{r^{(k)}\}$ to an element $(z, p) \in M(G)$ in the $M(G)$-norm. That is to say, the uniform convergence of the sequence $\{z^{(k)}(x,t)\}$ to the function $z(x,t)$ over the domain G and the uniform convergence of the sequence $\{p^{(k)}(t)\}$ to the function $p(t)$ over the segment $[0, t_1]$ occur as $k \to \infty$. In view of (2.2.9′), the functions $\{w^{(k)}(x,t)\}$ converge uniformly over G and the limiting function $w(x,t)$ will be related with $z(x,t)$ and $p(t)$ by (2.2.9). The latter can be derived from (2.2.9′) by passing to the limit. The functions $w(x,t)$ and $p(t)$ being continuous must satisfy (2.2.7)–(2.2.8). This proves the existence of the inverse problem solution. The uniqueness here follows from the contraction mapping principle. ■

The result obtained applies equally well, in its uniqueness aspect, to the question of uniqueness in the study of problems of recovering other coefficients of the equations concerned. If you wish to explore this more deeply, you might find it helpful first to study the problem of finding a matrix $D = D(t)$ of the system (2.2.1). Additional information is available on the behavior of n solutions of this system on the boundary of the domain Ω. With this, we are looking for the set of functions

$$\left\{v^{(1)}(x,t), \ldots, v^{(n)}(x,t)\right\}$$

and a matrix $D(t)$ from the system

$$\frac{\partial v^{(k)}}{\partial t} + K \frac{\partial v^{(k)}}{\partial x} + D v^{(k)} = g^{(k)}, \qquad 1 \le k \le n, \tag{2.2.19}$$

supplied by the boundary condition

$$v^{(k)}|_{\partial\Omega} = \varphi^{(k)}, \qquad 1 \le k \le n. \tag{2.2.20}$$

The inverse problem so formulated will be reduced to problem (2.2.1)–(2.2.2) once we pass to the augmented system related to the function

$$\mathbf{v} = \left(v^{(1)}, v^{(2)}, \ldots, v^{(n)}\right)$$

and the unknown vector

$$\mathbf{p} = \left(d_{11}, d_{12}, \ldots, d_{1n}, d_{21}, d_{22}, \ldots, d_{2n}, \ldots, d_{n1}, d_{n2}, \ldots, d_{nn}\right).$$

Assume that the inverse problem (2.2.19)–(2.2.20) has two solutions $(\mathbf{v}^{(1)}, \mathbf{p}^{(1)})$ and $(\mathbf{v}^{(2)}, \mathbf{p}^{(2)})$. Putting $\mathbf{v} = \mathbf{v}^{(2)} - \mathbf{v}^{(1)}$ we subtract the system (2.2.19) written for $\mathbf{v}^{(2)}$ from the same system but written for $\mathbf{v}^{(1)}$. The outcome of this is

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + \widehat{K} \dfrac{\partial \mathbf{v}}{\partial x} + \widehat{D}\,\mathbf{v} = \widehat{H}\,\mathbf{p}, \\ \mathbf{v}|_{\partial\Omega} = 0, \end{cases} \tag{2.2.21}$$

where $\widehat{K}$ is a diagonal **hypermatrix** with n blocks on the main diagonal each of which coincides with the matrix K, $\widehat{D}$ is a diagonal hypermatrix with n **blocks** on the main diagonal each of which coincides with the matrix $D^{(2)}$ corresponding to the vector $\mathbf{p}^{(2)}$ and $\widehat{H}$ is a hypermatrix consisting of $n \times n$-blocks H_{ij}, $1 \le i \le n$, $1 \le j \le n$. In the block $\widehat{H}_{ij}$ the jth row coincides with the vector $-\mathbf{v}^{(i)(1)}$ and the others are taken to be zero. Here also $\mathbf{p} = \mathbf{p}^{(2)} - \mathbf{p}^{(1)}$.

In what follows the main object of investigation is a pair of the functions $(\mathbf{v}, \mathbf{p})$, for which relations (2.2.21) occur. The determinant arising from the conditions of Theorem 2.2.1 is equal to

$$\left[\det \begin{pmatrix} \varphi_1^{(1)}(0,t) & \dots & \varphi_n 1^{(1)}(0,t) \\ \dots & \dots & \dots \\ \varphi_1^{(n)}(0,t) & \dots & \varphi_n 1^{(n)}(0,t) \end{pmatrix}\right]^s \cdot$$

$$\cdot \left[\det \begin{pmatrix} \varphi_1^{(1)}(L,t) & \dots & \varphi_n 1^{(1)}(L,t) \\ \dots & \dots & \dots \\ \varphi_1^{(n)}(L,t) & \dots & \varphi_n 1^{(n)}(L,t) \end{pmatrix}\right]^{n-s},$$

$$1 \le i \le n\,, \quad 1 \le k \le n\,,$$

where $\varphi^{(k)}$, $1 \le k \le n$, are the vectors of the boundary conditions (2.2.20). But in this respect a profound result has been derived from Theorem 2.2.1 with the following corollary.

Corollary 2.2.1 *Let $K \in C^1$, $\| K \| \le M$, $k_i < 0$ for $1 \le i \le s$, $k_i > 0$ for $s < i \le n$ and $| k_i(x,t) | \ge c > 0$, $1 \le i \le n$. One assumes, in addition, that $\det\big(\varphi_i^{(k)}(L,t)\big) \neq 0$ for $s = 0$, $\det\big(\varphi_i^{(k)}(0,t)\big) \neq 0$ and $\det\big(\varphi_i^{(k)}(L,t)\big) \neq 0$ for $0 < s < n$ and $\det\big(\varphi_i^{(k)}(0,t)\big) \neq 0$ for $s = n$. Then the inverse problem (2.2.19)–(2.2.20) can have at most one solution in the class of functions*

$$v^{(k)} \in C^1, \qquad 1 \le k \le n; \qquad D \in C\,.$$

Let us dwell on the question of existence of the inverse problem (2.2.19)–(2.2.20) solution. By employing the methods developed above the existence can be achieved for sufficiently small values of the variable t. Most of the relations established in the proof of Theorem 2.2.1 remain valid if the function v will be replaced by $v^{(k)}$, $1 \le k \le n$, and $g^{(k)}$ will stand in place of Hp. Integrating the equations of the system (2.2.19) along the characteristics yields

$$(2.2.22) \qquad v_i^{(k)}(x,t) - \Phi_i^{(k)}(x,t) = \int\limits_{\beta_i(x,t)}^{t} \big(-D\,v^{(k)} + g^{(k)}\big)\, d\tau\,,$$

$$(2.2.23) \qquad \varepsilon_i^{(k)}(t) - \Phi_i^{(k)}(\gamma_i\,,t) = \int\limits_{\beta_i(\gamma_i\,,t)}^{t} \big(-D\,v^{(k)} + g^{(k)}\big)\, d\tau\,.$$

The first step is to insert in the above calculations

$$w_i^{(k)}(x,t) = \frac{\partial}{\partial x}\, v_i^{(k)}(x,t)\,,$$

which are inverted by the formulae

$$v_i^{(k)}(x,t) = \int_{\gamma_i}^{x} w_i^{(k)}(\xi,t)\, d\xi + \varepsilon_i^{(k)}(t)\,. \tag{2.2.24}$$

The next step in this direction is to differentiate (2.2.22) with respect to x and (2.2.23) with respect to t. Via transform (2.2.24) it is possible to eliminate the functions $v_i^{(k)}$ from the resulting relations and derive the following equations:

$$\begin{aligned} w_i^{(k)}(x,t) &= F_i^{(k)}(x,t) \\ &\quad - \sum_{j=1}^{n} \int_{\beta_i(x,t)}^{t} d_{ij}(\tau)\, w_j^{(k)}\big(\xi_i(\tau;x,t),\tau\big)\, \frac{\partial \xi_i}{\partial x}\, d\tau \\ &\quad + \sum_{j=1}^{n} d_{ij}\big(\beta_i(x,t)\big)\, \Phi_j^{(k)}(x,t)\, \frac{\partial \beta_i(x,t)}{\partial x}\,, \end{aligned} \tag{2.2.25}$$

$$\begin{aligned} \sum_{j=1}^{n} d_{ij}(t)\varphi_j^{(k)}(\gamma_i t) &= \widetilde{F}_i^{(k)}(t) \\ &\quad + \sum_{j=1}^{n} d_{ij}\big(\beta_i(\gamma_i\,,t)\big)\, \Phi_j^{(k)}(\gamma_i,t)\, \frac{\partial \beta_i}{\partial t} \\ &\quad - \sum_{j=1}^{n} \int_{\beta_i(\gamma_i,\,t)}^{t} d_{ij}(\tau) w_j^{(k)}\big(\xi_i(\tau;\gamma_i,t),\tau\big) \frac{\partial \xi_i}{\partial \tau} d\tau, \end{aligned} \tag{2.2.26}$$

where

$$\begin{aligned} F_i^{(k)}(x,t) &= \frac{\partial \Phi_i^{(k)}(x,t)}{\partial x} + \sum_{j=1}^{n} \int_{\beta_i(x,t)}^{t} \frac{\partial g_i^{(k)}}{\partial x}\, \frac{\partial \xi_i}{\partial x}\, d\tau \\ &\quad - g_i^{(k)}\big(\alpha_i(x,t),\beta_i(x,t)\big)\, \frac{\partial \beta_i}{\partial x}(x,t)\,, \end{aligned}$$

$$\widetilde{F}_i^{(k)}(t) = \frac{\partial \Phi_i^{(k)}(\gamma_i\,, t)}{\partial t} + g_i^{(k)}(\gamma_i\,, t)$$

$$- g_i^{(k)}\big(\alpha_i(\gamma_i\,, t), \beta_i(\gamma_i\,, t)\big)\, \frac{\partial \beta_i}{\partial t}(\gamma_i\,, t)\,,$$

$$+ \int\limits_{\beta_i(\gamma_i\,, t)}^{t} \frac{\partial g_i^{(k)}}{\partial x}\, \frac{\partial \xi_i}{\partial t}\, d\tau - \varepsilon_i^{(k)\prime}(t)$$

and the functions $\Phi_i^{(k)}(x,t)$ and $\varepsilon_i^{(k)}(t)$ are of the same form as the functions $\Phi_i(x,t)$ and $\varepsilon_i(t)$ involved in (2.2.7)–(2.2.8) once written for the function $v(x,t) = v^{(k)}(x,t)$, $1 \le k \le n$.

When considered only in the specified domain G, the system of equations (2.2.25)–(2.2.26) is much more simpler, since $\beta_i(\gamma_i\,, t) \equiv 0$, $1 \le i \le n$. Replacing the unknown functions with the aid of substitutions

$$w_i^{(k)}(x,t) = z_i^{(k)}(x,t) + \sum_{j=1}^{n} d_{ij}\big(\beta_i(x,t)\big)\, \Phi_j^{(k)}(x,t)\, \frac{\partial \beta_i(x,t)}{\partial x} \tag{2.2.27}$$

we arrive at the system of the Volterra equations

$$\begin{aligned} z_i^{(k)}(x,t) = {} & F_i^{(k)}(x,t) + \sum_{j=1}^{n} \int\limits_{\beta_i(x,t)}^{t} a_{ij}(\tau) \\ & \times z_j^{(k)}(\xi_i\,, \tau)\, \frac{\partial \xi_i}{\partial x}\, d\tau \\ & - \sum_{j=1}^{n} \sum_{m=1}^{n} \int\limits_{\beta_i(x,t)}^{t} a_{ij}(\tau)\, a_{jm}\big(\beta_j(\xi_i\,, \tau)\big) \\ & \times \Phi_m^{(k)}(\xi_i\,, \tau)\, \frac{\partial \beta_i}{\partial x}\, \frac{\partial \xi_i}{\partial x}\, d\tau\,, \end{aligned} \tag{2.2.28}$$

$$\begin{aligned} \sum_{j=1}^{n} d_{ij}(t)\, \varphi_j^{(k)}(\gamma_i\,, t) = {} & \widetilde{F}_i^{(k)}(t) \\ & - \sum_{j=1}^{n} \int\limits_{\beta_i(\gamma_i\,, t)}^{t} d_{ij}(\tau)\, z_j^{(k)}(\xi_i\,, \tau) \frac{\partial \xi_i}{\partial \tau}\, d\tau \end{aligned} \tag{2.2.29}$$

$$-\sum_{j=1}^{n}\sum_{m=1}^{n}\int\limits_{\beta_i(\gamma_i\,,t)}^{t} d_{ij}(\tau)\, d_{jm}\big(\beta_j(\xi_i\,,\tau)\big)$$

$$\times\, \Phi_m^{(k)}(\xi_i\,,\tau)\,\frac{\partial\beta_j}{\partial x}\,\frac{\partial\xi_i}{\partial\tau}\,d\tau\,.$$

In the derivation of compatibility conditions one should consider the ith equation of the governing system (2.2.19) at the points $(\gamma_i, 0)$ and $(\delta_i, 0)$, $1 \le i \le n$. By $E_i^{(k)}(t)$ and $\chi_i^{(k)}(x)$ we denote the functions coinciding with $E_i(t)$ and $\chi_i(x)$ introduced above in establishing the compatibility conditions (2.2.12)–(2.2.13). The superscript k there indicates that the functions have been constructed for

$$v(x,t) \;=\; v^{(k)}(x,t)\,.$$

Equations (2.2.19) imply that

$$\varepsilon_i^{(k)'}(0) + k_i(\gamma_i\,,0)\,\chi_i^{(k)'}(\gamma_i) + \sum_{j=1}^{n} d_{ij}(0)\,\varphi_j^{(k)}(\gamma_i\,,0) = g_i^{(k)}(\gamma_i\,,0) \tag{2.2.30}$$

and

$$E_i^{(k)'}(0) + k_i(\delta_i\,,0)\,\chi_i^{(k)'}(\delta_i) + \sum_{j=1}^{n} d_{ij}(0)\,\varphi_j^{(k)}(\delta_i\,,0) = g_i^{(k)}(\delta_i\,,0)\,. \tag{2.2.31}$$

When the subscript i of the ingredients of (2.2.30) is held fixed, the preceding relations for the unknowns

$$a_{i1}(0), a_{i2}(0), \dots , a_{in}(0)$$

constitute a system of linear equations with the matrix coinciding with $\big(\varphi_j^{(k)}(0,0)\big)$ for $1 \le i \le s$ and $\big(\varphi_j^{(k)}(L,0)\big)$ for $s < i \le n$. Being concerned with invertible matrices we can find the elements a_{ij} for $1 \le i \le n$, $1 \le j \le n$. Consequently, the compatibility conditions are convenient to be presented by relations (2.2.31), whose elements $a_{ij}(0)$, $1 \le i \le n$, $1 \le j \le n$, should be replaced by their values known from equations (2.2.30).

Theorem 2.2.2 *Let $K \in C^1$, $\| K \| \le M$, $k_i < 0$ for $1 \le i \le s$, $k_i > 0$ for $s < i \le n$, and let $| k_i(x,t) | \ge c > 0$, $1 \le i \le n$. Let $\varphi^{(k)}(x,t) \in C$, $\varphi^{(k)}(0,t)$, $\varphi^{(k)}(L,t)$, $\varphi^{(k)}(x,0) \in C^1$; $g^{(k)}$, $g_x^{(k)} \in C$ and, in addition, $\det\big(\varphi_i^{(k)}(L,t)\big) \ne 0$ for $s = 0$, $\det\big(\varphi_i^{(k)}(0,t)\big) \ne 0$ and $\det\big(\varphi_i^{(k)}(L,t)\big) \ne 0$ for $0 < s < n$, $\det\big(\varphi_i^{(k)}(0,t)\big) \ne 0$ for $s = n$. One assumes that the compatibility conditions (2.2.31), whose ingredients $a_{ij}(0)$ are replaced by their values from equations (2.2.30), hold. Then there exist a time $T > 0$ such that for $t \le T$ the inverse problem (2.2.19)–(2.2.20) has a solution in the class of functions*

$$v^{(k)} \in C^1, \qquad 1 \le k \le n; \qquad D \in C.$$

Proof By exactly the same reasoning as in the proof of Theorem 2.2.1 it is convenient to operate in the domain

$$\Omega_T = \{(x,t)\colon\ (x,t) \in \Omega,\ t \le T\}$$

and the space $M(\Omega_T)$ of all bounded measurable functions

$$r = \{z^{(1)}(x,t), \dots, z^{(n)}(x,t),\ D(t)\}$$

with the norm

$$| r, M(\Omega_T) | = \max_{(x,t)\in\Omega_T} \{ \| z^{(1)}(x,t) \|, \dots, \| z^{(n)}(x,t) \|,\ \| D(t) \| \} .$$

Let an operator U be defined by integral terms in relations (2.2.28)–(2.2.29). We choose the "initial" element r_0 in such a way that the system of equations (2.2.28)–(2.2.29) can be written in the form (2.2.15), making it possible to solve the governing system by appeal to the successive approximations

$$r^{(0)} = r_0 , \qquad r^{(m)} = r_0 + U\, r^{(m-1)}, \qquad m = 1, 2, \dots .$$

Observe that the functions $z^{(k)(m)}(x,t)$ may have discontinuities of the first kind on the characteristics, while the functions $D^{(m)}(t)$ should be continuous. By merely setting $t = 0$ in (2.2.29) it is easily verified that the values $D^{(m)}(0)$ do not depend on m and coincide with the system (2.2.30) solution. With the aid of (2.2.27) we construct the approximations for $w^{(k)}(x,t)$, $1 \le k \le n$, and observe that the new functions turn out to be continuous in Ω_T by virtue of the compatibility conditions (2.2.31).

The current proof differs from that carried out in the preceding theorem, since the system (2.2.28)–(2.2.29) is nonlinear and its solution can be shown to exist, generally speaking, only for sufficiently small values of the variable t. This completes the proof. ■

It seems worthwhile to consider a particular case of problem (2.2.19)–(2.2.20) to help motivate what is done. It is required to find the functions $v(x,t)$ and $d(t)$ from the system of relations

$$
\left\{
\begin{array}{lll}
v_t + v_x = d(t)\, v\,, & 0 \le x \le 1\,, & t \ge 0\,, \\
v(x,0) = \varphi(x)\,, & 0 \le x \le 1\,, & \\
v(0,t) = 0\,, & t \ge 0\,, & \\
v(1,t) = 1\,, & t \ge 0\,, &
\end{array}
\right.
\tag{2.2.32}
$$

where

$$
\varphi(x) = \left\{
\begin{array}{ll}
0\,, & 0 \le x \le 1-\varepsilon\,, \\
\left(\dfrac{x+\varepsilon-1}{\varepsilon} \right)^2, & 1-\varepsilon < x \le 1\,,
\end{array}
\right.
$$

and $0 < \varepsilon < 1$. The function $\varphi(x)$ so constructed is continuously differentiable. By the replacement

$$
v(x,t) = u(x,t)\, \exp\left\{ \int\limits_0^t d(\tau)\, d\tau \right\}
$$

we obtain the equation

$$
u_t + u_x = 0\,,
$$

making it possible to derive for the inverse problem (2.2.32) solution the explicit formulae

$$
v(x,t) = \left\{
\begin{array}{lll}
0\,, & 0 \le x \le 1-\varepsilon\,, & \\
0\,, & x > 1-\varepsilon, & t \ge x-1+\varepsilon\,, \\
\left(\dfrac{x-t+\varepsilon-1}{t-\varepsilon} \right)^2, & x > 1-\varepsilon, & t < x-1+\varepsilon\,,
\end{array}
\right.
$$

$$
d(t) = \frac{2}{(t-\varepsilon)^2}\,,
$$

Unfortunately, this solution cannot be continuously extended to the domain $t \ge \varepsilon$ in spite of the fact that all the conditions of Theorem 2.2.2 are satisfied. This example shows that usually a solution of (2.2.19)–(2.2.20) exists only for sufficiently small values t during which we could make the interval of the solution existence as small as we like. We will not pursue analysis of this: the ideas needed to do so have been covered.

2.3 Inverse problems for hyperbolic equations of the second order

Linear hyperbolic partial differential equations of the second order find a wide range of applications in mathematical physics problems. As a rule, they are involved in describing oscillating and wave processes in elastic and electromagnetic mediums. Certain types of hyperbolic systems of the first order can also be reduced to the wave equation. On the other hand, a hyperbolic equation of the second order is, in turn, treated as a hyperbolic system of the first order. In preceding subsections much progress has been achieved for inverse problems with hyperbolic systems. Common practice involves the reduction of the wave equation to such a system.

We now consider in the strip

$$\Omega_T = \{(x,t)\colon\ x \in \mathbf{R},\ 0 \le t \le T\}$$

the **hyperbolic equation** of the second order

$$u_{tt} = a^2\, u_{xx} + b\, u_x + c\, u_t + du + F \tag{2.3.1}$$

with the supplementary Cauchy data

$$\begin{cases} u(x,0) = \varphi(x)\,, & x \in \mathbf{R}, \\ u_t(x,0) = \psi(x)\,, & x \in \mathbf{R}. \end{cases} \tag{2.3.2}$$

The subsidiary information about the problem (2.3.1)–(2.3.2) solutions is

$$u(x_i\,, t) = \chi_i(t)\,, \qquad 0 \le t \le T\,, \qquad 1 \le i \le n\,. \tag{2.3.3}$$

We begin by placing the problem statement for finding a function $F(x,t)$ from (2.3.1)–(2.3.3) under the approved decomposition

$$F(x,t) = \sum_{i=1}^{n} g_i(x,t)\, p_i(t) + h(x,t)\,, \tag{2.3.4}$$

where $g_i(x,t)$ and $h(x,t)$ are the known functions, while the unknown functions $p_i(t)$, $1 \le i \le n$, are sought. In what follows the coefficient $b(x,t)$ in equations (2.3.1) will be taken to be zero without loss of generality. Indeed, having performed the standard substitution

$$u(x,t) = v(x,t)\, \exp\left\{ -\frac{1}{2} \int\limits_{x_0}^{x} \frac{b(\xi,t)}{a^2(\xi,t)}\, d\xi \right\} \tag{2.3.5}$$

we can always come to the same result. For the sake of definiteness, let

$$x_1 < x_2 < \cdots < x_{n-1} < x_n \tag{2.3.6}$$

and let the function $a(x,t)$ be positive, bounded and twice continuously differentiable. Currently the object of investigation is the equation of the characteristics $\xi = \xi_i(\tau; x, t)$ passing through a point (x,t)

$$\begin{cases} \dfrac{d\xi_i}{d\tau} = \varepsilon_i\, a(\xi_i\,, \tau)\,, \\ \xi_i\,(t; x, t) = x\,, \end{cases} \tag{2.3.7}$$

where $i = 1,\ 2;\ \varepsilon_1 = -1$ and $\varepsilon_2 = 1$.

Theorem 2.3.1 *Let $a(x,t) > 0$, $|\,a(x,t)\,| \le M$, $a(x,t) \in C^2$, $b(x,t) = 0$, $\psi(x) \in C^1$ and let $\varphi(x)$, $\chi_i(t) \in C^2$, $1 \le i \le n$. One assumes, in addition, that $\varphi(x_i) = \chi_i(0)$ and $\psi(x_i) = \chi_i'(0)$ for $1 \le i \le n$. Let $c(x,t)$, $d(x,t)$, $g_i(x,t)$ and $h(x,t)$ be continuous along with their first x-derivatives in the domain Ω_T and let* $\det\big(g_i(x_j\,,t)\big) \ne 0$. *Then in the domain Ω_T there exists a solution of the inverse problem* (2.3.1)–(2.3.3) *in the class*

$$u \in C^2, \qquad p_i \in C, \qquad 1 \le i \le n\,,$$

and this solution is unique in the indicated class of functions.

Proof There is a need to emphasize the following fact. If the continuous functions p_1, p_2 and p_3 are known, then the Cauchy (direct) problem (2.3.1)–(2.3.2) will be uniquely solvable in the class of functions $u \in C^2(\Omega_T)$. This feature of the direct problem enables us to consider the inverse problem (2.3.1)–(2.3.3) in any subdomain $\Omega \subset \Omega_T$, for which the functions p_1, p_2 and p_3 can uniquely be recovered. More specifically, by Ω we mean a bounded closed domain, whose boundary consists of two straight lines $t = 0$ and $t = T$, and two graphs of the functions $x = \xi_1(t; x_n, T)$ and $x = \xi_2(t; x_1, T)$.

A possibility of this inverse problem to be localized is based on the assumptions that the functions p_1, p_2 and p_3 in question depend only on t. Let us show that in the specified domain Ω these functions can uniquely be recovered.

Equation (2.3.1) can be viewed as a system of differential equations related to the replacements $u_t = v$ and $u_x = w$. Writing this as a vector equality we arrive at

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a^2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x + \begin{pmatrix} 0 & 1 & 0 \\ d & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ F \\ 0 \end{pmatrix}.$$

By the replacements

$$r_1 = u\,, \qquad r_2 = \frac{1}{2}\left(\frac{v}{a} + w\right), \qquad r_2 = \frac{1}{2}\left(\frac{v}{a} - w\right)$$

we are led to the canonical form

$$r_t = K\,r_x + D\,r + \Phi\,,$$

where

$$r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix},$$

$$D = \frac{1}{2a}\begin{pmatrix} 0 & 2a^2 & 2a^2 \\ d & ac + aa_x - a_t & ac + aa_x - a_t \\ d & ac - aa_x - a_t & ac - aa_x - a_t \end{pmatrix}, \qquad \Phi = \frac{1}{2a}\begin{pmatrix} 0 \\ F \\ F \end{pmatrix}.$$

Additional information provides the validity of the relations

$$v(x_i\,, t) = u_t(x_i\,, t) = \chi_i'(t)\,, \qquad 0 \le i \le n\,,$$

yielding

$$\text{(2.3.8)} \qquad a(x_i\,,t)\left[r_2(x_i\,,t) + r_3(x_i\,,t)\right] = \chi_i'(t)\,, \qquad 1 \le i \le n\,.$$

Keeping the integral along the characteristic

$$\int\limits_{L_i(x,t)} (\varphi) = \int\limits_0^t \varphi\big(\xi_i(\tau; x, t), \tau\big)\,d\tau\,, \qquad i = 1,\,2\,,$$

and the new parameters

$$B = \frac{d}{2a}\,, \qquad C = \frac{ac + aa_x - a_t}{2a^2}\,,$$

$$E = \frac{ac - aa_x - a_t}{2a^2}\,, \qquad H = \frac{1}{2a}\,,$$

$$\mathbf{p} = (p_1\,, \dots, p_n)\,, \qquad \mathbf{g} = (g_1\,, \dots, g_n)\,, \qquad \mathbf{p}\,\mathbf{g} = \sum_{i=1}^n p_i\,g_i\,,$$

we prefer to deal with concise expressions which are good enough for our purposes. Other ideas are connected with integral equations which are capable of elucidating many of the facets of current problems. The first equation is obtained by integrating the equality $u_t = v$ as follows:

$$u(x,t) = \varphi(x) + \int_0^t v(x,\tau)\,d\tau\,. \tag{2.3.9}$$

Let us integrate the second and third equations of the system (2.3.8) along the corresponding characteristics and then add one to another. Multiplying the resulting expressions by $a(x,t)$ and taking into account the relation $a(r_1+r_3)=v$, we arrive at

$$\begin{aligned} v(x,t) = {} & R(x,t) + a(x,t) \\ & \times \left[\int\limits_{L_1(x,t)} (Bu + Cv + Hg\,\mathbf{p}) \right. \\ & \left. + \int\limits_{L_2(x,t)} (Bu + Ev + Hg\,\mathbf{p}) \right], \end{aligned} \tag{2.3.10}$$

where

$$\begin{aligned} R(x,t) = {} & a(x,t)\,\big[r_2(\xi_1\,,0) + r_3(\xi_2\,,0)\big] \\ & + a(x,t)\left[\int\limits_{L_1(x,t)} \left(\frac{h}{2a}\right) + \int\limits_{L_2(x,t)} \left(\frac{h}{2a}\right)\right]. \end{aligned}$$

Since

$$r_2(x,0) = \frac{1}{2\,a}\;\psi(x) + \frac{1}{2}\;\varphi'(x)$$

and

$$r_3(x,0) = \frac{1}{2\,a}\;\psi(x) - \frac{1}{2}\;\varphi'(x)\,,$$

one might expect that the function $R(x,t)$ is known. The remaining integral equations can be derived by merely inserting $x = x_i$, $1 \le i \le n$, in (2.3.10). By the same token,

$$\begin{aligned} \chi_i'(t) = {} & R(x_i\,,t) + a(x_i\,,t) \\ & \times \left[\int\limits_{L_1(x_i\,,t)} (Bu + Cv + Hg\,\mathbf{p}) \right. \\ & \left. + \int\limits_{L_2(x_i\,,t)} (Bu + Ev + Hg\,\mathbf{p}) \right]. \end{aligned} \tag{2.3.11}$$

Our further step is to differentiate relations (2.3.9)–(2.3.11). To avoid cumbersome expressions, it is reasonable to introduce the following notations for the function $\mathcal{F}(x,t)$ of two independent variables:

$$\mathcal{F}_i(\tau,x,t) = \frac{d}{dx}\,\mathcal{F}\big(\xi_i(\tau;x,t),\tau\big)\,, \tag{2.3.12}$$

$$\mathcal{F}_{i+2}(\tau,x,t) = \mathcal{F}(\xi_i\,,\tau)\,\frac{d\xi_i}{dx}\,, \tag{2.3.13}$$

$$\mathcal{F}_{i+4}(\tau,x,t) = \frac{d}{dt}\,\mathcal{F}(\xi_i\,,\tau)\,, \tag{2.3.14}$$

$$\mathcal{F}_{i+6}(\tau,x,t) = \mathcal{F}(\xi_i\,,\tau)\,\frac{d\xi_i}{d\tau}\,. \tag{2.3.15}$$

In formulae (2.3.12)–(2.3.15) the subscript i takes the values 1 and 2. The intervention of a new unknown function $z(x,t) = v_x(x,t) = u_{xt}(x,t)$ complements the notation of the integral along characteristics. If $\mathcal{F}(\tau,x,t)$ is an arbitrary function of three variables and $X(x,t)$ refers to each of the functions u, v, w, z and p, one trick we have encountered is to adopt

$$\int\limits_{L_i(x,t)} (\mathcal{F}\,X) = \int\limits_0^t \mathcal{F}(\tau,x,t)\,X\big(\xi_i(\tau;x,t),\tau\big)\,d\tau\,, \qquad i=1,\,2\,.$$

Differentiating relation (2.3.9) with respect to x yields

$$w(x,t) = \varphi'(x) + \int\limits_0^t z(x,\tau)\,d\tau\,. \tag{2.3.16}$$

One more relation, namely,

$$\begin{aligned} z(x,t) = \frac{\partial R(x,t)}{\partial x} + \frac{\partial a(x,t)}{\partial x}\Bigg[&\int\limits_{L_1(x,t)} (Bu + Cv + Hg\,\mathbf{p}) \\ &+ \int\limits_{L_2(x,t)} (Bu + Ev + Hg\,\mathbf{p})\Bigg] + a(x,t) \\ \times\Bigg[&\int\limits_{L_1(x,t)} (B_1\,u + C_1\,v + (Hg)_1\,\mathbf{p} + B_3\,w + C_3\,z) \\ &+ \int\limits_{L_2(x,t)} (B_2\,u + E_2\,v + (Hg)_2\,\mathbf{p} + B_4\,w + D_4\,z)\Bigg] \end{aligned} \tag{2.3.17}$$

is a result of differentiating equation (2.3.10) with respect to x.

Let us introduce one of the integration operators acting in accordance with the rule

$$I\big[\mathcal{F}(x,t)\big] = \int\limits_{x_1}^{x} \mathcal{F}(\xi,t)\, d\xi\,.$$

By definition,

$$u(x,t) = I\big[w(x,t)\big] + \chi_1(t)$$

and

$$v(x,t) = I\big[z(x,t)\big] + \chi_1'(t)$$

and equation (2.3.17) takes the form

$$\begin{aligned}
(2.3.18) \qquad z(x,t) = \psi(x,t) + \frac{\partial a(x,t)}{\partial x} \Bigg[\int\limits_{L_1(x,t)} (BIw + CIz + H\mathbf{g}\,\mathbf{p}) \\
+ \int\limits_{L_2(x,t)} (BIw + EIz + H\mathbf{g}\,\mathbf{p}) \Bigg] \\
+ a(x,t) \Bigg[\int\limits_{L_1(x,t)} (B_1\, Iw + C_1\, Iz \\
+ (H\mathbf{g})_1\, \mathbf{p} + B_3\, w + C_3\, z) \\
+ \int\limits_{L_2(x,t)} (B_2\, Iw + E_2\, Iz \\
+ (H\mathbf{g})_2\, \mathbf{p} + B_4\, w + E_4\, z) \Bigg] \,,
\end{aligned}$$

where

$$\begin{aligned}
\psi(x,t) = \frac{\partial R(x,t)}{\partial x} + \frac{\partial a(x,t)}{\partial x} \\
\times \Bigg[\int\limits_{L_1(x,t)} (B\chi_1 + C\chi_1')
\end{aligned}$$

$$+ \int\limits_{L_2(x,t)} (B\chi_1 + E\chi_1')\Bigg]$$

$$+ a(x,t)\Bigg[\int\limits_{L_1(x,t)} (B_1\,\chi_1 + C_1\,\chi_1')$$

$$+ \int\limits_{L_2(x,t)} (B_2\,\chi_1 + E_2\,\chi_1')\Bigg] .$$

Differentiating (2.3.11) with respect to t and denoting by $G_{ij}(t)$, $1 \le i \le n$, $1 \le j \le n$, the elements of the inverse of $\big(g_i(x_j\,,t)\big)$, the resulting equations are solved with respect to the values $p_i(t)$, $1 \le i \le n$, leading to the decompositions

$$p_i(t) = H_i(t) - \sum_{j=1}^{n} G_{ij}(t)\Bigg\{\frac{\partial a(x_j\,,t)}{\partial t} \tag{2.3.19}$$

$$\times\Bigg[\int\limits_{L_1(x_j,t)} (BIw + CIz + (H\mathbf{g})\,\mathbf{p})$$

$$+ \int\limits_{L_2(x_j,t)} (BIw + EIz + (H\mathbf{g})\,\mathbf{p})\Bigg]$$

$$+ a(x_j\,,t)\Bigg[\int\limits_{L_1(x_j,t)} (B_5\,Iw + C_5\,Iz$$

$$+ (H\mathbf{g})_5\,\mathbf{p} + B_7\,w + C_7\,z)$$

$$+ \int\limits_{L_2(x_j,t)} (B_6\,Iw + E_6\,Iz$$

$$+ (H\mathbf{g})_6\,\mathbf{p} + B_8\,w + E_9\,z)\Bigg]\Bigg\},$$

where

$$H_i(t) = \sum_{j=1}^{n} G_{ij}(t)\Bigg[\chi_i''(t) - \frac{\partial R(x_j\,,t)}{\partial t} - d(x_j\,,t)\,\chi_j(t)$$

$$- \left(a(x_j\,,t)\, c(x_j,t) - \frac{\partial a(x_j\,,t)}{\partial t} \right) \frac{\chi_j'(t)}{a(x_j\,,t)}$$

$$- \frac{\partial a(x_j\,,t)}{\partial t} \Big[\int\limits_{L_1(x_j,t)} (B\,\chi_1 + C\,\chi_1')$$

$$+ \int\limits_{L_2(x_j,t)} (B\,\chi_1 + E\,\chi_1') \Big]$$

$$- a(x_j\,,t) \Big[\int\limits_{L_1(x_j,t)} (B_5\,\chi_1 + C_5\,\chi_1')$$

$$+ \int\limits_{L_2(x_j,t)} (B_6\,\chi_1 + E_6\,\chi_1') \Big] \Big] \,.$$

Let us assure ourselves that the system (2.3.16), (2.3.18), (2.3.19) in the class w, z, $\mathbf{p} \in C$ is equivalent to the inverse problem (2.3.1)–(2.3.4) in the class $u \in C^2$, $\mathbf{p} \in C$. Assume that the functions w, z and p satisfy the system (2.3.16), (2.3.18), (2.3.19). Furthermore, setting

$$u(x,t) = \int\limits_{x_1}^{x} w(\xi,t)\; d\xi + \chi_1(t)\,, \quad v(x,t) = \int\limits_{x_1}^{x} z(\xi,t)\; d\xi + \chi_1'(t)$$

we integrate equation (2.3.16) over x. After scrutinising the compatibility conditions we deduce that $v = u_t$, $w = u_x$, $z = u_{xt}$, $v(x_1\,,t) = \chi_1'(t)$, $u(x_1\,,t) = \chi_1(t)$ and $u(x,0) = \varphi(x)$. Hence equality (2.3.9) holds true.

Observe that relations (2.3.19) are equivalent to those derived from (2.3.11) by differentiation. Just for this reason (2.3.11) can be recovered up to constants equal to zero by virtue of the compatibility conditions

$$\chi_1'(0) = R(x_i\,,0)\,, \qquad 1 \le i \le n\,.$$

Similarly, equation (2.3.10) is reconstructed from (2.3.18) up to a function depending on t. It turns out that a vanishing function happens to be at our disposal if we assume here $x = x_i$ and apply equality (2.3.11) to $i = 1$. Therefore, the system (2.3.9)–(2.3.11) is an implication of the system (2.3.16), (2.3.18), (2.3.19). Moreover,

$$v(x,0) \;=\; R(x,0) \;=\; \psi(x)\,.$$

Substituting $x = x_i$ into (2.3.10) and recalling (2.3.11) yield $v(x_i\,, t) = \chi_i'(t)$ for $1 \le i \le n$. Whence the compatibility conditions

$$\varphi(x_i) = \chi_i(,0)\,, \qquad 1 \le i \le n\,,$$

provide $u(x_i\,, t) = \chi_i(t)$, thus causing the occurrence of relations (2.3.2)–(2.3.3).

It remains to verify whether the function $u(x,t)$ satisfies equation (2.3.1). We proceed as usual. This amounts to inserting the new functions

$$r_1(x,t) = u(x,t)\,,$$

$$r_2(x,t) = R_2(x,t) + \int\limits_{L_1(x,t)} \left(Bu + Cv + H \mathbf{g}\,\mathbf{p} + \frac{h}{2a} \right), \tag{2.3.20}$$

$$r_3(x,t) = R_3(x,t) + \int\limits_{L_2(x,t)} \left(Bu + Ev + H \mathbf{g}\,\mathbf{p} + \frac{h}{2a} \right) \tag{2.3.21}$$

with

$$R_2(x,t) = \frac{1}{2\,a(\xi_1(0;x,t),0)}\; \psi(\xi_1(0;x,t)) + \frac{1}{2}\; \varphi'(\xi_1(0;x,t))\,,$$

$$R_3(x,t) = \frac{1}{2\,a(\xi_2(0;x,t),0)}\; \psi(\xi_2(0;x,t)) - \frac{1}{2}\; \varphi'(\xi_2(0;x,t))$$

and establishing the following relationships:

$$a\,(r_1 + r_2) = v\,, \qquad r_2(x,0) = \frac{1}{2\,a(x,0)}\; \psi(x) + \frac{1}{2}\; \varphi'(x)$$

and

$$r_3(x,0) = \frac{1}{2\,a(x,0)}\; \psi(x) - \frac{1}{2}\; \varphi'(x)\,.$$

Upon differentiating (2.3.20)–(2.3.21) along the corresponding characteristics it is easily seen that the functions r_1, r_2 and r_3 solve equations (2.3.8). Subtracting the third equation (2.3.8) from the second yields one useful relation

$$(r_2 - r_3)_t = a\,(r_2 + r_3)_x + a_x\,(r_2 + r_3)\,,$$

which, with v/a substituted for $r_2 + r_3$, reduces to

$$(r_2 - r_3)_t = v_x = z\,.$$

Since $w_t = z$ and

$$r_2(x,0) - r_3(x,0) = \varphi'(x) = w(x,0)\,,$$

we might have

$$r_2 - r_3 = w$$

and, therefore, the system of equations written initially is a corollary of (2.3.8) and can be derived from it by the inverse replacements

$$u = r_1\,, \qquad v = a\,(r_2 + r_3)\,, \qquad w = r_2 - r_3\,.$$

However, the second equation of this system represents an alternative form of writing equation (2.3.1). Thus, the equivalence between the system (2.3.16), (2.3.18), (2.3.19) and the inverse problem concerned is proved.

The system of equations (2.3.16), (2.3.18), (2.3.19) can be solved by the method of successive approximations. This can be done using the space $C(\Omega)$ of all vector functions having the form $\mathbf{a} = (w, z, \mathbf{p})$, where w and z are defined and continuous in the domain Ω and the vector $\mathbf{p}(t)$ of the dimension n possesses the same smoothness on the segment $[0, T]$. The norm on that space is defined by

$$\|\,\mathbf{a}\,\| = \max\Big\{\max_{\Omega}|\,w\,|,\ \max_{\Omega}|\,z\,|,\ \max_{\substack{[0,T]\\ 1\le i\le n}}|\,p_i\,|\Big\}\,. \tag{2.3.22}$$

We refer to the vector

$$\mathbf{a}_0 = \big(\varphi'(x), \psi(x,t), \mathbf{H}(t)\big)$$

and an operator L in the space $C(\Omega)$ to be defined by the group of uniform terms on the right-hand sides of relations (2.3.16), (2.3.18) and (2.3.19), by means of which the system of integral equations can be recast as

$$\mathbf{a} = \mathbf{a}_0 + L\,\mathbf{a}\,. \tag{2.3.23}$$

In the light of the theorem premises the coefficients of equations (2.3.16), (2.3.18) and (2.3.19) are really continuous and, therefore, bounded in the domain Ω. Taking into account the obvious inequality

$$\big|\,I[\mathcal{F}(x,t)]\,\big| \le (\beta - \alpha)\,\max_{\Omega}\,|\,\mathcal{F}(x,t)|$$

one can derive the usual estimates for the Volterra equations:

$$\|\,L^k\,\mathbf{a}\,\| \le \frac{(NT)^k}{k\,!}\,\|\,\mathbf{a}\,\|\,,$$

which can be justified by induction on k. The use of the contraction mapping principle implies the existence and uniqueness of the equation (2.2.3) solution. With the equivalence established above, the same will be valid for the inverse problem (2.3.1)–(2.3.4), thereby completing the proof of the theorem. ■

By means of substitution (2.3.5) from Theorem 2.3.1 we derive one useful corollary.

Corollary 2.3.1 *Let* $0 < a(x,t) < M$, $a(x,t) \in C^2$, $\psi(x) \in C^1$ *and let* $\varphi(x)$, $\chi_i(t) \in C^2$, $1 \le i \le n$. *One assumes, in addition, that* $\varphi(x_i) = \chi_i(0)$ *and* $\chi_i'(0) = \psi(x_i)$, $1 \le i \le n$. *Let the functions* $c(x,t)$, $d(x,t)$, $h(x,t)$ *and* $g_i(x,t)$, $1 \le i \le n$, *be continuous along with their first* x*-derivatives in the domain* Ω *and let* $b(x,t) \in C^2$, $\det\big(g_i(x_j\,,t)\big) \ne 0$. *Then there exists a solution of the inverse problem* (2.3.1)–(2.3.4) *in the class*

$$u \in C^2, \qquad p_i \in C\,, \qquad 1 \le i \le n\,,$$

and this solution is unique in the indicated class of functions.

The results obtained permit us to give a definite answer concerning the uniqueness of recovering other coefficients of equation (2.3.1). Two lines of research in the study of second order hyperbolic equations are evident in available publications in this area over recent years. Not much is known in the case of the combined recovery of the coefficients $a(t)$, $c(t)$, $d(t)$ and the function $u(x,t)$ satisfying the relations

$$\text{(2.3.24)} \qquad \begin{cases} u_{tt} = a^2 u_{xx} + c\,u_t + d\,u\,, \\ u(x,0) = \varphi(x)\,, \\ u_t(x,0) = \psi(x)\,, \\ u(x_i\,,t) = \chi_i(t)\,, \qquad 1 \le i \le 3\,, \end{cases}$$

where the variables x and t are such that the point (x,t) should belong to the domain Ω_T described at the very beginning of this section.

Let both collections

$$\big(u^{(1)}, a^{(1)}, c^{(1)}, d^{(1)}\big)$$

and

$$\big(u^{(2)}, a^{(2)}, c^{(2)}, d^{(2)}\big)$$

solve the inverse problem we have posed above. Setting $A^{(i)} = \big[a^{(i)}\big]^2$ and $v = u^{(2)} - u^{(1)}$ and subtracting relations (2.3.24) written for $u = u^{(1)}$ and $u = u^{(2)}$ one from another, we derive the system

$$\text{(2.3.25)} \qquad \begin{cases} v_{tt} = A^{(2)} v_{xx} + c^{(2)} v_t + d^{(2)} v + F\,, \\ v(x,0) = 0\,, \\ v_t(x,0) = 0\,, \\ v(x_i\,,t) = 0\,, \qquad 1 \le i \le 3\,, \end{cases}$$

where

$$F(x,t) = \left(d^{(2)} - d^{(1)}\right) u^{(1)} + \left(c^{(2)} - c^{(1)}\right) u_t^{(1)} + \left(A^{(2)} - A^{(1)}\right) u_{xx}^{(1)} .$$

It remains to note that relations (2.3.25) constitute what is called an inverse problem of the type (2.3.1)–(2.3.4) under the following agreements:

$$g_1(x,t) = u^{(1)}(x,t), \qquad g_2(x,t) = u_t^{(1)}(x,t), \qquad g_3(x,t) = u_{xx}^{(1)}(x,t),$$

$$p_1(t) = d^{(2)}(t) - d^{(1)}(t),\ p_2(t) = c^{(2)}(t) - c^{(1)}(t),\ p_3(t) = A^{(2)}(t) - A^{(1)}(t),$$

$$h(x,t) = 0, \qquad n = 3.$$

Moreover,

$$\begin{aligned} g_1(x_j, t) &= \chi_j(t), && 1 \le j \le 3, \\ g_2(x_j, t) &= \chi_j'(t), && 1 \le j \le 3, \\ g_3(x_j, t) &= u_{xx}^{(1)}(x_j, t), && 1 \le j \le 3. \end{aligned}$$

The last value can be expressed by (2.3.24) in terms of the functions $\chi_j(t)$ and their derivatives as follows:

$$u_{xx}^{(1)}(x_j, t) = \frac{\chi_j''(t)}{A^{(1)}(t)} - \frac{c^{(1)}(t)\chi_j'(t)}{A^{(1)}(t)} - \frac{d^{(1)}(t)\chi_j(t)}{A^{(1)}(t)}, \qquad 1 \le j \le 3.$$

Due to the determinant properties we thus have

$$\det\left(g_i(x_j, t)\right) = \frac{1}{A^{(1)}(t)}\ W(\chi_1, \chi_2, \chi_3),$$

where $W(\chi_1, \chi_2, \chi_3)$ is the Wronskian of the system

$$\left\{\chi_1(t), \chi_2(t), \chi_3(t)\right\}.$$

It is clear that the system (2.3.25) satisfying the conditions of Theorem 2.3.1 has no solutions other than a trivial solution. This is just the clear indication that a solution of the inverse problem (2.3.24) is unique. This profound result is established in the following assertion.

Corollary 2.3.2 *When $W(\chi_1, \chi_2, \chi_3) \neq 0$, a solution of problem* (2.3.24) *is unique in the class of functions*

$$u \in C^2, \quad u_{xxx} \in C, \quad a \in C^2, \quad a > 0, \quad c \in C, \quad d \in C.$$

A similar way of investigating can be approved in solving the inverse problem of finding the functions $u(x,t)$, $c(t)$ and $d(t)$ from one more system

$$\text{(2.3.26)} \qquad \begin{cases} u_{tt} = a^2 u_{xx} + c\, u_t + d\, u\,, \\ u(x,0) = \varphi(x)\,, \qquad u_t(x,0) = \psi(x)\,, \\ u(x_i\,,t) = \chi_i(t)\,, \qquad 1 \le i \le 2\,, \end{cases}$$

where $(x,t) \in \Omega_T$.

Corollary 2.3.3 *Let $a(x,t) \in C^2$ and $0 < a(x,t) \le M$. If the Wronskian $W(\chi_1, \chi_2) \neq 0$, then a solution of the inverse problem* (2.3.26) *is unique in the class of functions*

$$u \in C^2, \qquad c \in C\,, \qquad d \in C\,.$$

It is desirable to have at own disposal some recommendations and rules governing what can happen. The rest of the present chapter focuses on the problem of recovering a single coefficient $d = d(t)$ from the following relations over Ω_T:

$$\text{(2.3.27)} \qquad \begin{cases} u_{tt} = a^2 u_{xx} + +b\, u_x + c\, u_t + d\, u + F\,, \\ u(x,0) = \varphi(x)\,, \qquad u_t(x,0) = \psi(x)\,, \\ u(x_1\,,t) = \chi(t)\,. \end{cases}$$

Theorem 2.3.1 yields the uniqueness condition for the problem at hand.

Corollary 2.3.4 *Let $a(x,t) \in C^2$, $0 < a(x,t) \le M$, $b(x,t) \in C^2$ and let $c(x,t)$, $c_x(x,t) \in C$ and $\chi(t) \neq 0$. Then a solution of problem* (2.3.27) *is unique in the class of functions*

$$u \in C^2, \qquad d \in C\,.$$

Under the same assumptions one can prove the local solvability of problem (2.3.27) by employing the method developed in Theorem 2.3.1 for $b(x,t) \equiv 0$ (or otherwise recalling substitution (2.3.5)). The system of equations related to the functions

$$w(x,t) = u_x(x,t)$$

and

$$z(x,t) = u_{xt}(x,t)$$

can be derived in a similar manner. Relation (2.3.16) remains unchanged, whereas (2.3.18) should be replaced by

$$\begin{aligned}(2.3.18') \qquad z(x,t) = \tilde{\psi}(x,t) + \frac{\partial a(x,t)}{\partial x} &\times \Bigg[\int\limits_{L_1(x,t)} (BIw + CIz + B\chi) \\ &+ \int\limits_{L_2(x,t)} (BIw + EIz + B\chi) + \Bigg] + a(x,t) \\ &\times \Bigg[\int\limits_{L_1(x,t)} (B_1\, Iw + C_1\, Iz + B_3\, w + B_1\, \chi) \\ &+ \int\limits_{L_2(x,t)} (B_2\, Iw + E_2\, Iz + B_4\, w + E_4\, z + B_2\, \chi) \Bigg],\end{aligned}$$

where

$$\begin{aligned}\tilde{\psi}(x,t) = \frac{\partial \widetilde{R}(x,t)}{\partial x} + \frac{\partial a(x,t)}{\partial x} &\times \Bigg[\int\limits_{L_1(x,t)} (C\chi') + \int\limits_{L_2(x,t)} (E\chi') \Bigg] \\ &+ a(x,t) \Bigg[\int\limits_{L_1(x,t)} (C_1\, \chi') + \int\limits_{L_2(x,t)} (E_2\, \chi') \Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{R}(x,t) = a(x,t) \Bigg[r_2\big(\xi_1(0;x,t),0\big) &+ r_3\big(\xi_2(0;x,t),0\big) \\ &+ \int\limits_{L_1(x,t)} \Big(\frac{F}{2a}\Big) + \int\limits_{L_2(x,t)} \Big(\frac{F}{2a}\Big) \Bigg].\end{aligned}$$

Instead of (2.3.19) we arrive at the equation

$$
\begin{aligned}
(2.3.19') \qquad d(t) = d_0(t) - \frac{a_t(x_1\,,t)}{\chi(x,t)}
&\times \Bigg[\int\limits_{L_1(x_1,t)} (BIw + CIz + B\chi) \\
&+ \int\limits_{L_2(x_1,t)} (BIw + EIz + B\chi) \Bigg] \\
&- \frac{a(x_1\,,t)}{\chi(t)} \Bigg[\int\limits_{L_1(x_1,t)} (B_5\, Iw + C_5\, Iz \\
&+ B_7\, w + C_7\, z + B_5\, \chi) \\
&+ \int\limits_{L_2(x_1,t)} (B_6\, Iw + E_6\, Iz \\
&+ B_8\, w + E_8\, z + B_6\, \chi) \Bigg] ,
\end{aligned}
$$

where

$$
\begin{aligned}
d_0(t) = \frac{\chi''(t) - \widetilde{R}_t(x_1\,,t)}{\chi(t)} &- \frac{a(x_1\,,t)\, c(x_1\,,t) - a_t(x_1\,,t)\, \chi'(t)}{a(x_1\,,t)\, \chi(t)} \\
&- \frac{a_t(x_1\,,t)}{\chi(t)} \Bigg[\int\limits_{L_1(x_1,t)} (C\chi') + \int\limits_{L_2(x_1,t)} (E\chi') \Bigg] \\
&- \frac{a(x_1\,,t)}{\chi(t)} \Bigg[\int\limits_{L_1(x_1,t)} (C_5\, \chi') + \int\limits_{L_2(x_1,t)} (D_6\, \chi') \Bigg] .
\end{aligned}
$$

Evidently, relations (2.3.16), (2.3.18′) and (2.3.19′) constitute a system of the second kind nonlinear integral Volterra equations. If the coefficients of the preceding equations are continuous, this system possesses a unique continuous solution for all sufficiently small t. In concluding the chapter we give the precise formulation of this fact.

Corollary 2.3.5 *Let* $a(x,t) \in C^2$, $0 < a(x,t) \le M$; $b(x,t) \in C^2$; $c(x,t)$, $c_x(x,t)$, $F(x,t)$, $F_x(x,t) \in C$ *and let* $\varphi(x) \in C^2$, $\psi(x) \in C^1$ *and* $\chi(t) \in C^2$. *One assumes, in addition, that* $\varphi(x_1) = \chi(0)$, $\psi(x_1) = \chi'(0)$ *and* $\chi(t) \ne 0$. *Then, for sufficiently small* t, *there exists a solution of the inverse problem* (2.3.27) *in the class of functions*

$$u(x,t) \in C^2, \qquad d(t) \in C\,.$$

We have nothing worthwhile to add to such discussions, so will leave it at this.

Chapter 3

Inverse Problems for Equations of the Elliptic Type

3.1 Introduction to inverse problems in potential theory

The first section of this chapter deals with inverse problems in potential theory and places special emphasis on questions of existence, uniqueness and stability along with further development of efficient methods for solving them. As to the question of existence, we are unaware of any criterion providing its global solution. There are a number of the existence theorems "in the small" for inverse problems related to a body differing only slightly from a given one as it were. And even ,in that case the problems were not completely solved because of insufficient development of the theory of nonlinear equations capable of describing inverse problems. That is why, it is natural from the viewpoint of applications to preassume in most cases the existence of global solutions beforehand and pass to deeper study of the questions of uniqueness and stability. Quite often, solutions of inverse problems turn out to be nonunique, thus causing difficulties. It would be most interesting to learn about extra restrictions on solutions if we want to ensure their uniqueness. The main difficulty involved in proving uniqueness lies, as a rule, in the fact that the inverse problems of interest are equivalent to integral equations of the first kind with the Urysohn-type kernel for which the usual ways of solving are unacceptable. The problem of uniqueness is intimately connected with the problem of stability of the inverse problem solutions. For the inverse problems in view, because they are stated by means of first kind equations, arbitrarily small perturbations of the right-hand side function may, generally speaking, be responded by a finite variation of a solution. The requirement of well-posedness necessitates imposing additional restrictions on the behavior of a solution.

Special attention is paid throughout to the important questions of uniqueness and stability of solutions of inverse problems related to potentials of elliptic equations of the second order.

This section is of auxiliary character and introduces the basic notations necessary in the sequel. We begin by defining the potentials which are in common usage and list their main properties. Denote by $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ the points in the space $\mathbf{R}^n$ and by Ω a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$ of class C^2, $\bar{\Omega} = \Omega \cup \partial\Omega$. For an arbitrary vector field $\mathbf{w}$ of the class $C^1(\bar{\Omega})$ the following relation ascribed to Gauss and Ostrogradsky appears very useful in the future:

$$\int\limits_{\Omega} \operatorname{div} \mathbf{w}\; dy = \int\limits_{\partial\Omega} (\mathbf{w} \cdot \boldsymbol{\nu}_y)\; ds_y \,, \tag{3.1.1}$$

where ds_y is an $(n-1)$-dimensional surface element on $\partial\Omega$ and $\boldsymbol{\nu}_y$ is a unit external normal to the boundary $\partial\Omega$.

Let us consider a pair of the functions $u = u(y)$ and $v = v(y)$, each being of the class $C^2(\bar{\Omega})$. Substituting $\mathbf{w} = v \cdot \nabla u$ into (3.1.1) yields the **first Green formula**

$$\int\limits_{\Omega} v \cdot \Delta u\; dy + \int\limits_{\Omega} \nabla u \cdot \nabla v\; dy = \int\limits_{\partial\Omega} v \cdot \frac{\partial u}{\partial \nu_y}\; ds_y \,, \tag{3.1.2}$$

where the symbols ∇u and ∇v stand for the gradients of the functions u and v, respectively.

By successively interchanging the functions u and v in (3.1.2) and subtracting the resulting relation from (3.1.2) we derive the **second Green formula**

$$\int\limits_{\Omega} (v \cdot \Delta u - u \cdot \Delta v)\; dy = \int\limits_{\partial\Omega} \left(v \cdot \frac{\partial u}{\partial \nu_y} - u \cdot \frac{\partial v}{\partial \nu_y} \right) ds_y \,. \tag{3.1.3}$$

Recall that the Laplace equation $\Delta u = 0$ has the radially symmetric solution r^{2-n} for $n > 2$ and $\log \frac{1}{r}$ for $n = 2$, where r is the distance to a fixed point, say the origin of coordinates. Holding a point $x \in \Omega$ fixed we introduce the normalized **fundamental solution** of the **Laplace equation** by means of the relations

$$E(x,y) = E(|\,x-y\,|) = \begin{cases} \dfrac{1}{2\pi} \log \dfrac{1}{|\,x-y\,|}\,, & n=2\,, \\[2ex] \dfrac{1}{\omega_n\,(n-2)}\;\dfrac{1}{|\,x-y\,|^{n-2}}\,, & n \geq 3\,, \end{cases} \tag{3.1.4}$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of unit sphere of the space $\mathbf{R}^n$ and

$$\Gamma(\alpha) = \int\limits_0^{+\infty} t^{\alpha-1} \exp\{-t\}\; dt$$

is the **Euler gamma-function**.

Evidently, the function $E(x,y)$ is harmonic whenever $y \neq x$. However, because of the singularity at the point $y = x$, it is impossible to substitute the function E into the Green formula (3.1.3) in place of the function v. One way of proceeding is to "move" from the domain Ω to the domain $\Omega \setminus \overline{B(x,\varepsilon)}$, where $B(x,\varepsilon)$ is a ball of a sufficiently small radius ε with center x.

All this enables us to write down (3.1.3) for the domain $\Omega \setminus \overline{B(x,\varepsilon)}$ by substituting $E(x,y)$ for the function $v(y)$ and regarding the point x to be fixed. The usual manipulations may be of help in estimating the behavior of the integrals on $\partial B(x,\varepsilon)$ as $\varepsilon \to 0$. Adopting the above arguments for the different locations of x: $x \in \Omega$, $x \in \partial\Omega$ or $x \notin \bar{\Omega}$, we can derive the **third Green formula**

$$\int\limits_{\partial\Omega} \left(E(x,y)\;\frac{\partial u(y)}{\partial \nu_y} - u(y)\;\frac{\partial E(x,y)}{\partial \nu_y} \right) ds_y \tag{3.1.5}$$

$$- \int\limits_{\Omega} E(x,y)\,\Delta u(y)\; dy = \begin{cases} u(x)\,, & x \in \Omega\,, \\ \frac{1}{2}\,u(x)\,, & x \in \partial\Omega\,, \\ 0\,, & x \notin \bar{\Omega}\,. \end{cases}$$

Having no opportunity to touch upon this topic, we address the readers to Bitsadze (1966), (1976), Vladimirov (1971), Tikhonov and Samarskii (1963) and others.

Let us now introduce the potentials to be involved in further considerations. For any bounded and integrable in Ω function $\mu(y)$ we adopt the function

$$u(x,y) = \int\limits_{\Omega} E(x,y)\,\mu(y)\; dy$$

as the **potential of a volume mass** with **density** μ under the natural premise that $\mu(y) \neq 0$ almost everywhere in Ω. It is known that the volume potential u so defined obeys the following properties:

(3.1.6) if $\mu \in L_\infty(\Omega)$ then $u \in C^{1+h}(\mathbf{R}^n)$;

(3.1.7) if $\mu \in C^h(\bar{\Omega})$, $0 < h < 1$, then $\Delta u(x) = \begin{cases} -\mu(x), & x \in \Omega, \\ 0, & x \notin \bar{\Omega}. \end{cases}$

(3.1.8) As $|x| \to \infty$, $\begin{cases} \left(\log \dfrac{1}{|x|}\right)^{-1} u(x) \to M, & n = 2, \\ |x|^{n-2} u(x) \to M, & n \geq 3, \end{cases}$

where

$$M = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_\Omega \mu(y)\, dy, & n = 2, \\ \dfrac{1}{\omega_n (n-2)} \displaystyle\int_\Omega \mu(y)\, dy, & n \geq 3. \end{cases}$$

Here the symbol $C^h(\bar{\Omega})$ is used for the Hölder space formed by all continuous on $\bar{\Omega}$ functions satisfying Hölder's condition with exponent h, $0 < h < 1$. The norm on that space is defined by

$$\| u \|_{C^h(\bar{\Omega})} = \sup_{x \in \Omega} | u(x) | + H^h(u), \tag{3.1.9}$$

where $H^h(u)$ is Hölder's constant and, by definition,

$$H^h(u) = \sup_{x_1, x_2 \in \Omega} \Big\{ | u(x_1) - u(x_2) | \cdot | x_1 - x_2 |^{-h} \Big\}.$$

In addition, $C^{l+h}(\bar{\Omega})$, $l \in \mathbf{N}$, $0 < h < 1$, is a space comprising all the functions with the first l derivatives which are Hölder's continuous with exponent h.

The **potential** of a **simple layer** is given by the relation

$$v(x) = \int_{\partial\Omega} E(x, y)\, \rho(y)\, ds_y, \tag{3.1.10}$$

where the integrable **density** $\rho(y) \neq 0$ almost everywhere on $\partial\Omega$. The function $v(x)$ defined for $x \in \mathbf{R}^n \setminus \partial\Omega$ is twice continuuosly differentiable and satisfies the Laplace equation, that is, $\Delta\, v(x) = 0$ for $x \in \mathbf{R}^n \setminus \partial\Omega$. Moreover, in the case where $\rho \in L_\infty(\partial\Omega)$ the function v belongs to $C^h(\mathbf{R}^n)$

for any h, $0 < h < 1$. Note that if the function ρ is continuous on $\partial\Omega$, then for the simple layer potential the **jump formulae** are valid:

$$(3.1.11) \qquad \left(\frac{\partial v}{\partial \nu_{x_0}}\right)^{\pm}(x_0) = \pm\,\frac{\rho(x_0)}{2} + \int\limits_{\partial\Omega} \frac{\partial E(x_0, y)}{\partial \nu_{x_0}}\,\rho(y)\,ds_y\,,$$

where $\left(\frac{\partial v}{\partial \nu_{x_0}}\right)^{+}(x_0)$ and $\left(\frac{\partial v}{\partial \nu_{x_0}}\right)^{-}(x_0)$ denote the limits of $\frac{\partial v}{\partial \nu_{x_0}}$ as $x \to x_0$ $(x_0 \in \partial\Omega)$ taken along the external and internal normal $\boldsymbol{\nu}_{x_0}$ with respect to Ω, respectively.

3.2 Necessary and sufficient conditions for the equality of exterior magnetic potentials

This section focuses on establishing several preliminary assertions which will be used in the sequel.

Let finite domains Ω_α, $\alpha = 1, 2$, be bounded by piecewise smooth surfaces $\partial\Omega_\alpha$, $\bar{\Omega}_\alpha \subset D_0$, where D_0 is a bounded domain in the space $\mathbf{R}^n$ with a piecewise smooth boundary $\partial\Omega_\alpha$. The potentials of volume masses and the potentials of simple layers are defined as follows:

$$(3.2.1) \qquad u^\alpha(x) = u(x; \Omega_\alpha, \mu_\alpha) = \int\limits_{\Omega_\alpha} E(x, y)\,\mu_\alpha(y)\,dy$$

and

$$(3.2.2) \qquad v^\alpha(x) = v(x; \partial\Omega_\alpha, \rho_\alpha) = \int\limits_{\partial\Omega_\alpha} E(x, y)\,\rho_\alpha(y)\,dy\,.$$

Let real numbers β and γ be such that $\beta^2 + \gamma^2 \neq 0$. By a **generalized magnetic potential** we mean the function

$$(3.2.3) \qquad w^\alpha(x) = w(x; \Omega_\alpha, \partial\Omega_\alpha, \mu_\alpha, \rho_\alpha) = \beta\,u^\alpha(x) + \gamma\,v^\alpha(x).$$

If A_1 and A_2, $\bar{A}_\alpha \subset D_0$, $\alpha = 1, 2$, are open bounded sets, each being a union of a finite number of domains

$$(3.2.4) \qquad A_1 = \bigcup_{j=1}^{m_1} \Omega_1^j\,, \qquad A_2 = \bigcup_{j=1}^{m_2} \Omega_2^j\,,$$

where m_1 and m_2 are fixed numbers, $\partial\Omega_1^j$ and $\partial\Omega_2^j$ are piecewise smooth boundaries, ∂A_α, $\alpha = 1, 2$, is the boundary of A_α, we will replace Ω_α and $\partial\Omega_\alpha$ by A_α and ∂A_α everywhere in (3.2.1)–(3.2.3).

Let D be an arbitrary domain (in general, multiply connected) and let

$$D_0 \supset \bar{D} . \tag{3.2.5}$$

The symbol D_1 stands for a domain having a piecewise smooth boundary such that

$$D \supset \bar{D}_1 , \qquad \operatorname{mes}(\partial D_1 \cap \partial A_\alpha) = 0 , \qquad \alpha = 1, 2 . \tag{3.2.6}$$

Let $h(y)$ be a **regular** in D **solution of the Laplace equation**

$$\Delta h(y) = 0 , \qquad y \in D . \tag{3.2.7}$$

For the purposes of the present chapter we have occasion to use the functional $J(h)$ with the values

$$\begin{aligned} J(h) = \beta \Bigg[& \int\limits_{A_1\setminus(D_0\setminus\bar{D}_1)} \mu_1(y)\, h(y)\, dy \\ & - \int\limits_{A_2\setminus(D_0\setminus\bar{D}_1)} \mu_2(y)\, h(y)\, dy \Bigg] \\ & + \gamma \Bigg[\int\limits_{\partial A_1\setminus(D_0\setminus\bar{D}_1)} \rho_1(y)\, h(y)\, ds_y \\ & - \int\limits_{\partial A_2\setminus(D_0\setminus\bar{D}_1)} \rho_2(y)\, h(y)\, ds_y \Bigg] , \end{aligned} \tag{3.2.8}$$

where μ_α and ρ_α are bounded integrable functions.

Lemma 3.2.1 *If $h(y)$ is any of the regular in D solutions to equation* (3.2.7), *then the functional $J(h)$ specified by* (3.2.8) *admits the representation*

$$J(h) = - \int\limits_{\partial D_1} M_x\big[w(x); h(x)\big]\, ds_x \tag{3.2.9}$$

with

$$w(x) = w^1(x) - w^2(x)\,, \tag{3.2.10}$$

where $w^\alpha(x)$, $\alpha = 1, 2$, are the generalized magnetic potentials defined by (3.2.3) *and the domains D and D_1 satisfy conditions* (3.2.5)-(3.2.6). *Here the symbol $M_x[w; h]$ denotes the integrand on the right-hand side of the second Green formula* (3.1.3)*:*

$$M_x[w; h] = h(x)\,\frac{\partial w(x)}{\partial \nu_x} - w(x)\,\frac{\partial h(x)}{\partial \nu_x}\,.$$

Proof If $h(y)$ is any solution to equation (3.2.7), then formula (3.1.5) gives in combination with relations (3.2.5)–(3.2.6) the representation:

$$-\int\limits_{\partial D_1} M_x\big[E(x,y); h(x)\big]\;ds_x = \begin{cases} h(y)\,, & y \in D_1\,, \\ 0\,, & y \in D_0 \setminus \bar{D}_1\,, \end{cases} \tag{3.2.11}$$

where $E(x,y)$ is the fundamental solution (3.1.4) to the Laplace equation. Multiplying (3.2.11) by $\mu_\alpha(y)$ and integrating over A_α yield

$$\begin{aligned} &\int\limits_{A_\alpha\setminus(D_0\setminus\bar{D}_1)} \mu_\alpha(y)\,h(y)\;dy \\ &\qquad = \int\limits_{A_\alpha} \mu_\alpha(y)\left\{-\int\limits_{\partial D_1} M_x\big[E(x,y); h(x)\big]\;ds_x\right\}\,dy\,. \end{aligned} \tag{3.2.12}$$

Changing the order of integration (this operation is correct, since the integrals on the right-hand side of (3.2.12) have weak singularity; for more detail see Hunter (1953)) and retaining notation (3.2.1), we arrive at

$$\int\limits_{A_\alpha\setminus(D_0\setminus\bar{D}_1)} \mu_\alpha(y)\,h(y)\;dy = \int\limits_{\partial D_1} M_x\big[u^\alpha(x); h(x)\big]\;ds_x\,. \tag{3.2.13}$$

If (3.2.11) is multiplied by $\rho_\alpha(y)$ and subsequently integrated over ∂A_α with (3.2.6) involved, we thus have

$$\begin{aligned} \int\limits_{\partial A_\alpha\setminus(D_0\setminus\bar{D}_1)} \rho_\alpha(y)\,h(y)\;ds_y &= -\int\limits_{\partial A_\alpha}\rho_\alpha(y)\left\{\int\limits_{\partial D_1} M_x\big[E(x,y); h(x)\big]\;ds_x\right\}\,ds_y \\ &= -\int\limits_{\partial D_1} M_x\left[\left\{\int\limits_{\partial A_\alpha}\rho_\alpha(y)\,E(x,y)\;ds_y\right\}; h(x)\right]\,ds_x \\ &= -\int\limits_{\partial D_1} M_x\big[v^\alpha(x); h(x)\big]\;ds_x\,. \end{aligned} \tag{3.2.14}$$

From (3.2.13) and (3.2.14) it follows that

$$(3.2.15)\quad \beta \int\limits_{A_\alpha \setminus (D_0 \setminus \bar{D}_1)} \mu_\alpha(y)\, h(y)\; dy + \gamma \int\limits_{\Gamma_\alpha \setminus (D_0 \setminus \bar{D}_1)} \rho_\alpha(y)\, h(y)\; ds_y = - \int\limits_{\partial D_1} M_x \left[w^\alpha(x); h(x) \right]\; ds_x \, .$$

Finally, subtracting (3.2.15) with $\alpha = 1$ from (3.2.15) with $\alpha = 2$ we get the assertion of the lemma. ■

Of great importance is the functional

$$(3.2.16)\quad J(h, A_\alpha, \mu_\alpha, \partial A_\alpha, \rho_\alpha) = \beta \int\limits_{A_\alpha} \mu_\alpha(y)\, h(y)\; dy + \gamma \int\limits_{\partial A_\alpha} \rho_\alpha(y)\, h(y)\; ds_y \, ,$$

where μ_α and ρ_α are bounded measurable functions.

Lemma 3.2.2 *For the equality of exterior magnetic potentials*

$$(3.2.17)\quad \begin{gathered} w(x, A_1, \mu_1, \partial A_1, \rho_1) = w(x, A_2, \mu_2, \partial A_2, \rho_2)\, , \\ x \in D_0 \setminus (\bar{A}_1 \cup \bar{A}_2)\, , \end{gathered}$$

to be valid it is necessary and sufficient that functional (3.2.16) *satisfies the relation*

$$(3.2.18)\quad J(h, A_1, \mu_1, \partial A_1, \rho_1) = J(h, A_2, \mu_2, \partial A_2, \rho_2)\, ,$$

where $h(y)$ is any regular solution to the equation

$$(3.2.19)\quad \Delta h(y) = 0\, , \qquad y \in D\, .$$

Here D is an arbitrary domain for which the following inclusions occur:

$$(3.2.20)\quad D_0 \supset \bar{D} \supset D \supset (\bar{A}_1 \cup \bar{A}_2)\, .$$

Proof First of all observe that if (3.2.18) holds, then for $x \in D_0 \setminus \bar{D}$ and $y \in D$ with $h(y) = E(x, y)$ the combination of relation (3.2.18) and representation (3.2.16) gives (3.2.17).

Granted (3.2.17), we pass to a domain D_1 ordered with respect to inclusion:

$$D \supset \bar{D}_1 \supset D_1 \supset (\bar{A}_1 \cup \bar{A}_2)\,. \tag{3.2.21}$$

Any domains D and D_1, arising from (3.2.20)–(3.2.21), must satisfy (3.2.5)–(3.2.6) and, moreover,

$$\bar{A}_\alpha \cap \left(D_0 \setminus \bar{D}_1 \right) = \varnothing\,. \tag{3.2.22}$$

Therefore, the properties of the potential of a volume mass and those of the potential of a single layer along with (3.2.17) imply that in (3.2.9)–(3.2.10)

$$M_x\big[w(x); h(x)\big] = 0 \quad \text{for} \quad x \in \partial D_1\,. \tag{3.2.23}$$

Together (3.2.8), (3.2.9) and (3.2.22) lead to (3.2.18) and the lemma is completely proved. ■

In auxiliary lemmas we agree to consider

$$B = (A_1 \cup A_2) \setminus \bar{A}_0\,, \qquad A_0 = A_1 \cap A_2\,, \tag{3.2.24}$$

$$J(h, A_\alpha \setminus A_0, \mu_\alpha, \partial A_\alpha, \rho_\alpha) = \beta \int\limits_{A_\alpha \setminus A_0} \mu_\alpha(y)\, h(y)\; dy + \gamma \int\limits_{\partial A_\alpha} \rho_\alpha(y)\, h(y)\; ds_y\,. \tag{3.2.25}$$

Lemma 3.2.3 *For the equality*

$$w(x; A_1, \mu_1, \partial A_1, \rho_1) = w(x; A_2, \mu_2, \partial A_2, \rho_2)\,, \qquad x \in D_0 \setminus \bar{B}\,, \tag{3.2.26}$$

to be valid it is necessary and sufficient that the relations

$$\mu_1(y) = \mu_2(y) \qquad \textit{for} \quad y \in A_0 \qquad (\textit{if} \quad \beta \neq 0)\,, \tag{3.2.27}$$

$$J(h, A_1 \setminus A_0, \mu_1, \partial A_1, \rho_1) = J(h, A_2 \setminus A_0, \mu_2, \partial A_2, \rho_2) \tag{3.2.28}$$

hold, where $h(y)$ is any regular solution to the equation

$$\Delta h(y) = 0\,, \qquad y \in D\,, \tag{3.2.29}$$

and D is an arbitrary domain involved in the chain of inclusions

$$D_0 \supset \bar{D} \supset D \supset \bar{B}\,. \tag{3.2.30}$$

Here the set B and the functional J built into (3.2.28) *are given by formulae* (3.2.24)–(3.2.25).

Proof The statement of the lemma will be proved if we succeed in showing that (3.2.26) implies (3.2.27)–(3.2.28). The converse can be justified in just the same way as we did in Lemma 3.2.2.

From (3.2.24), (3.2.3) and (3.2.26) it follows that

$$w^1(x) = w^2(x)\,, \qquad x \in A_0\,, \tag{3.2.31}$$

and thereby we might have for the Laplace operator

$$\Delta w^1(x) = \Delta w^2(x)\,, \qquad x \in A_0\,. \tag{3.2.32}$$

On the other hand, the properties of the volume mass potential and the simple layer potential guarantee that

$$\Delta w^\alpha(x) = \beta\, \Delta u^\alpha(x) + \gamma\, \Delta v^\alpha(x) = -\beta\, \mu_\alpha(x)\,, \qquad x \in A_0\,, \tag{3.2.33}$$

where u^α, v^α and w^α have been specified by (3.2.1)–(3.2.3). Consequently, (3.2.32)–(3.2.33) are followed by (3.2.27) and one useful relation

$$\int\limits_{A_0} \mu_1(y)\, E(x,y)\; dy = \int\limits_{A_0} \mu_2(y)\, E(x,y)\; dy\,, \qquad x \in D_0\,. \tag{3.2.34}$$

Other ideas are connected with the transition to a domain D_1 having a piecewise smooth boundary ∂D_1 such that

$$D \supset \bar{D}_1 \supset D_1 \supset \bar{B}\,. \tag{3.2.35}$$

Any domains D and D_1 involved in (3.2.30) and (3.2.35) satisfy (3.2.5)–(3.2.6) during which

$$\partial A_\alpha \cap (D_0 \setminus \bar{D}_1) = \varnothing\,.$$

Under condition (3.2.6) associated with functional (3.2.8) representation (3.2.9)–(3.2.10) gives

$$J(h) = 0\,. \tag{3.2.36}$$

With the aid of relation (3.2.34) and the properties of the domains D

and D_1 that we have mentioned the functional $J(h)$ can be rewritten as

$$J(h) = \beta \Bigg[\int\limits_{A_1\setminus(A_0\cap\bar{D}_1)} \mu_1(y)\, h(y)\, dy - \int\limits_{A_2\setminus(A_0\cap\bar{D}_1)} \mu_2(y)\, h(y)\, dy \Bigg] + \gamma \Bigg[\int\limits_{\partial A_1} \rho_1(y)\, h(y)\, ds_y - \int\limits_{\partial A_2} \rho_2(y)\, h(y)\, ds_y \Bigg] \tag{3.2.37}$$

under the natural premise $\partial A_\alpha \cap (D_0 \setminus \bar{D}_1) = \varnothing$. Since the right-hand side of (3.2.36) is independent of D_1, we are led to (3.2.28) by merely choosing a sequence of domains $\{D_1^n\}_{n=1}^{\infty}$ satisfying (3.2.35) such that $\operatorname{mes}(A_0 \cap D_1^n) \to 0$ as $n \to \infty$. Thus, the lemma is completely proved. ■

Before giving further motivations, it will be convenient to introduce the new notations as they help avoid purely technical difficulties. Denote by $\bar{\Gamma}^e$ the boundary of the set $\bar{A}_1 \cup \bar{A}_2$. In the case $\Omega_1^j \neq \Omega_2^j$ we thus have

$$\begin{aligned} \Gamma_1^i &= \partial A_1 \cap \bar{A}_1 \cap \bar{A}_2\,, & \bar{\Gamma}_1^e &= \partial A_1 \setminus \Gamma_1^i\,, \\ \bar{\Gamma}_2^e &= \partial A_2 \cap \bar{\Gamma}^e\,, & \Gamma_2^i &= \partial A_\alpha \setminus \bar{\Gamma}_2^e\,. \end{aligned} \tag{3.2.38}$$

In what follows we accept $\bar{\Gamma}_\alpha^e = \partial A_\alpha$ for $\alpha = 1, 2$ if $A_1 = A_2$.

Let B_0 be any connected component of the open set $B = (A_1 \cup A_2) \setminus \bar{A}_0$ with ∂B_0 being the boundary of B_0 and set

$$\Gamma_\Delta^i = \Gamma_1^i \cap \Gamma_2^i\,. \tag{3.2.39}$$

Assume that the sets A_1 and A_2 are located in such a way that

$$\operatorname{mes}\big(\partial B_0 \cap \Gamma_\Delta^i\big) = 0 \tag{3.2.40}$$

for at least one of the domains B_0. Without loss of generality we may suppose that $B_0 \subset (\bar{A}_1 \setminus \bar{A}_0)$. Within notation (3.2.38), the set $\bar{\Gamma}_1^e \cup \Gamma_2^i$

represents the boundary of $\bar{A}_1 \setminus \bar{A}_0$. A simple observation may be of help as further developments occur:

$$\partial B_0 \cap \bar{\Gamma}_1^e = (\partial B_0)^e , \qquad \partial B_0 \cap \Gamma_2^i = (\partial B_0)^i . \tag{3.2.41}$$

With these relations established, it is plain to show that

$$\partial B_0 = (\partial B_0)^e \cup (\partial B_0)^i . \tag{3.2.42}$$

The following lemma is devoted to an arbitrary domain D ordered with respect to inclusion:

$$D_0 \supset \bar{D} \supset D \supset \bar{B}_0 . \tag{3.2.43}$$

Lemma 3.2.4 *Let* (3.2.40) *hold for the sets* A_α, $\alpha = 1, 2$. *One assumes, in addition, that the bounded functions* $\mu_\alpha(y)$ *and* $\rho_\alpha(y)$ *and the functions* $w^\alpha(x)$, $\alpha = 1, 2$, *defined by* (3.2.3) *coincide:*

$$w^1(x) = w^2(x) \quad \textit{for} \quad x \in D_0 \setminus \bar{B} . \tag{3.2.44}$$

Then

$$\beta \int\limits_{B_0} \mu_1(y)\, h(y)\; dy + \gamma \int\limits_{(\partial B_0)^e} \rho_1(y)\, h(y)\; ds_y = \gamma \int\limits_{(\partial B_0)^i} \rho_2(y)\, h(y)\; ds_y \tag{3.2.45}$$

for any solution $h(y)$ *of the Laplace equation regular in a domain* D *from inclusions* (3.2.43).

Proof One thing is worth noting here. As in Lemma 3.2.3 relation (3.2.45) can be derived with the aid of Lemma 3.2.1. But we prefer the direct way of proving via representation (3.1.5). This amounts to deep study of D_1, having a piecewise smooth boundary ∂D_1 and satisfying the conditions

$$D \supset \bar{D}_1 \supset D_1 \supset B_0 \, , D_1 \cap (B \setminus \bar{B}_0) = \varnothing \, , \partial D_1 \cap \partial B_0 = \partial B_0 \cap \Gamma_\Delta^i \, . \tag{3.2.46}$$

Formula (3.1.5) for any regular solution $h(y)$ to the equation

$$\Delta h(y) = 0 \, , \qquad y \in D \, , \tag{3.2.47}$$

implies that

$$(3.2.48) \qquad -\int\limits_{\partial D_1} M_x\big[E(x,y),h(x)\big]\,ds_x = \begin{cases} h(y)\,, & y \in D_1\,, \\ 0\,, & y \in D_0 \setminus \bar{D}_1\,. \end{cases}$$

Let us multiply (3.2.48) by $\mu_\alpha(x)$ and integrate then the resulting relation over the set $A_\alpha \setminus A_0$. Changing the order of integration and retaining notations (3.2.39)–(3.2.43), we arrive at

$$(3.2.49) \quad -\int\limits_{\partial D_1} M_x\Bigg[\bigg(\int\limits_{A_1\setminus A_0} E(x,y)\,\mu_1(y)\,dy\bigg);h(x)\Bigg]\,ds_x = \int\limits_{B_0} h(y)\,\mu_1(y)\,dy$$

and

$$(3.2.50) \qquad -\int\limits_{\partial D_1} M_x\Bigg[\bigg(\int\limits_{A_2\setminus A_0} E(x,y)\,\mu_2(y)\,dy\bigg);h(x)\Bigg]\,ds_x = 0\,.$$

Furthermore, let (3.2.48) be multiplied by $\rho_\alpha(y)$ and integrated over ∂A_α with regard to (3.2.39)–(3.2.43). Since $B_0 \subset (A_1 \setminus \bar{A}_1)$, we thus have

$$(3.2.51) \quad -\int\limits_{\partial D_1} M_x\Bigg[\bigg(\int\limits_{\partial A_1} E(x,y)\,\rho_1(y)\,ds_y\bigg);h(x)\Bigg]\,ds_x = \int\limits_{(\partial B_0)^e} \rho_1(y)\,h(y)\,ds_y$$

and

$$(3.2.52) \quad -\int\limits_{\partial D_1} M_x\Bigg[\bigg(\int\limits_{\partial A_2} E(x,y)\,\rho_2(y)\,ds_y\bigg);h(x)\Bigg]\,ds_x = \int\limits_{(\partial B_0)^i} \rho_2(y)\,h(y)\,ds_y\,.$$

Multiplying (3.2.49) and (3.2.51) by β and γ, respectively, one can add the results, whose use permits us to obtain the relation

$$(3.2.53) \quad -\int\limits_{\partial D_1} M_x\big[w(x, A_1 \setminus A_0, \mu_1, \partial A_1, \rho_1);h(x)\big]\,ds_x = \beta\int\limits_{B_0} h(y)\,\mu_1(y)\,dy + \gamma\int\limits_{(\partial B_0)^e} \rho_1(y)\,h(y)\,ds_y\,.$$

Likewise, it follows from (3.2.50) and (3.2.52) that

$$(3.2.54) \qquad -\int\limits_{\partial D_1} M_x\big[w(x, A_2 \setminus A_0, \mu_2, \partial A_2, \rho_2); h(x)\big]\, ds_x$$

$$= \gamma \int\limits_{(\partial B_0)^i} \rho_2(y)\, h(y)\, ds_y\,.$$

In conformity with (3.2.44) the proof of Lemma 4.2.3 serves as a basis for (3.2.34), thereby justifying that the combination of (3.2.34) and (3.2.45) gives

$$(3.2.55) \qquad w(x; A_1 \setminus A_0, \mu_1, \partial A_1, \rho_1) = w(x; A_2 \setminus A_0, \mu_2, \partial A_2, \rho_2) \quad \text{for} \quad x \in D_0 \setminus \bar{B}\,.$$

Therefore, the left-hand sides of (3.2.53) and (3.2.54) are equal by virtue of (3.2.55) and the properties of the potentials of volume masses and simple layers. Thus, the equality of the right-hand sides of (3.2.53)–(3.2.54) is established. This proves the assertion of the lemma. ■

In what follows we shall need the concept of **generalized solution** to the Laplace equation in the sense of Wiener (for more detail see Keldysh and Lavrentiev (1937), Keldysh (1940)). In preparation for this, we refer to the boundary $\partial\Omega$ of a domain Ω (in general, multiply connected) such that a neighborhood of any point of $\partial\Omega$ contains the points of the set $\mathbf{R}^n \setminus \bar{\Omega}$. Any domain Ω enabling the solvability of the Dirichlet problem for the Laplace equation with any continuous boundary data falls into the category of standard domains.

Let a function $f(x)$ be continuous and defined on the boundary $\partial\Omega$ of a domain Ω, which is multiply connected and bounded. The intention is to use a continuous function $\varphi(x)$ defined everywhere in the space $\mathbf{R}^n$ and identical with $f(x)$ on $\partial\Omega$ (for more detail see Keldysh and Lavrentiev (1937)). In what follows we involve a sequence of domains

$$D_1,\, D_2,\, \dots,\, D_m,\, \dots$$

with boundaries

$$\partial D_1,\, \partial D_2,\, \dots,\, \partial D_m,\, \dots,$$

containing the closed set $\Omega \cup \partial\Omega$ and converging to the domain Ω, so that from a certain number m_0 and on any closed subset of $\mathbf{R}^n \setminus \bar{\Omega}$ will be out D_{m_0}. We may assume that the components of ∂D_m are analytical with

$\bar{D}_{m+1} \subset D_m$ for any m. The symbol $h_{m\varphi}$ indicates the solution of the Dirichlet problem for the Laplace equation in D_m with the boundary data $\varphi|_{\partial D_m}$. Following the papers of Keldysh and Lavrentiev (1937), Keldysh (1940) one succeeds in showing that the sequence of functions

$$h_{1\varphi}, h_{2\varphi}, \dots, h_{m\varphi}, \dots$$

converges in the closed domain $\bar{\Omega}$ and $\{h_{m\varphi}\}_{m=1}^{\infty}$ converges uniformly over a closed subdomain $\bar{\Omega}' \subset \Omega$. The limiting function $h_f(x)$ satisfies the Laplace equation without concern for how the domains D_m and the function $\varphi(x)$ will be chosen.

Definition 3.2.1 *The function $h_f(x)$ constructed is said to be a generalized solution of the Dirichlet problem for the Laplace equation in the domain Ω with the boundary data $f(x)$ continuous on $\partial\Omega$.*

Let the domain Ω and its boundary $\partial\Omega$ be given in Definition 3.2.1 and let $\mu(y)$ be a summable bounded function.

Lemma 3.2.5 *Let the density $\mu(y)$ be such that $u(x, \Omega, \mu) = 0$ for $x \in \mathbf{R}^n \setminus \bar{\Omega}$. Then any generalized solution h_f of the Dirichlet problem in the domain Ω satisfies the relation*

$$\int_{\Omega} \mu(y)\, h_f(y)\, dy = 0\,. \tag{3.2.56}$$

Proof The main idea behind proof is to extract a sequence of domains

$$D_1, D_2, \dots, D_m, \dots, \qquad \bar{D}_{m+1} \subset D_m\,,$$

containing $\Omega \cup \partial\Omega$ and having the analytic boundaries ∂D_m. As stated above, there exists a sequence of solutions of the Dirichlet problem for the Laplace equation, when the boundary data are prescribed by a sequence of the continuous on $\partial\Omega$ functions, say

$$\{h_{m\varphi}\}_{m=1}^{\infty}, \qquad h_{m\varphi} \longrightarrow h_f\,, \qquad m \to \infty\,.$$

Because the function $h_{m\varphi}$ is harmonic in the domain $\bar{D}_{m+1}$, Lemma 3.2.2 implies that

$$\int_{\Omega} \mu(y)\, h_{m\varphi}(y)\, dy = 0\,. \tag{3.2.57}$$

As far as the function $f(x)$ is bounded, the extended function $\varphi(x)$ can be so chosen as to satisfy

$$|\varphi|, \ |f| \le M \equiv \text{const} \quad \text{in} \quad \mathbf{R}^n ,$$

yielding

$$|h_{m\varphi}|, \ |h_f| \le M \qquad \text{in} \quad \bar{\Omega} \quad \text{for any} \quad m = 1, 2, \ldots .$$

This is due to the principle of maximum modulus. The function $\mu(y)$ being bounded in $\bar{\Omega}$ provides the validity of the estimate

$$|\mu(y)\, h_{m\varphi}| \le c_1 \equiv \text{const} ,$$

which is uniform in m. Therefore, by the Lebesque theorem the limit relation

$$\lim_{m\to\infty} \int_{\Omega} \mu(y)\, h_{m\varphi}(y)\, dy = \int_{\Omega} \mu(y)\, h_f(y)\, dy \tag{3.2.58}$$

takes place. From (3.2.57) it follows that the left-hand side of (3.2.58) equals zero. Thus, (3.2.56) is true and thereby the lemma is completely proved. ■

To assist the readers in applications, we are going to show how the assertions of Lemmas 3.2.2–3.2.4 can be extended to cover the generalized solution h_f of the Dirichlet problem for the Laplace equation by using Lemma 3.2.4 as one possible example. True, it is to be shown for the problem

$$\begin{cases} \Delta h(x) = 0 , & x \in B_0 , \\ h(x) = \varphi(x) , & x \in \partial B_0 , \end{cases} \tag{3.2.59}$$

where B_0 arose from (3.2.39)–(3.2.41), that any continuous on ∂B_0 function φ can be put in correspondence with a function h_φ. Being a generalized solution of the Dirichlet problem (3.2.59) in the sense of Wiener, the function h_φ is subject to the relation

$$\Delta h_\varphi(x) = 0 , \qquad x \in B_0 , \tag{3.2.60}$$

and takes the values $\varphi(x)$ at regular points of the boundary ∂B_0, what means that

$$\lim_{x\to x_0} h_\varphi(x) = \varphi(x_0) , \quad x_0 \in \partial B_0 \quad (x_0 \ \text{ is a regular point}) ,$$

irrelevant to the choice of a continuous function φ (see Landis (1971)).

Note that the generalized solution h_φ is identical with the solution of the Dirichlet problem (3.2.59) itself, if any.

Lemma 3.2.6 *Under the conditions of Lemma* 3.2.4 *any generalized solution* h_f *of the Dirichlet problem* (3.2.59) *satisfies the relation*

$$\beta \int_{B_0} h_\varphi(y)\,\mu_1(y)\;dy + \gamma \int_{(\partial B_0)^e} h_\varphi(y)\,\rho_1(y)\;ds_y = \gamma \int_{(\partial B_0)^i} h_\varphi(y)\,\rho_2(y)\;ds_y\,. \tag{3.2.61}$$

The proof of Lemma 3.2.6 is omitted here, since it is similar to Lemma 3.2.4 with minor refinements identical with those of Lemma 3.2.5.

3.3 The exterior inverse problem for the volume potential with variable density for bodies with a "star-shaped" intersection

We cite here a simplified version of the statement of an inverse problem of finding the shape of a body from available values of its volume potential. Let D_0 be an arbitrary domain in the space $\mathbf{R}^n$ enclosing the origin of coordinates and let Ω be an open bounded set with boundary $\partial\Omega$ such that $\bar{\Omega} \subset D_0$. In the general case the set Ω is representable as the union of a finite number of domains Ω^j with piecewise boundaries $\partial\Omega^j$ in conformity with (3.2.4). Special investigations involve a pair of functions with the following properties:

(1) the function $\mu(x)$ is measurable and bounded in D_0;

(2) the function $h(x)$ is harmonic everywhere except for the origin of coordinates or a certain bounded domain D^*, $\bar{D}^* \subset D_0$.

In each such case the function $h(x)$ is assumed to behave at infinity as the fundamental solution $E(x,0)$ of the Laplace equation.

In dealing with the functions h and μ the inverse problem for the potential of volume masses

$$u(x;\Omega,\mu) = \int_{\Omega} E(x,y)\,\mu(y)\;dy = h(x)$$

consists of finding the domain Ω enclosing the origin of coordinates or, correspondingly, the domain D^*, $\bar{D}^* \subset \Omega$.

This section examines the uniqueness of solution of the aforementioned inverse problem. In other words, the main goal of our study is to find out the conditions under which the equality of the exterior volume potentials implies the coincidence of the emerging domains.

We now proceed to a more rigorous statement of the uniqueness problem. Let Ω_α, $\alpha = 1, 2$, be unknown open bounded sets with boundaries $\partial\Omega_\alpha$. The set $\partial\Omega_\alpha$ is the boundary of $\mathbf{R}^n \setminus \bar{\Omega}_\alpha$ and $\bar{\Omega}_\alpha \subset D_0$, where D_0 is a certain domain in the space $\mathbf{R}^n$. The volume mass potentials $u^\alpha(x)$ are defined by (3.2.1) with a common density $\mu(y)$ as follows:

$$u^\alpha(x) = u(x; \Omega_\alpha, \mu) = \int\limits_{\Omega_\alpha} E(x, y)\, \mu(y)\, dy .$$

Problem 1 It is required to formulate the conditions under which the equality of exterior volume potentials

$$\int\limits_{\Omega_1} E(x, y)\, \mu(y)\, dy = \int\limits_{\Omega_2} E(x, y)\, \mu(y)\, dy \quad \text{for} \quad x \in D_0 \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2)$$

implies the coincidence of Ω_1 and Ω_2.

Denote by s^e the boundary of the set $\bar{\Omega}^e = \bar{\Omega}_1 \cup \bar{\Omega}_2$. Under the natural premise $\Omega_1 \neq \Omega_2$ one can readily show that $s^e = s_1^e \cup s_2^e$ within the notations

$$\begin{aligned} s_1^i &= \partial\Omega_1 \cap \bar{\Omega}_1 \cap \bar{\Omega}_2 , & s_1^e &= \partial\Omega_1 \setminus s_1^i , \\ s_2^e &= \partial\Omega_2 \cap s^e , & s_2^i &= \partial\Omega_2 \setminus s_2^e . \end{aligned} \tag{3.3.1}$$

When $\Omega_1 = \Omega_2$ we put $s_\alpha^e = \partial\Omega_\alpha$ for $\alpha = 1, 2$. It is worth noting here two things. First, some of the sets s_α^e, s_α^i may be empty. Second, notation (3.3.1) coincides with (3.2.38) in the case where $A_\alpha = \Omega_\alpha$. In the sequel it will be always preassumed that the boundary $\partial\Omega_\alpha$, $\alpha = 1, 2$, is piecewise smooth.

The symbol $\mathbf{R}_y$ is used for the vector directed from the origin of coordinates O to a point y ($n \geq 2$). Let

$$r = |\, y \,| = |\, \mathbf{R}_y \,| .$$

We might attempt the function $\mu(y)$ in the form

$$\mu(y) = \xi(y)\, \delta(y) , \tag{3.3.2}$$

where

(a) the function $\delta(y)$ (in general, of nonconstant sign) is continuously differentiable and satisfies the condition
$$\frac{\partial \delta}{\partial r} = 0 ;$$

(b) $\xi(y) > 0$ and $\dfrac{\partial}{\partial r}\Big(r^n \xi\Big) > 0$ for all $y \in \bar{\Omega}_\alpha$.

Theorem 3.3.1 *Let the origin of coordinates O be enclosed in the set $\bar{\Omega}_1 \cap \bar{\Omega}_2$ and*

(1) *the radius vector $\mathbf{R}_y$ obey the inequality*

$$(\mathbf{R}_y, \mathbf{n}_y) \geq 0 \quad \text{for} \quad y \in s_1^i,\ s_2^i, \tag{3.3.3}$$

where $\mathbf{n}_y$ is a unit external normal to the boundary $\partial\Omega_\alpha$, $\alpha = 1, 2$, and $(\mathbf{R}_y, \mathbf{n}_y)$ signifies the scalar product of the vectors $\mathbf{R}_y$ and $\mathbf{n}_y$;

(2) *the exterior volume potentials of the Laplace equation generated by the domains Ω_α, $\alpha = 1, 2$, with density μ of class (3.3.2) satisfy the equality*

$$u(x; \Omega_1, \mu) = u(x; \Omega_2, \mu) \quad \text{for} \quad x \in \mathbf{R}^n \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2). \tag{3.3.4}$$

Then $\Omega_1 = \Omega_2$.

Proof We agree to consider $\bar{\Omega}_0 = \bar{\Omega}_1 \cap \bar{\Omega}_2$ with $\operatorname{mes} \Omega_0 \neq 0$. We spoke above about Ω^e denoting the domain bounded by the surface s^e, for which the relation $\bar{\Omega}^e = \bar{\Omega}_1 \cup \bar{\Omega}_2$ takes place.

Together condition (3.3.4) and Lemma 3.2.2 with $\beta = 1$ and $\gamma = 0$ imply the relation

$$\int_{\Omega_1} \mu(y)\, h(y)\, dy - \int_{\Omega_2} \mu(y)\, h(y)\, dy = 0 \tag{3.3.5}$$

for any function $h(y)$ harmonic in the domain $D \supset (\bar{\Omega}_1 \cap \bar{\Omega}_2)$.

In further reasoning the contradiction arguments may be of help in achieving the final aim. We are first interested in the case where $\delta(y) = 1$, which admits comparatively simple proof. To put it differently, the positive function $\mu(y) \in C^1(\bar{\Omega}_1 \cap \bar{\Omega}_2)$ must satisfy the inequality

$$\frac{\partial}{\partial r}\left(r^n \mu\right) > 0, \qquad r \neq 0, \qquad y \in \bar{\Omega}_1 \cup \bar{\Omega}_2, \qquad n \geq 2. \tag{3.3.6}$$

By merely setting $h = 1$ in (3.3.5) it is easily verified that the masses of the bodies Ω_1 and Ω_2 are equal in that case. Therefore, due to the positiveness of the density neither of the domains Ω_α will be strictly inside another. Because of this fact, another conclusion can be drawn for the domains Ω_1 and Ω_2 with different connectedness that their mutual location together with (3.3.3) guarantees that either of the sets s_α^e, $\alpha = 1, 2$, will be nonempty.

If for one reason or another it is known that a function $H(y)$ is harmonic in D, then so is the function

$$h = \sum_{k=1}^{n} y_k \frac{\partial H}{\partial y_k} . \tag{3.3.7}$$

After that, substituting (3.3.7) into (3.3.5) yields the relation

$$\int_{\Omega_1} \mu(y) \left[\sum_{k=1}^{n} y_k \frac{\partial H}{\partial y_k} \right] dy - \int_{\Omega_2} \mu(y) \left[\sum_{k=1}^{n} y_k \frac{\partial H}{\partial y_k} \right] dy = 0 , \tag{3.3.8}$$

which can be rewritten as

$$\begin{aligned} &\int_{\Omega_1} \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu(y)\, y_k\, H) \right] dy \\ &\quad - \int_{\Omega_2} \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu(y)\, y_k\, H) \right] dy \\ &\quad - \int_{\Omega_1} H \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu(y)\, y_k) \right] dy \\ &\quad + \int_{\Omega_1} H \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu(y)\, y_k) \right] dy = 0 . \end{aligned} \tag{3.3.9}$$

Other ideas are connected with transformations of the first and second volume integrals of (3.3.9) into the surface ones. By such manipulations we arrive at

$$\begin{aligned} &\int_{\partial\Omega_1} H\, \mu \left[\sum_{k=1}^{n} y_k \cos\, (\widehat{\mathbf{y}_k ; \mathbf{n}_y}) \right] ds_y \\ &\quad - \int_{\partial\Omega_2} H\, \mu \left[\sum_{k=1}^{n} y_k \cos\, (\widehat{\mathbf{y}_k ; \mathbf{n}_y}) \right] ds_y \end{aligned}$$

$$- \int_{\Omega_2} H \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu(y)\, y_k) \right] dy$$

$$+ \int_{\Omega_2} H \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu(y)\, y_k) \right] dy = 0 \, .$$

All this enables us to write down the equation

$$J(H) = 0 \tag{3.3.10}$$

for

$$J(H) = \int_{\partial\Omega_1} H\, \mu\,(\mathbf{R}_y, \mathbf{n}_y)\; ds_y$$
$$- \int_{\partial\Omega_2} H\, \mu\,(\mathbf{R}_y, \mathbf{n}_y)\; ds_y$$
$$- \int_{\Omega_1 \setminus \Omega_0} H \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy$$
$$+ \int_{\Omega_2 \setminus \Omega_0} H \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy \, .$$

Let the function $f(y)$ be defined on the surface s^e by the relations

$$f(y) = \begin{cases} 1 \, , & y \in s_1^e \, , \\ 0 \, , & y \in s_2^e \, , \end{cases} \tag{3.3.11}$$

under the natural premise $\operatorname{mes} s_\alpha^e \neq 0$, $\alpha = 1, 2$.

The function so constructed is aimed at extending relation (3.3.10) to involve the function H_f being harmonic in Ω^e and taking the values $f(y)$ on the boundary surface s^e except for a set of zero surface measure, by means of a sequence of surface patches $\left\{ (s_1^e)_k \right\}_{k=1}^{\infty}$ such that $\overline{(s_1^e)_k} \subset (s_1^e)_{k+1}$ and $\varepsilon_k \to 0$ as $k \to \infty$, where

$$\varepsilon_k = \operatorname{mes} \left[s_1^e \setminus (s_1^e)_k \right] . \tag{3.3.12}$$

Putting

$$f_k(y) = \begin{cases} 1 \, , & y \in (s_1^e)_k \, , \\ 0 \, , & y \in s_2^e \, , \; s_1^e \setminus (s_1^e)_{k+1} \, , \\ \gamma_k(y) \, , & y \in (s_1^e)_{k+1} \setminus (s_1^e)_k \, , \end{cases} \tag{3.3.13}$$

we might select a monotonically increasing sequence of functions $\{f_k(y)\}_{k=1}^{\infty}$ being continuous on s^e such that

$$f_{k+1}(y) \geq f_k(y) .$$

Here $\gamma_k(y)$ refers to a continuous function whose values range from 0 to 1.

For every continuous on s^e function $f_k(y)$ the generalized solution H_{f_k} of the Dirichlet problem is introduced to carry out more a detailed exploration. Since

$$f_{k+1}(y) \geq f_k(y) , \qquad y \in s^e ,$$

the principle of maximum modulus with respect to the domain $\bar{\Omega}^e$ implies that

$$(3.3.14) \qquad H_{f_{k+1}}(y) \geq H_{f_k}(y) , \qquad y \in \bar{\Omega}_1 \cup \bar{\Omega}_2 .$$

In so doing, $| H_{f_k} | < 1$.

As a matter of fact, $\{H_{f_k}\}_{k=1}^{\infty}$ is an increasing sequence of functions which are bounded in $\bar{\Omega}^e$ and harmonic in Ω^e. By Harnack's theorem this sequence converges to a function H_f uniformly over Ω^e and the limiting function appears to be harmonic in Ω^e. The convergence $H_{f_k}(y) \to H_f(y)$ occurs for all $y \in \bar{\Omega}_1 \cup \bar{\Omega}_2$. Using the results ascribed to Keldysh and Lavrentiev (1937), Keldysh (1940) we see that the function H_{f_k} takes the values $f_k(y)$ on s^e at any point of the stability boundary. Due to this property the construction of $f_k(y)$ guarantees that the limiting function $H_f(y)$ takes the values $f(y)$ on the boundary s^e except for a set of zero measure.

When k is held fixed, the sequence $\{H_{m\varphi_k}\}_{m=1}^{\infty}$, by means of which we have defined the function H_{f_k} in Section 1.2, converges in the closed domain $\bar{\Omega}^e$ and $| H_{m\varphi_k} | < 1$ for $m = 1, 2, \ldots$. Hence the Lebesque theorem on the passage to the limit yields

$$\lim_{m \to \infty} J(H_{m\varphi_k}) = J(H_{f_k})$$

for any fixed k.

The function H_{f_k}, in turn, converges to H_f in the closed domain $\bar{\Omega}^e$ and $| H_{f_k} | < 1$ for $k = 1, 2, \ldots$. On the same grounds as before, we find by the Lebesque theorem that

$$\lim_{k \to \infty} J(H_{f_k}) = J(H_f) .$$

Because of (3.3.10),

$$J(H_{m\varphi_k}) = 0$$

for any fixed k and $m = 1, 2, \ldots$. Consequently, $J(H_{f_k}) = 0$ for any k, so that

$$J(H_f) = 0, \tag{3.3.15}$$

where the function H_f is harmonic in Ω^e and takes the boundary values equal to $f(y)$ from (3.3.11) almost everywhere. Moreover,

$$0 < H_f < 1. \tag{3.3.16}$$

In continuation of such an analysis we refer to the functional $J(H_f)$ with the values

(3.3.17)

$$\begin{aligned} J(H_f) = & \int\limits_{s_1^e \cup s_1^i} H_f\, \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & - \int\limits_{s_2^e \cup s_1^i} H_f\, \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & - \int\limits_{\Omega_1 \setminus \Omega_0} H_f \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy \\ & + \int\limits_{\Omega_2 \setminus \Omega_0} H_f \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy . \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial}{\partial r} (r^n \mu) &= r^{n-1} \left(n\, \mu + r\, \frac{\partial \mu}{\partial r} \right) \\ &= r^{n-1} \sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k), \end{aligned} \tag{3.3.18}$$

condition (3.3.6) assures us of the validity of the inequality

$$\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) > 0, \quad \text{when} \quad r \neq 0 \quad \text{for } y \in \bar{\Omega}_1 \cup \bar{\Omega}_2 . \tag{3.3.19}$$

In the estimation of the quantity $J(H_f)$ with the aid of (3.3.16), (3.3.19) and (3.3.17) we derive in passing the inequality

$$\begin{aligned} J(H_f) > & \int\limits_{s_1^e \cup s_1^i} H_f\, \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & - \int\limits_{s_2^e \cup s_2^i} H_f\, \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & - \int\limits_{\Omega_1 \setminus \Omega_0} 1 \cdot \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy \\ & + \int\limits_{\Omega_2 \setminus \Omega_0} H_f \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy, \end{aligned}$$

so that we arrive at one useful relation

$$\begin{aligned} J(H_f) > & \int\limits_{s_1^e \cup s_1^i} H_f\, \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & - \int\limits_{s_2^e \cup s_1^i} H_f\, \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & - \int\limits_{s_1^e} \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & + \int\limits_{s_2^i} \mu(y)\, (\mathbf{R}_y, \mathbf{n}_y)\, ds_y \\ & + \int\limits_{\Omega_2 \setminus \Omega_0} H_f \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy. \end{aligned}$$

The preceding manipulations are based on the well-known decomposition

$$\int\limits_{\Omega_1 \setminus \Omega_0} \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} \, (\mu \, y_k) \right] dy = \int\limits_{s_1^e} \mu(y) \, (\mathbf{R}_y \, , \mathbf{n}_y) \; ds_y$$

$$- \int\limits_{s_2^i} \mu \, (\mathbf{R}_y \, , \mathbf{n}_y) \; ds_y \, ,$$

where $\mathbf{n}_y$ is a unit external normal to the boundary $\partial\Omega_\alpha$ for $\alpha = 1, 2$.

In accordance with what has been said, the function H_f takes the boundary values $f(y)$ almost everywhere on $s^e = s_1^e \cup s_2^e$. In view of this, substituting the data of (3.3.11) into the preceding inequality yields

$$\begin{aligned} J(H_f) > & \int\limits_{s_1^i} H_f \, \mu(y) \, (\mathbf{R}_y \, , \mathbf{n}_y) \; ds_y \\ & + \int\limits_{s_2^i} (1 - H_f) \, \mu(y) \, (\mathbf{R}_y \, , \mathbf{n}_y) \; ds_y \\ & + \int\limits_{\Omega_2 \setminus \Omega_0} H_f \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} \, (\mu \, y_k) \right] dy \, . \end{aligned} \tag{3.3.20}$$

Putting these together with relations (3.3.15) and (3.3.19) we deduce that

$$\int\limits_{\Omega_2 \setminus \Omega_0} H_f \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} \, (\mu \, y_k) \right] dy > 0 \, .$$

Since $\mu(y) > 0$ for any $y \in \bar{\Omega}_1 \cup \bar{\Omega}_2$, the combination of the second condition of the theorem with (3.3.16) gives

$$\int\limits_{s_1^i} H_f \, \mu(y) \, (\mathbf{R}_y \, , \mathbf{n}_y) \; ds_y \geq 0 \, ,$$

$$\int\limits_{s_2^i} (1 - H_f) \, \mu(y) \, (\mathbf{R}_y \, , \mathbf{n}_y) \; ds_y \geq 0 \, ,$$

meaning $J(H_f) > 0$, which disagrees with (3.3.15). Thus, the theorem in its first part is proved.

The situation in which the density $\mu(y)$ happens to be of nonconstant sign and satisfies (3.3.2) with items (a) and (b) comes second. In case (b) condition (3.3.4) written for an arbitrary function $H(y)$ harmonic in D implies relation (3.3.10), that is,

$$J(H) = 0\,, \tag{3.3.21}$$

where

$$\begin{aligned} J(H) = & \int_{\partial\Omega_1} H(y)\,\mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y \\ & - \int_{\partial\Omega_2} H(y)\,\mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y \\ & - \int_{\Omega_1\setminus\Omega_0} H(y)\left[\sum_{k=1}^{n}\frac{\partial}{\partial y_k}(\mu\,y_k)\right]dy \\ & + \int_{\Omega_2\setminus\Omega_0} H(y)\left[\sum_{k=1}^{n}\frac{\partial}{\partial y_k}(\mu\,y_k)\right]dy\,. \end{aligned}$$

The well-established decomposition

$$J(H) = J_{\mu^+}(H) + J_{\mu^-}(H)$$

applies equally well to the following members: the first term $J_{\mu^+}(H)$ comprises those parts of integrals in (3.3.21) which are taken over $\partial\Omega_\alpha$ and $\Omega_\alpha\setminus\Omega_0$, where $\mu(y) > 0$ for all $y \in \bar{\Omega}_1 \cup \bar{\Omega}_2$. The second term $J_{\mu^-}(H)$ corresponds to those parts of integrals in (3.3.21) which are taken over $\partial\Omega_\alpha$ and $\Omega_\alpha\setminus\Omega_0$, where $\mu(y) \le 0$ for all $y \in \bar{\Omega}_1 \cup \bar{\Omega}_2$.

Let

$$\begin{aligned} \partial\Omega_{\mu^+} &= \big\{y\in\partial\Omega,\ \mu(y)>0\big\}\,; & \partial\Omega_{\mu^-} &= \big\{y\in\partial\Omega,\ \mu(y)\le 0\big\}\,; \\ \Omega_{\mu^+} &= \big\{y\in\Omega,\ \mu(y)>0\big\}\,; & \Omega_{\mu^-} &= \big\{y\in\Omega,\ \mu(y)\le 0\big\}\,. \end{aligned} \tag{3.3.22}$$

Bearing in mind (3.3.1) and the way notation (3.3.22) has been introduced above, it is possible to produce a number of the new symbols. In particular, it is fairly common to use

$$s^e_{\alpha\mu^+} = \big\{\, y\in s^e_\alpha\colon\ \mu(y)>0\,\big\}\,, \qquad \alpha = 1,2\,.$$

Within these notations, we introduce the function $f(y)$ defined on the surface $s^e = s_1^e \cup s_2^e$ by the relations

$$
(3.3.23) \qquad f(y) = \begin{cases} 1 & \text{for} \quad y \in s^e_{1\mu^+},\ s^e_{2\mu^-}, \\ 0 & \text{for} \quad y \in s^e_{1\mu^-},\ s^e_{2\mu^+}. \end{cases}
$$

Under the premises of the theorem it follows from the foregoing that $f(y) \neq$ const for $y \in s^e$. As in the first part of the proof it is necessary to extend relation (3.3.10) in order to involve the solution $H_f(y)$ of the Dirichlet problem for the Laplace equation. The boundary values taken by $H_f(y)$ on s^e coincide with those from (3.2.3) almost everywhere. In this line, we obtain

$$
(3.3.24) \qquad J(H_f) = 0\,,
$$

where

$$
J(H_f) = J_{\mu^+}(H_f) + J_{\mu^-}(H_f)
$$

and

$$
\begin{aligned}
(3.3.25) \qquad J_{\mu^+}(H_f) = & \int\limits_{s^e_{1\mu^+}} 1 \cdot \mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y \\
& + \int\limits_{s^i_{1\mu^+}} H_f(y)\,\mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y \\
& - \int\limits_{s^i_{2\mu^+}} H_f(y)\,\mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y \\
& - \int\limits_{(\Omega_1 \setminus \Omega_0)_{\mu^+}} H_f(y) \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k}(\mu\, y_k) \right] dy \\
& + \int\limits_{(\Omega_2 \setminus \Omega_0)_{\mu^+}} H_f(y) \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k}(\mu\, y_k) \right] dy\,;
\end{aligned}
$$

$$J_{\mu^-}(H_f) = \int\limits_{s^i_{1\mu^-}} H_f(y)\,\mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y \tag{3.3.26}$$

$$- \int\limits_{s^e_{2\mu^-}} 1\,\cdot\,\mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y$$

$$- \int\limits_{s^i_{2\mu^-}} H_f(y)\,\mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y$$

$$- \int\limits_{(\Omega_1\setminus\Omega_0)_{\mu^-}} H_f(y)\left[\sum_{k=1}^n \frac{\partial}{\partial y_k}(\mu\,y_k)\right]dy$$

$$+ \int\limits_{(\Omega_2\setminus\Omega_0)_{\mu^-}} H_f(y)\left[\sum_{k=1}^n \frac{\partial}{\partial y_k}(\mu\,y_k)\right]dy\,.$$

Conditions (3.3.2) in terms of (3.3.22) imply the chain of inequalities

$$\left[\sum_{k=1}^n \frac{\partial}{\partial y_k}(\mu\,y_k)\right] > 0 \quad \text{for}\;\; y\in(\Omega_\alpha\setminus\Omega_0)_{\mu^+}\,,\qquad \alpha=1,2; \tag{3.3.27}$$

$$\left[\sum_{k=1}^n \frac{\partial}{\partial y_k}(\mu\,y_k)\right] < 0 \quad \text{for}\;\; y\in(\Omega_\alpha\setminus\Omega_0)_{\mu^-}\,,\qquad \alpha=1,2. \tag{3.3.28}$$

In view of the bounds $0 < H_f < 1$ for any $y\in\Omega^e$, we are led by relations (3.3.2), (3.3.27) and (3.3.28) to the estimates

$$- \int\limits_{(\Omega_1\setminus\Omega_0)_{\mu^+}} H_f(y)\left[\sum_{k=1}^n \frac{\partial}{\partial y_k}(\mu\,y_k)\right]dy \tag{3.3.29}$$

$$\geq - \int\limits_{s^e_{1\mu^+}} \mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y$$

$$+ \int\limits_{s^i_{2\mu^+}} \mu(y)\,(\mathbf{R}_y\,,\mathbf{n}_y)\;ds_y\;,$$

$$\int\limits_{(\Omega_2 \setminus \Omega_0)_{\mu^-}} H_f(y) \left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k) \right] dy \tag{3.3.30}$$

$$\geq \int\limits_{s^e_{2\mu^-}} \mu(y)\, (\mathbf{R}_y \,, \mathbf{n}_y)\; ds_y$$

$$- \int\limits_{s^i_{1\mu^-}} \mu(y)\, (\mathbf{R}_y \,, \mathbf{n}_y)\; ds_y \,.$$

From (3.3.27)–(3.3.30) it follows that

$$J_{\mu^+}(H_f) \geq \left[\int\limits_{s^i_{1\mu^+}} H_f(y)\, \mu(y)\, (\mathbf{R}_y \,, \mathbf{n}_y)\; ds_y \right] \tag{3.3.31}$$

$$+ \left[\int\limits_{s^i_{2\mu^+}} (1 - H_f(y))\, \mu(y)\, (\mathbf{R}_y \,, \mathbf{n}_y)\; ds_y \right]$$

$$+ \left[\int\limits_{(\Omega_2 \setminus \Omega_0)_{\mu^+}} H_f(y) \sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k)\; dy \right],$$

$$J_{\mu^-}(H_f) \geq \left[\int\limits_{s^i_{1\mu^-}} (H_f(y) - 1)\, \mu(y)\, (\mathbf{R}_y \,, \mathbf{n}_y)\; ds_y \right] \tag{3.3.32}$$

$$+ \left[- \int\limits_{s^i_{2\mu^-}} H_f(y)\, \mu(y)\, (\mathbf{R}_y \,, \mathbf{n}_y)\; ds_y \right]$$

$$+ \left[- \int\limits_{(\Omega_1 \setminus \Omega_0)_{\mu^-}} H_f(y) \sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\mu\, y_k)\; dy \right].$$

Taking into account (3.3.3), the properties of the function $\mu(y)$ specified by (3.3.2) and the properties of the function $H_f(y)$ revealed in estimates

(3.3.23), we deduce that every term in square brackets on the right-hand sides of (3.3.31)–(3.3.32) is nonnegative and, moreover, at least one of them is strictly positive. Thus, we should have $J(H_f) > 0$, violating (3.3.24). The obtained contradiction proves the assertion of the theorem. ■

Remark 3.3.1 Condition (3.3.3) of Theorem 3.3.1 is satisfied if $\bar{\Omega}_1 \cap \bar{\Omega}_2$ is a "star-shaped" set with respect to the point $O \in \bar{\Omega}_1 \cap \bar{\Omega}_2$. For each such set, $\mathbf{R}^n \setminus (\bar{\Omega}_1 \cap \bar{\Omega}_2)$ appears to be a one-component set. In particular, when either of the sets $\bar{\Omega}_1$ and $\bar{\Omega}_2$ is "star-shaped" with respect to a common point O, we thus have (3.3.3).

3.4 Integral stability estimates for the inverse problem of the exterior potential with constant density

As we have already mentioned in preliminaries to this chapter, the question of uniqueness of inverse problem solutions is intimately connected with their stability. The general topological criterion of stability ascribed to Tikhonov (1943) and based on the corresponding uniqueness theorems implies the qualitative stability tests.

In this section several stability estimates for the inverse problem of the exterior potential for $n \geq 3$ will be derived in the class of "non-star-shaped" bodies that consists of the so-called "absolutely star-ambient" and "absolutely projectively-ambient" bodies including those with "star-shaped" intersections and, correspondingly, with boundaries having intersections only at two points by a straight line parallel to a known direction.

We denote by $u(x; A_\alpha) \equiv u(x; A_\alpha, 1)$ the volume potential of the body A_α with unit density. Throughout this section, we retain the notations given by formulae (3.2.1)–(3.2.4), (3.2.38) and (3.3.1) and attempt the fundamental solution of the Laplace equation in the form (3.1.4). In particular, we agree to consider

$$E(x, y) = \frac{1}{4\pi} \, \frac{1}{r_{xy}} \tag{3.4.1}$$

for $n = 3$ and $r_{xy} = |\, x - y \,|$.

As we will see a little later, it will be convenient to deal with

$$w(x) = w(x; A_1) - w(x; A_2)\,, \tag{3.4.2}$$

where

$$\begin{aligned} w(x; A_\alpha) &= 2\,\gamma_1\, u(x; A_\alpha) \\ &\quad - \sum_{k=1}^{n} \big(\gamma_1 x_k + \beta_1 q_k\big)\, \frac{\partial}{\partial x_k}\, u(x; A_\alpha)\,, \end{aligned} \tag{3.4.3}$$

γ_1 and β_1 are real numbers, $\gamma_1^2+\beta_1^2 \neq 0$, and $\mathbf{q}=(q_1,\dots,q_n)$ is a constant vector.

For $A_1 \neq A_2$ the traditional tool for carrying out this work involves

$$F^e = \int\limits_{\bar{\Gamma}_1^e} |\Phi(y)|\, ds_y + \int\limits_{\bar{\Gamma}_2^e} |\Phi(y)|\, ds_y \,, \tag{3.4.4}$$

$$F^i = \int\limits_{\Gamma_1^i} |\Phi(y)|\, ds_y + \int\limits_{\Gamma_2^i} |\Phi(y)|\, ds_y \,, \tag{3.4.5}$$

where the function

$$\Phi(y) = \left(\gamma_1 \mathbf{R}_y + \beta_1 \mathbf{q},\, \mathbf{n}_y\right) \tag{3.4.6}$$

is adopted as the scalar product of the vectors $\gamma_1\mathbf{R}_y+\beta_1\mathbf{q}$ and $\mathbf{n}_y$. Here $\mathbf{n}_y$ denotes a unit exterior normal to the boundary ∂A_α, $\alpha=1,2$. In what follows we accept $A_\alpha=\Omega_\alpha$ unless otherwise is explicitly stated, where Ω_α is a simply connected domain with a piecewise smooth boundary $\partial\Omega_\alpha$. Also, either of the sets $\Omega_0=\Omega_1\cap\Omega_2$ and $\mathbf{R}^n\setminus(\bar{\Omega}_1\cup\bar{\Omega}_2)$ is supposed to be simply connected.

Theorem 3.4.1 *Let $U(x;\Omega_\alpha)$ be the volume mass potentials of domains Ω_α, $n\geq 3$, $\alpha=1,2$, whose constant density is equal to* 1. *One assumes, in addition, that the potentials $U(x,\Omega_\alpha)$ can harmonically be extended from $\mathbf{R}^n\setminus\bar{\Omega}_\alpha$ for $x\in\mathbf{R}^n\setminus D^*$, where D^* is a simply connected domain with a smooth boundary ∂D^*, $\overline{D^*}\subset\Omega^e$. Then, within notations* (3.4.2)–(3.4.6), *the estimate*

$$F^e - F^i \leq c_1 \max_{x\in\partial D^*} \left| \frac{\partial}{\partial n_x}\, w(x)\right| \tag{3.4.7}$$

is valid with $c_1=\mathrm{const}>0$ depending only on the configuration of the boundary ∂D^* and $\left[\dfrac{\partial}{\partial n_x}\, w(x)\right]\Big|_{\partial D^*}$ denoting the external normal derivative at a point $x\in\partial D^*$ of the function $w(x)$, which has been harmonically extended to the boundary ∂D^*.

Proof Let D and D_1 be arbitrary domains with piecewise smooth boundaries ∂D and ∂D_1 such that

$$D\supset \bar{D}_1\supset D_1\supset(\bar{\Omega}_1\cup\bar{\Omega}_2)\,.$$

A simply connected domain $D^{*\prime}$ with a smooth boundary $\partial D^{*\prime}$ is much involved in further reasoning to avoid cumbersome calculations under the agreements that $D_1 \supset \overline{D^{*\prime}}$ and none of the singular points of both functions $u(x;\Omega_\alpha)$, $\alpha = 1, 2$, lies within the domain $D^{*\prime}$.

One way of proceeding is to refer to the functional

$$J(H) = \int\limits_{\partial\Omega_1} H(y)\,\Phi(y)\; ds_y - \int\limits_{\partial\Omega_2} H(y)\,\Phi(y)\; ds_y\,, \tag{3.4.8}$$

where $H(y)$ is an arbitrary harmonic in D function, $\Phi(y)$ has been defined in (3.4.6) and $\partial\Omega_\alpha$ is the boundary of $\Omega_\alpha \subset \mathbf{R}^n$, $n \geq 3$, $\alpha = 1, 2$. Before we undertake the proof of the theorem, a preliminary lemma will be introduced.

Lemma 3.4.1 *The functional $J(H)$ specified by (3.4.8) admits the estimate*

$$|\,J(H)| \leq c_2 \max_{y\in\partial D_1} |\,H(y)\,| \max_{x\in\partial D^{*\prime}} \left|\,\frac{\partial}{\partial n_x}\; w(x)\right|, \tag{3.4.9}$$

where $c_2 = \mathrm{const} > 0$ depends on the boundary $\partial D^{*\prime}$.

Proof Indeed, any function $H(y)$ harmonic in the domain $D \supset \bar{D}_1$ can be represented in the form

$$\int\limits_{\partial D_1} M_x\;[E(x,y); H(x)]\; ds_x = \begin{cases} H(y)\,, & y \in D_1\,, \\ 0\,, & y \in \mathbf{R}^n \setminus \bar{D}_1\,, \end{cases} \tag{3.4.10}$$

where $E(x,y)$ stands for the fundamental solution of the Laplace equation for $n \geq 3$. The expression for $M[u;v]$ is as follows:

$$M[u;v] = v(x)\;\frac{\partial u}{\partial n_x} - \frac{\partial v}{\partial n_x}\; u(x) \tag{3.4.11}$$

with $\mathbf{n}_x$ denoting a unit external normal to ∂D_1 at point x. Furthermore, we will use as a tool in achieving important results a functional and a harmonic function which carry out the following actions:

$$\bar{J}(h) = \int\limits_{\Omega_1} h(y)\; dy - \int\limits_{\Omega_2} h(y)\; dy \tag{3.4.12}$$

and

$$h(y) = \sum_{k=1}^{n} \frac{\partial}{\partial y_k}\;[(\gamma_1 y_k + \beta_1 q_k)\,H(y)]\,. \tag{3.4.13}$$

Here $H(y)$ is harmonic in D and the numbers γ_1, β_1 and q_k have been defined in (3.4.6). Via representation (3.4.10) for the function $H(y)$ we might have

$$\begin{aligned}
(3.4.14)\qquad \int\limits_{\Omega_\alpha} h(y)\,dy &= \int\limits_{\Omega_\alpha} \sum_{k=1}^{n} \frac{\partial}{\partial y_k}\,\left[(\gamma_1 y_k + \beta_1 q_k)\,H(y)\right]\,dy \\
&= \int\limits_{\Omega_\alpha} \sum_{k=1}^{n} \frac{\partial}{\partial y_k}\,\Big\{(\gamma_1 y_k + \beta_1 q_k) \\
&\quad \times \int\limits_{\partial D_1} M_x\left[E(x,y); H(x)\right]\,ds_x\Big\}\,dy \\
&= \int\limits_{\partial D_1} M_x\Bigg[\Bigg\{\int\limits_{\Omega_\alpha} \sum_{k=1}^{n} \frac{\partial}{\partial y_k}(\gamma_1 y_k + \beta_1 q_k) \\
&\quad \times E(x,y)\,dy\Bigg\};\, H(x)\Bigg]\,ds_x
\end{aligned}$$

for $\alpha = 1, 2$. Using the well-established decomposition for $n \geq 3$

$$\begin{aligned}
(3.4.15)\quad \sum_{k=1}^{n} \frac{\partial}{\partial y_k}\,&\Big[(\gamma_1 y_k + \beta_1 q_k)\,E(x,y)\Big] \\
&= 2\,\gamma_1\,E(x,y) - \sum_{k=1}^{n} (\gamma_1 y_k + \beta_1 q_k)\,\frac{\partial}{\partial x_k}\,E(x,y)
\end{aligned}$$

we rewrite the expression in curly brackets from the last formula (3.4.14) as

$$(3.4.16)\qquad \int\limits_{\Omega_\alpha} \sum_{k=1}^{n} \frac{\partial}{\partial y_k}(\gamma_1 y_k + \beta_1 q_k)\,E(x,y)\,dy = w(x;\Omega_\alpha)\,,$$

where the function $w(x;\Omega_\alpha)$ has been defined in (3.4.3). Therefore, with the aid of (3.4.14) and (3.4.16) we establish for any pair of the functions $h(y)$ and $H(y)$ built into (3.4.13) the relation

$$(3.4.17)\qquad \int\limits_{\Omega_\alpha} h(y)\,dy = \int\limits_{\partial D_1} M_x\big[\,w(x;\Omega_\alpha); H(x)\big]\,ds_x\,.$$

Some progress will be achieved if the function $h(y)$ will be taken in the form (3.4.13) with further substitution into (3.4.12). As a final result we get the equality

$$\bar{J}(H) = J(H), \tag{3.4.18}$$

where the functional $J(H)$ has been defined by (3.4.8). On the other hand, (3.4.12), (3.4.17) and (3.4.18) imply that

$$J(H) = \int_{\partial D_1} M_x\big[w(x); H(x)\big]\, ds_x \tag{3.4.19}$$

with $w(x)$ being still subject to (3.4.2)–(3.4.3).

Via representation (3.4.19) one can derive an upper bound for the absolute value of the functional $J(H)$. In the light of the premises of the lemma the function $w(x)$ is harmonically extendable from $\mathbf{R}^n \setminus \bar{\Omega}_\alpha$ to the domain $\mathbf{R}^n \setminus D^{*\prime}$. When solving the exterior boundary value Neymann problem, one can find on the surface $\partial D^{*\prime}$ the density $\omega(y)$ of the simple layer potential

$$v(x) = \int_{\partial D^{*\prime}} \omega(y)\, E(x,y)\, ds_y\,, \tag{3.4.20}$$

for which

$$\left[\frac{\partial}{\partial n_x}\, v(x)\right]^- = \frac{\partial}{\partial n_x}\, w(x) \quad \text{for} \quad x \in \partial D^{*\prime}, \tag{3.4.21}$$

where $\left[\dfrac{\partial}{\partial n_x} v(x)\right]^-$ denotes the limiting value of

$$\frac{\partial}{\partial n_x}\, v(x') \quad \text{as} \quad x' \to x \quad \big(x \in \partial D^{*\prime}\big)$$

along a unit external normal $\mathbf{n}_x$ to $D^{*\prime}$. In conformity with the uniqueness of a solution of the exterior boundary value Neymann problem ($n \geq 3$), relation (3.4.21) yields

$$w(x) = v(x) \quad \text{for} \quad x \in \mathbf{R}^n \setminus D^{*\prime}, \tag{3.4.22}$$

showing the notation $w(x)$ to be a sensible one. Consequently, (3.4.19), (3.4.20) and (3.4.22) serve to motivate the representations

$$(3.4.23) \qquad J(H) = \int\limits_{\partial D_1} M_x \Bigg[\int\limits_{\partial D^{*\prime}} \omega(y)\, E(x,y)\; ds_y;\; H(x) \Bigg]\; ds_x$$

$$= \int\limits_{\partial D^{*\prime}} \omega(y) \Bigg\{ \int\limits_{\partial D_1} M_x \big[E(x,y);\; H(x) \big]\; ds_x \Bigg\}\; ds_y\,.$$

With (3.4.10) in view, the latter becomes much more simpler:

$$(3.4.24) \qquad J(H) = \int\limits_{\partial D^{*\prime}} \omega(y)\, H(y)\; ds_y\,.$$

Thus, the functional $J(H)$ can be estimated as follows:

$$(3.4.25) \qquad |\, J(H)\,| \leq \max_{y \in \partial D^{*\prime}} |\,H(y)| \int\limits_{\partial D^{*\prime}} |\,\omega(y)\,|\; ds_y$$

$$\leq \max_{y \in \partial D_1} |\,H(y)| \int\limits_{\partial D^{*\prime}} |\,\omega(y)\,|\; ds_y\,.$$

On the other hand, we should take into account that the function $\omega(y)$ solves the integral Fredholm equation of the second kind

$$(3.4.26) \qquad (I - T)\,\omega = f$$

on the basis of (3.4.20)–(3.4.21) and the formulae for the jump of the normal derivative of the potential of the simple layer. Here I is, as usual, the identity operator. The operator T and the function f act in accordance with the following rules:

$$(3.4.27) \qquad T\,\omega = \int\limits_{\partial D^{*\prime}} 2\, \frac{\partial E(x,y)}{\partial n_x}\, \omega(y)\; ds_y\,;$$

$$f(x) = -2\, \frac{\partial w(x)}{\partial n_x}\,, \qquad x \in \partial D^{*\prime}.$$

On the strength of the uniqueness of a solution of the exterior Neymann problem ($n \geq 3$) the inverse operator $(I - T)^{-1}$ should be bounded in the space of all continuous functions. Thus, (3.4.26) and (3.4.27) imply the inequality

$$(3.4.28) \qquad \max_{y \in \partial D^{*\prime}} |\,\omega(y)\,| \leq c_2 \max_{x \in \partial D^{*\prime}} \left| \frac{\partial}{\partial n_x}\, w(x) \right|, \qquad c_2 \equiv \text{const} > 0\,.$$

Estimate (3.4.9) is an immediate implication of (3.4.26) and (3.4.28) and this proves the assertion of the lemma. ■

We return to proving the theorem by appeal to a function $f(y)$ defined on the surface $s^e = s_1^e \cup s_2^e$ by the relations

$$f(y) = \begin{cases} -\operatorname{sign} \Phi(y) & \text{for} \quad y \in s_1^e\,, \\ \operatorname{sign} \Phi(y) & \text{for} \quad y \in s_2^e\,. \end{cases} \tag{3.4.29}$$

By means of the function $f(y)$ one can produce a generalized solution $H_f(y)$ of the Dirichlet problem for the Laplace equation in such a way that $H_f(y)$ will be harmonic within Ω^e, $|H_f(y)| \leq 1$ will occur for $y \in \Omega^e$ and the boundary values of $H_f(y)$ on the boundary s^e will coincide almost everywhere with $f(y)$ involved in (3.4.29). All tricks and turns remain unchanged as in the proof of Theorem 3.3.1. The way its result is used here is to select a sequence of functions $\{H_{m\varphi_k}\}_{m=1}^{\infty}$, $|H_{m\varphi_k}| \leq 1$, which makes it possible to find a function H_{f_k} with relevant properties:

$$|H_{f_k}| \leq 1 \quad \text{and} \quad H_{f_k} \quad \text{converges to} \quad H_f \quad \text{in} \quad \Omega^e .$$

Under the conditions of the theorem we are now in a position to consider the domain D^*, $\overline{D^*} \subset \Omega^e$, instead of the domain $D^{*\prime}$, arising from the preceding lemma. Since $\bar{D}^* \subset \Omega^e$, estimate (3.4.9) implies that

$$J(H_{m\varphi_k}) \leq |J(H_{m\varphi_k})| \leq c_2 \max_{x \in \partial D^*} \left| \frac{\partial}{\partial n_x} \, w(x) \right| . \tag{3.4.30}$$

Holding k fixed and passing to the limit in (3.4.30) as $m \to \infty$, we derive the estimates

$$J(H_{f_k}) \leq |J(H_{f_k})| \leq c_2 \max_{x \in \partial D^*} \left| \frac{\partial}{\partial n_x} \, w(x) \right| , \tag{3.4.31}$$

which after another passage to the limit as $k \to \infty$ look as follows:

$$J(H_f) \leq |J(H_f)| \leq c_2 \max_{x \in \partial D^*} \left| \frac{\partial}{\partial n_x} \, w(x) \right| . \tag{3.4.32}$$

On the other hand, the boundary values of H_f on s^e have been defined by (3.4.29), so that (3.4.8) gives

$$\begin{aligned} \int\limits_{s_1^e} |\Phi(y)| \, ds_y + \int\limits_{s_2^e} |\Phi(y)| \, ds_y \\ + \int\limits_{s_1^i} H_f(y) \, \Phi(y) \, ds_y \\ - \int\limits_{s_1^i} H_f(y) \, \Phi(y) \, ds_y = J(H_f) . \end{aligned} \tag{3.4.33}$$

Since $| H_f | \leq 1$, inequality (3.4.33) is followed by

$$\int\limits_{s_1^e} |\Phi(y)|\, ds_y + \int\limits_{s_2^e} |\Phi(y)|\, ds_y - \int\limits_{s_1^e} |\Phi(y)|\, ds_y - \int\limits_{s_2^i} |\Phi(y)|\, ds_y \leq J(H_f)\,. \tag{3.4.34}$$

Now estimate (3.4.7) is an immediate implication of (3.4.32) and (3.4.34) with notations (3.4.4)–(3.4.6). This completes the task of motivating the desired estimate. ■

We give below several corollaries to Theorem 3.4.1 that furnish the justification for what we wish to do.

Corollary 3.4.1 *Let $\Omega_1 \neq \Omega_2$. If there exist a point O, numbers γ_1, β_1 and a vector $\mathbf{q}$ such that*

$$\int\limits_{s_1^i} |\Phi(y)|\, ds_y + \int\limits_{s_2^i} |\Phi(y)|\, ds_y < \int\limits_{s_1^e} |\Phi(y)|\, ds_y + \int\limits_{s_2^e} |\Phi(y)|\, ds_y\,, \tag{3.4.35}$$

then the exterior potentials $u(x;\Omega_\alpha)$ of a constant density cannot coincide, that is, the set $\mathbf{R}^n \setminus \left(\bar{\Omega}_1 \cup \bar{\Omega}_2\right)$ encloses a point $\hat{x}$ such that

$$u(\hat{x};\Omega_1) \neq u(\hat{x};\Omega_2)\,. \tag{3.4.36}$$

Remark 3.4.1 In the case where $\gamma_1 = 1$ and $\beta_1 = 0$ we might have $\Phi(y) = (\mathbf{R}_y, \mathbf{n}_y)$ and

$$\int\limits_{s_1^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y + \int\limits_{s_2^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y \leq \int\limits_{s_1^e} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y + \int\limits_{s_2^e} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y \tag{3.4.37}$$

instead of condition (3.4.35).

Note that Corollary 3.4.1 is still valid if (3.4.37) is replaced by

$$\tag{3.4.38} \int_{s_1^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y + \int_{s_2^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y < \int_{s_1^e} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y + \int_{s_2^e} |(\mathbf{R}_y, \mathbf{n}_y)|\, ds_y .$$

It is obvious that (3.4.37) follows from (3.4.38). We claim that (3.4.38) holds if, in particular, the set $\bar{\Omega}_1 \cap \bar{\Omega}_2$ is "star-shaped" with respect to a certain point $O \in \bar{\Omega}_1 \cap \bar{\Omega}_2$, both sets s_α^e, $\alpha = 1, 2$, are not empty with $\operatorname{mes}(s_1^i \cap s_2^i) = 0$. To avoid generality for which we have no real need, we will consistently confine ourselves to a domain $D \subset \mathbf{R}^n$ bounded by the surface ∂D. Then the volume of D can be expressed by

$$\operatorname{mes} D = \int_D dy = \frac{1}{n} \int_D \left[\sum_{k=1}^n \frac{\partial y_k}{\partial y_k} \right] dy = \frac{1}{n} \int_{\partial D} (\mathbf{R}_y, \mathbf{n}_y)\, ds_y ,$$

yielding

$$\tag{3.4.39} \operatorname{mes} D = \frac{1}{n} \int_{\partial D} (\mathbf{R}_y, \mathbf{n}_y)\, ds_y .$$

In particular, if the set $\bar{\Omega}_1 \cap \bar{\Omega}_2$ is "star-shaped" with respect to at least one point $O \in \bar{\Omega}_1 \cap \bar{\Omega}_2$, $\operatorname{mes}(s_1^i \cap s_2^i) = 0$ and $\operatorname{mes} s_\alpha^e \neq 0$, $\alpha = 1, 2$, then $(\mathbf{R}_y, \mathbf{n}_y) \geq 0$ on s_α^i, $\alpha = 1, 2$. Because of (3.4.39), the meaning of relation (3.4.38) is that we should have

$$\operatorname{mes}(\bar{\Omega}_1 \cup \bar{\Omega}_2) > \operatorname{mes}(\bar{\Omega}_1 \cap \bar{\Omega}_2) .$$

Remark 3.4.2 If $\gamma_1 = 0$ and $\beta_1 = 1$, then $\Phi(y) = (\mathbf{q}, \mathbf{n}_y)$ and condition (3.4.35) can be replaced by

$$\tag{3.4.40} \int_{s_1^i} |(\mathbf{q}, \mathbf{n}_y)|\, ds_y + \int_{s_2^i} |(\mathbf{q}, \mathbf{n}_y)|\, ds_y \leq \int_{s_1^e} |(\mathbf{q}, \mathbf{n}_y)|\, ds_y + \int_{s_2^e} |(\mathbf{q}, \mathbf{n}_y)|\, ds_y .$$

The preceding inequality holds true if, in particular,

$$\operatorname{mes}(s_1^i \cap s_2^i) = 0$$

and the intersection of the set $s_1^i \cup s_2^i$ by a straight line, parallel to the vector $\mathbf{q}$, contains at most two points or two whole segments.

Remark 3.4.3 A geometric interpretation for conditions (3.4.37) and (3.4.40) is connected with introduction of a surface piece

$$\hat{s} = \hat{s}^1 \cup \hat{s}^2 \cup \ldots \cup \hat{s}^m$$

such that a ray originating from the point O intersects every part $\hat{s}^\gamma$ either at one point or by one whole segment. Let

$$\| V_{K(\hat{s})} \| = | V_{K(\hat{s}^1)} | + | V_{K(\hat{s}^2)} | + \cdots + | V_{K(\hat{s}^m)} | ,$$

where $| V_{K(\hat{s}^j)} |$ means the absolute value of the volume of the cone constructed over the piece $\hat{s}^j$ with vertex O. Within this notation, condition (3.4.37) can be rewritten as

$$\| V_{K(s_2^i)} \| + \| V_{K(s_2^i)} \| < \| V_{K(s_2^e)} \| + \| V_{K(s_2^e)} \| . \tag{3.4.41}$$

Likewise, let the piece $\hat{s} = \hat{s}^1 \cup \ldots \cup \hat{s}^m$ be such that every part $\hat{s}^\gamma$ can uniquely be projected onto a plane $N \perp \mathbf{q}$ by straight lines parallel to the vector $\mathbf{q}$. If so, it is reasonable to try to use the quantity

$$\| \sigma_{\pi(\hat{s})} \| = | \sigma_{\pi(\hat{s}^1)} | + | \sigma_{\pi(\hat{s}^2)} | + \cdots + | \sigma_{\pi(\hat{s}^m)} | , \tag{3.4.42}$$

where $| \sigma_{\pi(\hat{s}^j)} |$ means the absolute value of the surface area of $\pi(\hat{s}^j)$, the projection of the dimension $n - 1$ of $\hat{s}^j$ onto the plane N. Therefore, condition (3.4.40) becomes

$$\| \sigma_{\pi(s_2^i)} \| + \| \sigma_{\pi(s_2^i)} \| < \| \sigma_{\pi(s_2^e)} \| + \| \sigma_{\pi(s_2^e)} \| . \tag{3.4.43}$$

Under condition (3.4.41) the bodies Ω_1 and Ω_2 fall within the category of **"absolutely star-ambient" domains**. The bodies Ω_1 and Ω_2 are said to be **"absolutely projectively-outwards-ambient"** provided condition (3.4.43) holds.

If, in particular, under the initial conditions the set $\bar{\Omega}_1 \cap \bar{\Omega}_2$ is "star-shaped" with respect to an inner point, then the domains Ω_α are referred to as "absolutely star-ambient". Such domains Ω_α turn out to be "absolutely projectively-outwards-ambient" when a straight line, parallel to the vector $\mathbf{q}$, will intersect the boundary of the set $\Omega_1 \cap \Omega_2$ at most at two points.

Remark 3.4.4 Because of (3.4.7), we might have

$$0 \leq F^e - F^i$$

for any "absolutely star-ambient" or "absolutely projectively-ambient" domains.

Other stability estimates given below are asserted by Theorem 3.4.1.

Corollary 3.4.2 *Let a set P contain all singular points of the potentials $u(x;\Omega_\alpha)$, $\alpha = 1, 2$, and let*

$$\bar{P} \subset D^* \subset \overline{D^*} \subset \Omega^e .$$

If

$$|u(x;\Omega_1) - u(x;\Omega_2)| < \varepsilon , \qquad x \in \partial D^* , \tag{3.4.44}$$

then

$$F^e - F^i \le c_3 \frac{\varepsilon}{l^2} , \tag{3.4.45}$$

where $c_3 = c_3(\partial D^*) \equiv$ const > 0 and $l =$ dist $(\partial D^*, s^e)$ denotes the distance between ∂D^* and s^e.

Indeed, the function $u(x) \equiv u(x;\Omega_1) - u(x;\Omega_2)$ is harmonic in the domain $\mathbf{R}^n \setminus \bar{P}$ and, consequently, is uniformly bounded in a domain G ordered with respect to inclusion: $\mathbf{R}^n \setminus D \subset G \subset \bar{G} \subset \mathbf{R}^n \setminus \bar{P}$. Therefore, estimate (3.4.45) follows from (3.4.7) and (3.4.44) on the basis of the well-known estimates for the derivatives of the function u.

Assuming mes $(\partial\Omega_1 \cap \partial\Omega_2) = 0$ and retaining notations (3.4.40) and (3.4.42), we recast estimate (3.4.7) as

$$\gamma_1 \left[\| V_{K(\hat{s}^e)} \| - \| V_{K(\hat{s}^i)} \| \right] + \beta \left[\| \sigma_{\pi(\hat{s}^e)} \| - \| \sigma_{\pi(\hat{s}^i)} \| \right]$$

$$\le c_1 \max_{x \in \partial D^*} \left[\frac{\partial w(x)}{\partial n_x} \right] .$$

Our next step is to define for the domains Ω_1 and Ω_2 the **distance function** by means of the relation

$$\text{dist}\,(\Omega_1, \Omega_2) = \text{mes}\,(\Omega_1 \ominus \Omega_2) , \tag{3.4.46}$$

where $\Omega_1 \ominus \Omega_2 = (\Omega_1 \cup \Omega_2) \setminus (\Omega_1 \cap \Omega_2)$ designates the **symmetric difference** between the domains Ω_1 and Ω_2. Once equipped with the distance function defined by (3.4.46), the union of domains becomes a metric space. For more detail we refer the readers to Sobolev (1988).

Corollary 3.4.3 *If $\Omega_0 = \Omega_1 \cap \Omega_2$ is a "star-shaped" set with respect to a point $O \in \Omega_1 \cap \Omega_2$ and all the conditions of Theorem* 3.4.1 (*or Corollary* 3.4.2) *hold, then the estimates*

$$d(\Omega_1, \Omega_2) \le c_4 \max_{x \in \partial D^*} \left| \frac{\partial w(x)}{\partial n_x} \right| \tag{3.4.47}$$

and

$$d(\Omega_1, \Omega_2) \le c_5 \frac{\varepsilon}{l^2} \tag{3.4.48}$$

are true.

Proof Indeed, if the domain Ω_0 is supposed to be "star-shaped" with respect to a point $O \in \Omega_0$, then on account Remark 3.4.1

$$F^e - F^i \geq n \operatorname{mes}(\Omega_1 \ominus \Omega_2)$$

and $F^i < F^e$, so that estimates (3.4.47) and (3.4.48) follow from (3.4.7) and (3.4.44). ■

Corollary 3.4.4 *If under the premices of Corollary* 3.4.2 *any straight line, parallel to the vector* $\mathbf{q}$, *may intersect the boundary of the set* $\Omega_1 \cap \Omega_2$ *either at most at two points or by two whole segments, then*

$$\operatorname{mes}\big(\Omega_1 \ominus \Omega_2\big) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

Proof We proceed to establish this relation by inserting $\gamma_1 = 0$ and $\beta_1 = 0$. Since any straight line parallel to $\mathbf{q}$ may intersect s_1^i and s_2^i at most at two points, the inequality $F^e - F^i \geq 0$ is certainly true. On account of Remark 3.4.2 estimate (3.4.45) implies the above corollary. ■

Remark 3.4.5 Under the conditions of Theorem 3.4.1 we might have

$$F^e - F^i \leq \int\limits_{\partial D^*} |\, w(y)\,|\; ds_y\,, \tag{3.4.49}$$

where the function $\omega(y)$ and the function $\dfrac{\partial}{\partial n_x}\, w(x)$ for $x \in \partial D^*$ are related by the integral equation

$$-\frac{1}{2}\,\omega(x) + \int\limits_{\partial D^*} \frac{\partial}{\partial n_x}\, E(x,y)\,\omega(y)\; ds_y = \frac{\partial}{\partial n_x}\, w(x)\,. \tag{3.4.50}$$

A preliminary step in establishing the preceding relationship is to pass to the limit in (3.4.25) with respect to k and m in order to demonstrate that the functional $J(H_f)$ satisfies the estimate

$$|\, J(H_f)\,| \leq \int\limits_{\partial D^*} |\, w(y)\,|\; ds_y\,,$$

whose use permits us to show that (3.4.49) becomes an implication of (3.4.34).

It is worth noting here two things. With the aid of relations (3.4.49)–(3.4.50) one can derive various estimates of the integral type. Some of them are more sharper than (3.4.7) and (3.4.47).

Being concerned with the difference of the potentials

$$u(x) = u(x, \Omega_1) - u(x, \Omega_2) \tag{3.4.51}$$

and the sphere S_R of radius R, we will assume that either of the domains $\bar{\Omega}_\alpha$, $\alpha = 1, 2$, lies entirely within the sphere S_R and the subsidiary Cauchy data can be estimated as follows:

$$|\, u(x) \,|_{G(R)} \le \varepsilon_1 \,, \qquad \left| \frac{\partial u}{\partial n_x} \right|_{G(R)} \le \varepsilon_1 \,, \tag{3.4.52}$$

where $G(R)$ is a part of S_R. Relying on the well-known estimates for a solution to the integral equation (3.4.50) in terms of the right-hand function and exploiting some facts from Lavrentiev (1956), (1957), we can write out explicitly a function $\psi(\varepsilon_1)$ giving the upper bound for the right-hand side of (3.4.49) such that $\psi(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$.

Similar remarks are still valid with regard to inequalities (3.4.7) and (3.4.45), whose right-hand sides are estimated in terms of the Cauchy data (3.4.52) like $c_6\, \varphi(\varepsilon_1)$. For example, if ε emerged from (3.4.44), each such number admits the expansion $\varepsilon = \text{const}\ \varphi(\varepsilon_1)$, where ε_1 arose from estimates (3.4.52) and $\varphi(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. Also, it is possible to represent $\varphi(\varepsilon_1)$ in the explicit form.

Other ideas are connected with the derivation of stability estimates. Let open bounded sets A_1 and A_2 be so chosen as to keep the notations given by (3.2.38). For the potentials $u(x; A_\alpha)$ of the sets A_α with unit density it would be possible to get stability estimates and uniqueness theorems similar to those stated above.

As an example we cite below an assertion similar to Corollary 3.4.1.

Theorem 3.4.2 *Let there exist a point O, the numbers γ_1 and β_1, $\gamma_1^2 + \beta_1^2 \neq 0$, and a constant vector $\mathbf{q}$ such that*

$$\int\limits_{\Gamma_1^i} |\, \Phi(y) \,|\; ds_y + \int\limits_{\Gamma_2^i} |\, \Phi(y) \,|\; ds_y < \int\limits_{\bar{\Gamma}_1^e} |\, \Phi(y) \,|\; ds_y + \int\limits_{\bar{\Gamma}_2^e} |\, \Phi(y) \,|\; ds_y \,. \tag{3.4.53}$$

If the potentials $u(x; A_\alpha)$ satisfy the equality

$$u(x; A_1) \;=\; u(x; A_2) \quad \textit{for} \quad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2) \,, \tag{3.4.54}$$

then $A_1 = A_2$.

Proof As usual, this amounts to taking into consideration the union $\bar{A}_1 \cup \bar{A}_2$. When that set is not connected, we initiate another construction

$$A^* \subset \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2),$$

for which $\overline{A^*} \cap (\bar{A}_1 \cup \bar{A}_2)$ consists of a finite number of points. The set A^* can be regarded as a union of domains with piecewise boundaries except for a finite number of points. Moreover, by construction, the set $\bar{A}_1 \cup \bar{A}_2 \cup \overline{A^*}$ will be connected.

Set

$$\bar{B}_1 = \bar{A}_1 \bigcup \overline{A^*}, \qquad \bar{B}_2 = \bar{A}_2 \bigcup \overline{A^*}. \tag{3.4.55}$$

Let us denote by $u(x; A^*)$ the potential of the set A^* with density $\mu \equiv 1$. Condition (3.4.53) implies that

$$u(x; A_1) + u(x; A^*) = u(x; A_2) + u(x; A^*) \quad \text{for} \quad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2 \cup \overline{A^*}),$$

so that we obtain

$$u(x; B_1) = u(x; B_2) \quad \text{for} \quad x \in \mathbf{R}^n \setminus (\bar{B}_1 \cup \bar{B}_2). \tag{3.4.56}$$

Since $\bar{B}_1 \cup \bar{B}_2$ is a connected closed set, Lemma 3.2.2 yields the relation

$$\int_{B_1} h(y)\, dy = \int_{B_2} h(y)\, dy, \tag{3.4.57}$$

which is valid for an arbitrary function h harmonic in $D \supset \bar{B}_1 \cup \bar{B}_2$. In view of (3.4.55), the last equality becomes

$$\int_{A_1} h(y)\, dy = \int_{A_2} h(y)\, dy. \tag{3.4.58}$$

As in the proof of Theorem 3.4.1 we need the function

$$f(y) = \begin{cases} -\operatorname{sign} \Phi(y) & \text{for} \quad y \in \bar{\Gamma}_1^e, \\ \operatorname{sign} \Phi(y) & \text{for} \quad y \in \bar{\Gamma}_2^e, \\ 0 & \text{for} \quad y \in \partial A^*, \end{cases} \tag{3.4.59}$$

and insert in place of h

$$h(y) = \sum_{k=1}^{n} \frac{\partial}{\partial y_k} (\gamma_1 y_k + \beta_1 q_k)\, H.$$

Adopting the arguments of Theorem 3.4.1 we arrive at the relation

$$\int_{A_1} H_f\, \Phi(y)\; ds_y - \int_{A_2} H_f\, \Phi(y)\; ds_y = 0\,,$$

which can be rewritten for later use as

$$(3.4.60)\qquad \int_{\bar\Gamma_1^e} H_f\, \Phi(y)\; ds_y - \int_{\bar\Gamma_2^e} H_f\, \Phi(y)\; ds_y$$

$$= \int_{\Gamma_2^i} H_f\, \Phi(y)\; ds_y - \int_{\Gamma_1^i} H_f\, \Phi(y)\; ds_y\,.$$

Since $|\, H_f \,| \leq 1$ on $\bar A_1 \cup \bar A_2$, the boundary data (3.4.59) and relation (3.4.60) assure us of the validity of the estimate

$$\int_{\bar\Gamma_1^e} |\,\Phi(y)\,|\; ds_y + \int_{\bar\Gamma_2^e} |\,\Phi(y)\,|\; ds_y \leq \int_{\Gamma_1^i} |\,\Phi(y)\,|\; ds_y + \int_{\Gamma_2^i} |\,\Phi(y)\,|\; ds_y\,,$$

which contradicts (3.4.53) and thereby proves the theorem. ■

3.5 Uniqueness theorems for the harmonic potential of "non-star-shaped" bodies with variable density

Unlike Section 3.3 we will prove in the sequel the uniqueness theorems for the volume potential of the Laplace equation

$$(3.5.1)\qquad \Delta h = 0$$

related to "non-star-shaped" bodies. Also, in contrast to Section 3.4 the available densities do not have constant signs.

Let A_α, $\alpha = 1, 2$, be open bounded sets satisfying conditions (3.2.38).

Theorem 3.5.1 *If there exists at least one constant vector* $\mathbf{q}$, *for which the following conditions hold:*

(1) *either of the sets* $(s_1^e)^j (j = 1, \dots, j_1)$ *and* $(s_2^e)^j\ (\,j = 1, \dots, j_2\,)$ *is not empty and any straight line, parallel to the vector* $\mathbf{q}$, *may intersect the set* $\Gamma_1^i \cup \Gamma_2^i$ *either at most at two points or by two whole segments;*

(2) *the function* $\mu(y)$ *is subject to the relation* $\partial\mu(y)/\partial y_n = 0$*, where the axis* Oy_n *is directed along the vector* $\mathbf{q}$ *and the function* $\mu(y)$ *does not have, in general, constant sign for all* $y \in \bar{A}_1 \cup \bar{A}_2$*;*

(3) *the harmonic potentials* $u(x; A_\alpha, \mu)$ *with density* μ *possess the property*

$$u(x; A_1, \mu) = u(x; A_2, \mu) \quad \textit{for} \quad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2)\,; \tag{3.5.2}$$

then $A_1 = A_2$.

Proof To avoid cumbersome calculations, we confine ourselves to the case $A_\alpha = \Omega_\alpha$, which admits comparatively simple proof. Condition (3) just formulated and Lemma 3.2.2 with $\beta = 1$ and $\gamma = 0$ imply that any function $h(y)$ being harmonic in a domain D, $D \supset (\bar{\Omega}_1 \cup \bar{\Omega}_2)$, complies with the relation

$$\int_{\Omega_1} \mu(y)\, h(y)\; dy - \int_{\Omega_2} \mu(y)\, h(y)\; dy = 0\,. \tag{3.5.3}$$

The meaning of this is that $h(y)$ represents a regular solution to equation (3.5.1) over D. To do the same amount of work, the function h will be taken to be

$$h(y) = \sum_{k=1}^{n} q_k\, \frac{\partial H}{\partial y_k}\,, \qquad \mathbf{q} = (q_1, \dots, q_n)\,,$$

where $H(y)$ is an arbitrary harmonic in D function. Upon substituting the last expression into (3.5.3) we see that condition 2) of the present theorem may be of help in transforming the volume integrals to the surface ones, whose use permits us to establish the relation

$$\int_{\partial\Omega_1} \mu(y)\, H(y)\, (\mathbf{q}, \mathbf{n}_y)\; ds_y - \int_{\partial\Omega_2} \mu(y)\, H(y)\, (\mathbf{q}, \mathbf{n}_y)\; ds_y = 0\,, \tag{3.5.4}$$

where $(\mathbf{q}, \mathbf{n}_y)$ is the scalar product of the vectors $\mathbf{q}$ and $\mathbf{n}_y$. Here, as usual, $\mathbf{n}_y$ is a unit external normal to $\partial\Omega_\alpha$, $\alpha = 1, 2$, at point y.

Good judgment in the selection of notations

$$s_1^+ = \{\, y \in \partial\Omega_1,\, [\mu(y)\;, (\mathbf{q}, \mathbf{n}_y)] > 0 \}\,; \qquad s_1^- = \partial\Omega_1 \setminus s_1^+\,;$$

$$s_2^- = \{\, y \in \partial\Omega_2,\, [\mu(y)\, (\mathbf{q}, \mathbf{n}_y)] < 0 \}\,; \qquad s_2^+ = \partial\Omega_2 \setminus s_2^-\,;$$

with saving (3.3.1) may improve the clarity of the exposition. In particular, we are led by relation (3.5.4) to

$$\int\limits_{s_1^{e+} \cup s_1^{e-} \cup s_1^{i+} \cup s_1^{i-}} H(y)\,\mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y - \int\limits_{s_2^{e+} \cup s_2^{e-} \cup s_2^{i+} \cup s_2^{i-}} H(y)\,\mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y = 0\,. \tag{3.5.5}$$

What is more, it is supposed that a function $f(y)$ is defined on the surface s^e by means of the relations

$$f(y) = \begin{cases} 1 & \text{for} \quad y \in s_1^{e+},\, s_2^{e-}\,, \\ 0 & \text{for} \quad y \in s_1^{e-},\, s_2^{e+}\,, \end{cases} \tag{3.5.6}$$

provided that $f(y) \neq \text{const}$ for $y \in s^e$.

As in the proof of Theorem 3.3.1 relation (3.5.5) can be extended to cover a generalized solution H_f of the Dirichlet problem for the Laplace equation. In that case the boundary values of the function H_f will coincide almost everywhere on the boundary s^e with f involved in (3.5.6). Under such a formalization, (3.5.5) and (3.5.6) together imply that

$$J(H_f) = 0\,, \tag{3.5.7}$$

where

$$\begin{aligned} J(H_f) = & \int\limits_{s_1^{e+}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \\ & + \int\limits_{s_1^{i+}} H_f\,\mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \\ & + \int\limits_{s_1^{i-}} H_f\,\mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \\ & - \int\limits_{s_2^{e-}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \end{aligned} \tag{3.5.8}$$

$$- \int_{s_2^{i+}} H_f\, \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y$$

$$- \int_{s_2^{i-}} H_f\, \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y\,.$$

Under the agreements $f(y) \neq \text{const}$ and $0 < h_f < 1$ for $y \in \Omega^e$, we are now in a position to evaluate the integral

$$J_1 = \int_{s_1^{i+}} H_f\, \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y - \int_{s_2^{i-}} H_f\, \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y\,. \tag{3.5.9}$$

Indeed, by construction, $\left[\mu(y)\,(\mathbf{q}, \mathbf{n}_y)\right] > 0$ on s_1^{i+} and $\left[\mu(y)\,(\mathbf{q}, \mathbf{n}_y)\right] \leq 0$ on s_2^{i-} and, therefore, $J_1 \geq 0$.

The next step is the estimation of another integral

$$J_2 = \int_{s_1^{e+}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y - \int_{s_2^{i+}} H_f\, \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y\,. \tag{3.5.10}$$

In what follows y' will denote the point with components $(y_1, y_2, \ldots, y_{n-1}, 0)$ and dy' will stand for a volume element of the dimension $n-1$. If $(s_1^{e+})'$ is a piece of the surface s_1^{e+}, for which there exists a unique projection onto a hyperplane N perpendicular to the vector $\mathbf{q}$, then it does so to the axis Oy_n and (3.5.10) can be rewritten as

$$\begin{aligned} J_2 = & \int_{s_1^{e+} \setminus (s_2^{e+})'} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \\ & + \int_{(s_1^{e+})'} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \\ & - \int_{s_2^{i+}} H_f\, \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y\,. \end{aligned} \tag{3.5.11}$$

In the case where $\operatorname{mes} \pi(s_2^{i+}) \neq 0$ we thus have

$$\int_{s_1^{e+} \setminus (s_1^{e+})'} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \geq 0$$

and

$$\int\limits_{(s_1^{e+})'} \mu(y)\,(\mathbf{q},\mathbf{n}_y)\,ds_y - \int\limits_{s_2^{i+}} H_f\,\mu(y)\,(\mathbf{q},\mathbf{n}_y)\,ds_y$$

$$= \int\limits_{\pi(s_2^{i+})} \mu(y')[1-H_f]\,dy' > 0\,.$$

The same procedure works for the integral $J_2 > 0$ under the constraint $\operatorname{mes}\pi(s_2^{i+}) \neq 0$. By exactly the same reasoning as before we can deduce that $J_3 > 0$ for the integral

$$J_3 = \int\limits_{s_1^{i-}} H_f\,\mu(y)\,(\mathbf{q},\mathbf{n}_y)\,ds_y - \int\limits_{s_2^{e-}} \mu(y)\,(\mathbf{q},\mathbf{n}_y)\,ds_y \tag{3.5.12}$$

under the natural premise $\operatorname{mes}\pi(s_2^{i-}) \neq 0$.

With the obvious decomposition $J = J_1 + J_2 + J_3$ in view, we obtain the inequality $J(H_f) > 0$ for the case where

$$\operatorname{mes}\pi(s_1^i \cup s_2^i) \neq 0\,.$$

If it is not so, that is, $\operatorname{mes}\pi(s_1^i \cup s_2^i) = 0$, we can state under the conditions of the theorem that $\operatorname{mes} s_\alpha^e \neq 0$ for every set s_α^e, $\alpha = 1, 2$. Then at least one of the integrals J_α, $\alpha = 1, 2$, should be strictly positive. But this fact is not consistent with (3.5.7). We have a contradiction and finish the proof in the case where $f(y) \not\equiv \text{const}$ for $y \in s^e$. On the contrary, let $f \equiv 1$ for $y \in s^e$, it being understood that $\mu(y) > 0$, $s^e = s_1^{e+} \cup s_2^{e-}$, $s_1^i = s_1^{i-}$ and $s_2^i = s_2^{i+}$. It follows from the foregoing that the condition $\partial\mu/\partial y_n = 0$ assures us of the validity of the relation

$$\frac{\partial u}{\partial x_n} = -\int\limits_{\partial\Omega_1} \mu(y)\,(\mathbf{q},\mathbf{n}_y)\,E(x,y)\,ds_y + \int\limits_{\partial\Omega_2} \mu(y)\,(\mathbf{q},\mathbf{n}_y)\,E(x,y)\,ds_y \quad \text{for} \quad x \in \Omega^e \tag{3.5.13}$$

with

$$u(x) = u(x;\Omega_1,\mu) - u(x;\Omega_2,\mu)\,.$$

Furthermore, the combination of (3.5.13) and (3.5.4) with $H(y)|_{y\in s^e} = E(x,y)|_{y\in s^e}$ gives

$$\frac{\partial u}{\partial x_n} = \int\limits_{s_1^{i-}} \mu(y)\,(\mathbf{q},\mathbf{n}_y)\,G_0(x,y)\;ds_y - \int\limits_{s_2^{i+}} \mu(y)\,(\mathbf{q},\mathbf{n}_y)\,G_0(x,y)\;ds_y \quad \text{for} \quad x\in\Omega^e, \tag{3.5.14}$$

where

$$H(y) - E(x,y) = G_0(x,y)$$

is Green's function. This serves as a basis that for $x \in \Omega^e$ we would have $\partial u/\partial x_n > 0$, violating the relation $\partial u/\partial x_n = 0$, valid for all $x \in \mathbf{R}^n \setminus \bar{\Omega}^e$, $\bar{\Omega}^e = \bar{\Omega}_1 \cup \bar{\Omega}_2$. The obtained contradiction proves the assertion of the theorem. ■

3.6 The exterior contact inverse problem for the magnetic potential with variable density of constant sign

This section is devoted to the problem of finding the shape of a contact body from available values of its magnetic potential represented as a sum of potentials generated by a volume mass and a simple layer.

Recall that, in conformity with (3.2.3), the magnetic potential is defined to be

$$w^\alpha(x) = w(x; A_\alpha, \partial A_\alpha, \mu_\alpha, \rho_\alpha)$$
$$= \beta \int\limits_{A_\alpha} E(x,y)\,\mu_\alpha(y)\;dy + \gamma \int\limits_{\partial A_\alpha} E(x,y)\,\rho_\alpha(y)\;ds_y\,,$$

where β and γ are real numbers such that $\beta^2 + \gamma^2 \neq 0$ and $E(x,y)$ is the fundamental solution of the Laplace equation. The symbol A_α will stand for an open set satisfying the standard requirements imposed in Section 3.2. Throughout the entire section, we retain the notations of Section 3.2.

The main goal of our study is to show the uniqueness of a solution of the exterior inverse problem for the magnetic potentials of **contact bodies** with variable densities of constant signs. Although the new important theorems will be postponed until the sequel to this section, let us stress that these imply, as a corollary, more general results for the volume potential with $\beta = 1$ and $\gamma = 0$ in (3.2.3) as opposed to the propositions of the preceding sections saying about **noncontact bodies**.

Following established practice we introduce in agreement with (3.2.24)

$$B = (A_1 \cup A_2) \setminus \bar{A}_0 \,, \qquad A_0 = A_1 \cap A_2 \,.$$

The next theorem asserts the uniqueness of the solution when the magnetic potentials coincide outside the domains A_α as well as inside their intersection; meaning the validity of two equalities between exterior and interior magnetic potentials, respectively.

Theorem 3.6.1 *Let for arbitrary domains A_α, $\alpha = 1, 2$, imposed above the functions $\mu_\alpha \in C^1(\bar{A}_\alpha)$ and $\rho_\alpha \in C(\partial A_\alpha)$ must be nonnegative. If the functions $w^\alpha(x)$, $\alpha = 1, 2$, with positive coefficients β and γ, $\beta^2 + \gamma^2 \neq 0$, satisfy the equality*

$$w^1(x) = w^2(x) \quad \text{for} \quad x \in \mathbf{R}^n \setminus \bar{B} \,, \tag{3.6.1}$$

then

$$A_1 = A_2 \tag{3.6.2}$$

and

$$\begin{aligned} \mu_1(x) &= \mu_2(x) \,, \quad x \in A_1 \,, \quad \text{if} \quad \beta \neq 0 \,; \\ \rho_1(x) &= \rho_2(x) \,, \quad x \in \partial A_1 \,, \quad \text{if} \quad \gamma \neq 0 \,. \end{aligned} \tag{3.6.3}$$

Proof Let $A_1 \neq A_2$. Observe that for such sets A_α all the conditions of Section 3.2 are satisfied. This is certainly true for (3.2.40), making it possible to rely on Lemma 3.2.4 from which it follows that any regular in a domain D solution $h(y)$ to the Laplace equation satisfies equality (3.2.45) taking now the form

$$J(H) = 0 \,, \tag{3.6.4}$$

where

$$J(H) = \beta \int\limits_{B_0} h(y)\, \mu_1(y)\, dy + \gamma \int\limits_{(\partial B_0)^e} \rho_1(y)\, h(y)\, ds_y - \gamma \int\limits_{(\partial B_0)^i} \rho_2(y)\, h(y)\, ds_y \,. \tag{3.6.5}$$

Before giving further motivations, it will be sensible to define on ∂B_0 (see (3.2.39)–(3.2.42)) the function $f(y)$ by means of the relations

$$f(y) = \begin{cases} 1 & \text{if} \quad y \in (\partial B_0)^e \,, \\ 0 & \text{if} \quad y \in (\partial B_0)^i \,. \end{cases} \tag{3.6.6}$$

Because of (3.6.1), $f(y) \not\equiv$ const for $y \in \partial B_0$. As in the proof of Theorem 3.3.1 equality (3.6.4) can be extended to involve a generalized solution $h_f(y)$ of the Dirichlet problem with the boundary data (3.6.6). The function $h_f(y)$ is just a regular solution to the equation

$$(\Delta h_f)(y) = 0 \quad \text{for} \quad y \in B_0$$

and takes the values of the function $f(y)$ specified by (3.6.6) at almost all points of ∂B_0, so that

$$J(h_f) = 0 \,, \tag{3.6.7}$$

where, by definition (3.6.6),

$$J(h_f) = \beta \int\limits_{B_0} h_f(y)\, \mu_1(y)\; dy + \gamma \int\limits_{(\partial B_0)^e} \rho_1(y)\, h(y)\; ds_y \,. \tag{3.6.8}$$

The **Hopf principle** asserts that a nonconstant solution $h_f(y)$ cannot attain a negative relative minimum in the interior of the domain and

$$0 < h_f < 1 \quad \text{for almost all} \quad y \in B_0 \,. \tag{3.6.9}$$

From (3.6.8) and (3.6.9) it follows that $J(h_f > 0)$, which disagrees with (3.6.7) and provides support for the view that

$$A_1 = A_2 = A_0 \,. \tag{3.6.10}$$

Using notations (3.2.24), (3.2.3) together with condition (3.6.1) we find that

$$w^1(x) = w^2(x) \quad \text{for} \quad x \in A_0 \,.$$

Applying the Laplace operator to the preceding equality yields

$$(\Delta w^1)(x) = (\Delta w^2)(x) \quad \text{for} \quad x \in A_0 \,. \tag{3.6.11}$$

The properties of the potentials of volume masses and simple layers provide sufficient background for the relationships

$$(\Delta w^\alpha)(x) = -\beta\, \mu_\alpha(x) \quad \text{for} \quad x \in A_\alpha \,, \quad \alpha = 1, 2 \,. \tag{3.6.12}$$

When $\beta \neq 0$, (3.6.10) and (3.6.12) are followed by

$$\mu_1(x) = \mu_2(x) \quad \text{for} \quad x \in A_0 = A_1 = A_2 \,. \tag{3.6.13}$$

For $\gamma \neq 0$ the combination of (3.6.10) and (3.6.13) gives

$$v^1(x) = v^2(x) \quad \text{for} \quad x \in \mathbf{R}^n \setminus A_1 \,. \tag{3.6.14}$$

In conformity with (3.6.10),

$$v^\alpha(x) = \int\limits_{\partial A_1} \rho_\alpha(y)\, E(x,y)\; ds_y \,. \tag{3.6.15}$$

The formula for the jump of the derivative of the simple layer potential $v^\alpha(x)$ allows us to deduce that

$$\left(\frac{\partial v^\alpha}{\partial \nu_{x_0}} \right)^+ - \left(\frac{\partial v^\alpha}{\partial \nu_{x_0}} \right)^- = \rho_\alpha(x_0) \,, \qquad x \in \partial A_1 \,, \tag{3.6.16}$$

where $\left(\dfrac{\partial v^\alpha}{\partial \nu_{x_0}} \right)^\pm$ stand for the limiting values of $\dfrac{\partial v^\alpha(x)}{\partial \nu_{x_0}}$ as $x \to x_0$, $x_0 \in \partial A_1$, along the interior and exterior normal $\boldsymbol{\nu}_{x_0}$ to the set A_1, respectively ($x \in A_1$ and $x \in \mathbf{R}^n \setminus \bar{A}_1$). Within these notations, relation (3.6.14) implies that

$$\left(\frac{\partial v^1}{\partial \nu_{x_0}} \right)^\pm = \left(\frac{\partial v^2}{\partial \nu_{x_0}} \right)^\pm , \tag{3.6.17}$$

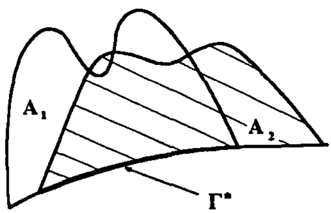
so that (3.6.17) with $\gamma \neq 0$ assures us of the validity of the equality $\rho_1(x_0) = \rho_2(x_0)$ at any point $x_0 \in \partial A_1$. This proves the assertion of the theorem. ■

Definition 3.6.1 *Two sets A_1 and A_2 are said to be* **externally contact** *in the sense of Prilepko (1968b) if the boundary of any connected component of the set $A_0 = A_1 \cap A_2$ (mes $A_0 \neq 0$) contains an $(n-1)$-dimensional part $\bar{\Gamma}_*$ (mes $\Gamma_* \neq 0$) being common with boundary of a certain component of the set $\mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2)$.*

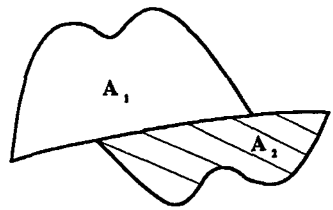
Here the sets A_α, $\alpha = 1, 2$, are again supposed to satisfy (3.2.40). We state a number of results with respect to

$$\bar{w}^\alpha(x) = \beta\, u(x; A_\alpha, \mu) + \gamma\, v(x; \partial A_\alpha, \rho) \,, \tag{3.6.18}$$

where γ and β are real numbers such that $\beta^2 + \gamma^2 \neq 0$, that furnish the justification for what we wish to do.



A_1 and A_2 are externally contact



A_1 and A_2 are not externally contact

Theorem 3.6.2 *Let A_α, $\alpha = 1, 2$, be externally contact sets. If nonnegative functions $\mu \in C^1(\bar{A}_\alpha)$ and $\rho \in C(\partial A_\alpha)$ and nonnegative numbers β and γ are such that the equality*

$$\bar{w}^1(x) = \bar{w}^2(x) , \qquad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2) , \tag{3.6.19}$$

holds, then $A_1 = A_2$.

As a preliminary the following assertion is introduced.

Lemma 3.6.1 *Let externally contact sets A_α, $\alpha = 1, 2$, be in line with* (3.2.40)*. If the functions $\mu \in C^1(\bar{A}_\alpha)$ and $\rho \in C(\partial A_\alpha)$ involved in* (3.6.18) *admit a representation of the type* (3.6.19)*, which appears below, then the equality*

$$\bar{w}^1(x) = \bar{w}^2(x)$$

is valid for all $x \in A_0$.

Proof As before, it will be convenient to have at our disposal the function

$$\bar{w}(x) = \bar{w}^1(x) - \bar{w}^2(x) , \tag{3.6.20}$$

for which the properties of the potentials under consideration imply that

$$(\Delta \bar{w})(x) = 0 \quad \text{for} \quad x \in \mathbf{R}^n \setminus \bar{B} , \tag{3.6.21}$$

where the set B has been defined by (3.2.24). By the same token,

$$\bar{w}(x) \in C^{0+h}(\mathbf{R}^n) \tag{3.6.22}$$

with any h, $0 < h < 1$.

In what follows the symbol Γ_* will stand for the collection of all interior points of the set $\bar{\Gamma}_*$, $\bar{\Gamma}_* \subset \partial A_1 \cap \partial A_2$. Since the sets A_α, $\alpha = 1, 2$, are externally contact, the open set Γ_* is nonempty and can be decomposed into connected, open and piecewise smooth surfaces. As such, the boundary ∂A_0 of the set A_0 contains the set Γ_*.

By successively applying (3.2.18), the formulae of the jump for the derivative of the simple layer potential $v(x; A_\alpha, \rho)$ and the properties of the volume mass potential $u(x; A_\alpha, \mu)$ we establish the relationship

$$\left(\frac{d\bar{w}^\alpha}{d\nu_{x_0}} \right)^+ - \left(\frac{d\bar{w}^\alpha}{d\nu_{x_0}} \right)^- = \gamma\, \rho(x_0), \qquad x \in \Gamma_*, \tag{3.6.23}$$

where the symbols

$$\left(\frac{d\bar{w}^\alpha}{d\nu_{x_0}} \right)^\pm$$

denote the limiting values of

$$\frac{d\bar{w}^\alpha(x)}{d\nu_{x_0}}$$

as $x \to x_0$, $x_0 \in \Gamma_*$, along the normal $\boldsymbol{\nu}_{x_0}$ in the cases when $x \in A_0$ and $x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2)$, respectively.

We next consider an open ball $Q = Q_{x_0}$ with center $x_0 \in \Gamma_*$ such that $\bar{Q} \cap (\bar{B} \setminus \Gamma_*) = 0$. As usual, ∂Q designates the boundary of Q. It is convenient to introduce further entries defined by

$$(\partial Q)^+ = \partial Q \cap A_0\,, \qquad \Gamma'_* = Q \cap \Gamma_* \tag{3.6.24}$$

and

$$\{ M_x[\bar{w}(x); E(y,x)] \}^+ = E(y,x) \left[\frac{d\bar{w}(x)}{d\nu_x} \right]^+ - \frac{\partial E(y,x)}{\partial \nu_x}\, [\bar{w}(x)]^+ . \tag{3.6.25}$$

From (3.1.5), (3.6.21) and the last notation it follows that

$$\int\limits_{(\partial Q)^+} M_x[\bar{w}(x); E(y,x)]\, ds_x + \int\limits_{\Gamma'_*} \{ M_x[\bar{w}(x); E(y,x)] \}^+\, ds_x = \begin{cases} \bar{w} & \text{for } \; y \in Q \cap A_0\,, \\ 0 & \text{for } \; y \in Q \cap \left(\mathbf{R}^n \setminus \bar{A}_1 \cup \bar{A}_2 \right). \end{cases} \tag{3.6.26}$$

It is straightforward to verify that the combination of (3.6.20)–(3.6.26) gives

$$(3.6.27) \qquad \bar{w}(y) = 0\,, \qquad y \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2)\,,$$

whose use permits us to find by formula (3.6.23) that

$$(3.6.28) \qquad \left(\frac{d\bar{w}}{d\nu_{x_0}} \right)^+ = 0 \quad \text{if} \quad x_0 \in \Gamma_* \,.$$

With the aid of (3.6.22) and (3.6.28) we deduce from relation (3.6.26) that

$$(3.6.29) \qquad \big\{ M_x[\bar{w}(x); E(y,x)] \big\}^+ = 0 \quad \text{for} \quad x_0 \in \Gamma_* \,.$$

Substitution of (3.6.29) into (3.6.26) yields

$$(3.6.30) \qquad \int\limits_{(\partial Q)^+} M_x[\bar{w}(x); E(y,x)]\, ds_x = \begin{cases} \bar{w}(y) & \text{for} \quad y \in Q \cap A_0\,, \\ 0 & \text{for} \quad y \in Q \cap \big(\mathbf{R}^n \setminus \bar{A}_1 \cup \bar{A}_2 \big)\,. \end{cases}$$

We note in passing that the function

$$(3.6.31) \qquad F(y) = \int\limits_{(\partial Q)^+} M_x[\bar{w}(x); E(y,x)]\, ds_x$$

is just a regular solution to the equation

$$(3.6.32) \qquad (\Delta F)(y) = 0 \quad \text{for} \quad y \in \mathbf{R}^n \setminus (\partial Q)^+.$$

Relations (3.6.30) and (3.6.31) are followed by

$$(3.6.33) \qquad F(y) = 0 \quad \text{for} \quad y \in Q \cap \big(\mathbf{R}^n \setminus \bar{A}_1 \cup \bar{A}_2 \big)\,,$$

which means that the function $F(y)$ and its normal derivative are equal to zero on Γ'_*. Due to the uniqueness of the Cauchy problem solution relations (3.6.32)–(3.6.33) imply that

$$(3.6.34) \qquad F(y) = 0 \quad \text{for} \quad y \in Q \cap A_0 \,.$$

On the other hand, (3.6.30) yields

$$(3.6.35) \qquad F(y) = \bar{w}(y) \quad \text{for} \quad y \in Q \cap A_0 \,.$$

Putting these together with (3.6.34) we conclude that

$$\bar{w}(y) = 0 \quad \text{for} \quad y \in Q \cap A_0 . \tag{3.6.36}$$

It is worth bearing in mind here the uniqueness of the solution of the Cauchy problem for (3.6.21). Because of this fact, it follows from the foregoing that

$$\bar{w}(y) = 0 \quad \text{for} \quad y \in A_0 , \tag{3.6.37}$$

so that the combination of (3.6.20) and (3.6.37) gives the equality

$$\bar{w}^1(x) = \bar{w}^2(x) \quad \text{for} \quad y \in A_0 , \tag{3.6.38}$$

thereby justifying the assertion of the lemma. ■

A simple observation that (3.6.19) and (3.6.38) together imply the equality

$$\bar{w}^1(x) = \bar{w}^2(x) \quad \text{for} \quad y \in \mathbf{R}^n \setminus \bar{B} \tag{3.6.39}$$

may be useful in the further development. Whence it is clear that all the conditions of Theorem 3.6.1 are satisfied in such a setting. In view of this, equality (3.6.2) is true and this proves the assertion of the theorem. ■

Corollary 3.6.1 *If for externally contact sets A_α, $\alpha = 1, 2$, the volume mass potentials $u(x; A_\alpha, \mu)$ of a given nonnegative density $\mu \in C^1(\bar{A}_\alpha)$ satisfy the condition*

$$u(x; A_1, \mu) = u(x; A_2, \mu) \quad \textit{for} \quad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2) ,$$

then $A_1 = A_2$.

Corollary 3.6.2 *If for externally contact sets A_α with boundaries ∂A_α, $\alpha = 1, 2$, the simple layer potentials $v(x; \partial A_\alpha, \mu)$ of a nonnegative density $\rho \in C(\partial A_\alpha)$ satisfy the condition*

$$v(x; \partial A_1, \rho) = v(x; \partial A_2, \rho) \quad \textit{for} \quad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2) ,$$

then $\partial A_1 = \partial A_2$.

3.7 Integral equation for finding the density of a given body via its exterior potential

In this section we state the theorems on the solution uniqueness for the problem of finding the density of a given body from the available information about its exterior potential. In passing we try to write out explicitly its solutions for certain configurations of domains and bodies and offer one possible way of deriving the explicit formulae.

In a common setting an open set A is supposed to be a union of a finite number of bounded domains Ω_j with piecewise smooth boundaries $\partial\Omega_j$, on which the function

$$u(x; A, \mu_\alpha) = \int_A E(x,y)\, \mu_\alpha(y)\, dy\,, \qquad \alpha = 1, 2\,, \tag{3.7.1}$$

represents the volume mass potential of a given set A with density $\mu_\alpha(y) \neq 0$ for almost all $y \in A$. As before, the function $E(x,y)$ is used for the fundamental solution of the Laplace equation.

Before giving further motivations, it will be sensible to describe the class of functions $\mu_\alpha(y) \in C^1(\bar{A})$, $\alpha = 1, 2$, satisfying the condition

$$\mu_\alpha(y) = \eta(y)\, \nu_\alpha(y)\,, \tag{3.7.2}$$

where

(a) $\nu_\alpha(y)$ is continuously differentiable on $\bar{A}$ and $\partial\nu_\alpha/\partial y_{k_1} = 0$, where a positive integer k_1 is kept fixed;

(b) $\eta(y) > 0$ and either for $y \in \bar{A}$

$$\partial\eta(y)/\partial y_{k_1} \geq 0$$

or for $y \in \bar{A}$

$$\partial\eta(y)/\partial y_{k_1} \leq 0\,.$$

Theorem 3.7.1 *The equality $\mu_1(y) = \mu_2(y)$ holds for all $y \in A$ when the functions $\mu_\alpha(y)$ happen to be of class* (3.7.2) *and the condition*

$$u(x; A, \mu_1) = u(x; A, \mu_2)\,, \qquad x \in \mathbf{R}^n \setminus \bar{A}\,, \tag{3.7.3}$$

is satisfied for the exterior potentials of a given set A.

Proof Although the complete theory we have presented could be recast in this case, we confine ourselves to the set A consisting of a single bounded domain Ω, whose boundary $\partial\Omega$ is piecewise smooth.

From condition (3.7.3) it seems clear that any function $h(y)$ being harmonic in a domain $D \supset \bar{\Omega}$ is subject to the relation

$$\int_{\Omega} (\mu_1 - \mu_2)\, h(y)\, dy = 0 . \tag{3.7.4}$$

Here we apply Lemma 3.2.2 to $A = A_1 = A_2$ and $\beta = 1$, $\gamma = 0$.

Furthermore, set $\mu(y) = \mu_1(y) - \mu_2(y)$ and assume to the contrary that $\mu(y) \neq 0$ for $y \in \bar{\Omega}$. From item (a) of condition (3.7.2) and (3.7.4) it is easily seen that the function

$$\nu(y) = \nu_1(y) - \nu_2(y)$$

does not have constant sign on the boundary $\partial\Omega$. Indeed, if the function $\nu(y)$ is, for example, positive on the boundary $\partial\Omega$, then so is the function $\nu(y)$ in the domain Ω, since

$$\partial\nu / \partial y_{k_1} = 0 .$$

Therefore, the function $\mu(y)$ would be strictly positive in Ω. But this disagrees with (3.7.4) due to the fact that the function $h(y)$ can be replaced there by a positive solution to the Laplace equation, by means of which the left-hand side of (3.7.4) becomes strictly positive. Hence,

$$\operatorname{sign} \nu(y) \not\equiv \text{const}$$

and, thereby, on the boundary $\partial\Omega$

$$\operatorname{sign} \mu(y) \not\equiv \text{const} .$$

Under the natural premise $\nu(y) \not\equiv 0$ we insert in (3.7.4) the function $h(y) = \partial H(y)/\partial y_{k_1}$, where $H(y)$ is a harmonic function in $D \supset \bar{\Omega}$, by means of which the following relation is attained:

$$\int_{\partial\Omega} H(y)\, \mu(y)\, (\mathbf{q}, \mathbf{n}_y)\, ds_y - \int_{\Omega} H(y)\, \frac{\partial \mu(y)}{\partial y_{k_1}}\, dy = 0 . \tag{3.7.5}$$

Here $(\mathbf{q}, \mathbf{n}_y)$ is the scalar product of a unit vector $\mathbf{q}$ along the Oy_{k_1}-axis and a vector $\mathbf{n}_y$, which coincides with a unit external normal at point $y \in \partial\Omega$. In this context, it is of interest several possible cases.

Case 1. Let

$$\partial\eta(y)/\partial y_{k_1} \geq 0$$

for $y \in \bar{\Omega}$ and let a function $f(y)$ defined on $\partial\Omega$ be such that

$$f(y) = \operatorname{sign}\mu(y) \quad \text{for} \quad y \in \partial\Omega\,. \tag{3.7.6}$$

Observe that

$$f(y) \not\equiv \text{const}$$

on $\partial\Omega$. Let (3.7.5) be extended to involve in subsequent reasonings the function $H_f(y)$, which is harmonic in the domain Ω and takes values (3.7.6) almost everywhere on $\partial\Omega$. In this line,

$$J(H_f) = 0\,, \tag{3.7.7}$$

where

$$J(H_f) = \int\limits_{\partial\Omega} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y - \int\limits_{\Omega'} H_f(y)\,\frac{\partial\mu(y)}{\partial y_{k_1}}\;dy \tag{3.7.8}$$

and

$$\Omega' = \left\{\, y\in\Omega\colon\ \frac{\partial\eta(y)}{\partial y_{k_1}} > 0 \right\}.$$

It is clear that the set Ω' so constructed is a subset of Ω and contains only those points for which $\partial\eta(y)/\partial y_{k_1} > 0$. To simplify the rest of the proof, we will assume that Ω' is a domain with a piecewise boundary $\partial\Omega'$ and $\operatorname{mes}\Omega' \neq 0$.

Since $|\,H_f(y)\,| < 1$ for any $y \in \Omega$, item (a) of condition (3.7.2) and the configuration of the domain Ω' imply the inequality

$$\int\limits_{\Omega'} H_f(y)\,\frac{\partial\mu(y)}{\partial y_{k_1}}\;dy \;<\; \int\limits_{\partial\Omega'} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y\,. \tag{3.7.9}$$

For $\Omega' = \Omega$ the preceding formulae (3.7.8)–(3.7.9) give $J(H_f) > 0$. If $\Omega' \subset \Omega$ and any straight line, parallel to a vector $\mathbf{q}$, may intersect either of the boundaries $\partial\Omega$ and $\partial\Omega'$ at most at two points, then the combination of (3.7.8)–(3.7.9) gives

$$\begin{aligned} J(H_f) &\geq \left[\,\int\limits_{\partial\Omega} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y - \int\limits_{\partial\Omega'} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y\right] \\ &= \left[\,\int\limits_{S+} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y - \int\limits_{S'+} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y\right] \\ &\quad + \left[-\int\limits_{S'-} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y - \int\limits_{S-} |\,\mu(y)\,|\,(\mathbf{q},\mathbf{n}_y)\;ds_y\right], \end{aligned} \tag{3.7.10}$$

where the new sets

$$S^+ = \{y \in \partial\Omega\colon\ (\mathbf{q}, \mathbf{n}_y) > 0\}; \qquad S^- = \{y \in \partial\Omega\colon\ (\mathbf{q}, \mathbf{n}_y) \le 0\};$$

$$S'^+ = \{y \in \partial\Omega'\colon\ (\mathbf{q}, \mathbf{n}_y) > 0\}; \qquad S'^- = \{y \in \partial\Omega'\colon\ (\mathbf{q}, \mathbf{n}_y) \le 0\}$$

make our exposition more transparent. From item (a) of condition (3.7.2) it follows that every expression in square brackets on the right of (3.7.10) is nonnegative and at least one of them is strictly positive. This provides support for the view that $J(H_f) > 0$.

When no additional restriction on the structure of the boundaries $\partial\Omega'$ and $\partial\Omega$ is imposed, a minor adaptation of fragmentation implies the inequality

$$\int\limits_{\partial\Omega} |\mu(y)|\,(\mathbf{q}, \mathbf{n}_y)\; ds_y - \int\limits_{\partial\Omega'} |\mu(y)|\,(\mathbf{q}, \mathbf{n}_y)\; ds_y > 0\,,$$

so that $J(H_f) > 0$. Adopting similar arguments we are led to the same inequality for other possible configurations of Ω'. But the strict positiveness of $J(H_f)$ disagrees with (3.7.7). This proves the assertion of the theorem in the first case.

Case 2. When $\partial\eta(y)/\partial y_{k_1} \le 0$ for $y \in \bar\Omega$, it will be convenient to introduce the new sets

$$\begin{aligned} S^+ &= \{y \in \partial\Omega\colon\ \mu(y)(\mathbf{q}, \mathbf{n}_y) > 0\}; \quad \Omega_{\nu^+} = \{y \in \Omega\colon\ \nu(y) \ge 0\}; \\ S^- &= \{y \in \partial\Omega\colon\ \mu(y)(\mathbf{q}, \mathbf{n}_y) \le 0\}; \quad \Omega_{\nu^-} = \{y \in \Omega\colon\ \nu(y) < 0\}; \end{aligned} \tag{3.7.11}$$

and the function $f(y)$ defined on the boundary by means of the relations

$$f(y) = \begin{cases} 1 & \text{for} \quad y \in S^+, \\ 0 & \text{for} \quad y \in S^-. \end{cases} \tag{3.7.12}$$

It is worth noting here that $\nu \equiv 0$ for $f(y) \equiv \text{const}$ on the boundary $\partial\Omega$.

As before, equality (3.7.5) can be extended in any convenient way. The function H_f being harmonic in Ω and having the boundary values (3.7.12) almost everywhere on $\partial\Omega$ applies equally well to such an extension, thus causing

$$J(H_f) = 0\,, \tag{3.7.13}$$

where

$$J(H_f) = \int\limits_{S^+} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\, ds_y - \int\limits_{\Omega_{\nu^+}} H_f(y)\, \frac{\partial \mu(y)}{\partial y_{k_1}}\, dy - \int\limits_{\Omega_{\nu^-}} H_f(y)\, \frac{\partial \mu(y)}{\partial y_{k_1}}\, dy\,. \tag{3.7.14}$$

We are only interested in special investigations for the case $\Omega_{\nu^-} = \Omega'_{\nu^-}$, where

$$\Omega'_{\nu^-} = \left\{ y \in \Omega_{\nu^-}\colon\ \frac{\partial \eta(y)}{\partial y_{k_1}} < 0 \right\}, \tag{3.7.15}$$

since the others can be treated in a similar manner.

By virtue of (3.7.11) and (3.7.15) and the properties of the function $\mu(y)$ we find that

$$\begin{aligned} &\frac{\partial \mu(y)}{\partial y_{k_1}} > 0 \quad \text{for} \quad y \in \Omega_{\nu^-}\,, \\ &\frac{\partial \mu(y)}{\partial y_{k_1}} \leq 0 \quad \text{for} \quad y \in \Omega_{\nu^+}\,. \end{aligned} \tag{3.7.16}$$

We put for the boundary $\partial\Omega_{\nu^-}$ of Ω_{ν^-}

$$S^+_{\nu^-} = \{ y \in \partial\Omega_{\nu^-}\colon\ (\mathbf{q}, \mathbf{n}_y) > 0 \}\,;$$

$$S^-_{\nu^-} = \{ y \in \partial\Omega_{\nu^-}\colon\ (\mathbf{q}, \mathbf{n}_y) \leq 0 \}\,.$$

As far as $0 < H_f < 1$ for any $y \in \Omega$, it is not difficult to establish the chain of relations

$$\begin{aligned} - \int\limits_{\Omega_{\nu^-}} H_f(y)\, \frac{\partial \mu(y)}{\partial y_{k_1}}\, dy &> - \int\limits_{\partial\Omega_{\nu^-}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\, ds_y \\ &= - \int\limits_{s^+_{\nu^-}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\, ds_y - \int\limits_{s^-_{\nu^-}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\, ds_y\,, \end{aligned} \tag{3.7.17}$$

yielding

$$J(H_f) \geq \left[\int_{S^+} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y - \int_{S^-_{\nu^-}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \right] + \left[- \int_{S^+_{\nu^+}} \mu(y)\,(\mathbf{q}, \mathbf{n}_y)\; ds_y \right] + \left[- \int_{\Omega_{\nu^+}} H_f(y)\, \frac{\partial \mu(y)}{\partial y_{k_1}}\, .dy \right] \tag{3.7.18}$$

Here we used also (3.7.17). With the aid of relations (3.7.16) and (3.7.11) and the properties of the functions $\nu(y)$ and $\eta(y)$ we deduce that each of the terms in square brackets on the right-hand side of (3.7.18) is nonnegative. Moreover, at least one of those terms is strictly positive. In view of this, we would have $J(H_f) > 0$, which disagrees with (3.7.13). The obtained contradiction proves the assertion of the theorem. ■

We now focus our attention on one more possible case in connection with spherical coordinates (ρ, θ) of $y \in \mathbf{R}^n$ $(n \geq 2)$, where ρ means the length of the radius-vector of y. The symbol θ is used as a common notation for all angular coordinates in the space $\mathbf{R}^n$. This follows established practice and does not lead to any ambiguity. Of special interest is the class of functions $\mu_\alpha(y) \in C^1(\bar{A})$, $\alpha = 1, 2$, satisfying the condition

$$\mu_\alpha(y) = \xi(y)\, \delta_\alpha(y)\,, \tag{3.7.19}$$

where

(a) $\delta_\alpha(y)$ $(\alpha = 1, 2)$ is continuosly differentiable for $y \in \bar{A}$ and $\partial \delta_\alpha / \partial \rho = 0$;

(b) $\xi(y) > 0$ and either $\partial(\rho^n \xi)/\partial \rho \geq 0$ or $\partial(\rho^n \xi)/\partial \rho \leq 0$ for $y \in \bar{A}$, where ρ is the length of the radius-vector of a point y with the origin $O \in \mathbf{R}^n \setminus \bar{A}$.

If under the conditions of item (b) the origin $O \in \bar{A}$, then $\xi(y)$ is supposed to be positive along with $\partial(\rho^n \xi)/\partial \rho \geq 0$ for $\rho < \rho_0$ and $\partial(\rho^n \xi)/\partial \rho \geq 0$ for $\rho \geq \rho_0$, where ρ_0 is kept fixed, sufficiently small and positive.

Theorem 3.7.2 *If the functions $\mu_\alpha(y)$ satisfy condition* (3.7.19) *and the exterior potentials of the set A are such that*

$$u(x;A,\mu_1) = u(x;A,\mu_2) \quad \textit{for} \quad x \in \mathbf{R}^n \setminus \bar{A}, \tag{3.7.20}$$

then $\mu_1 = \mu_2$ for $y \in A$.

Proof The proof of this theorem is similar to that carried out in Theorem 3.7.1. In this line, let $A = \Omega$. Consider the case when $\xi(y) > 0$ and $\partial(\rho^n \xi)/\partial\rho > 0$ for $y \in \bar{\Omega}$. By virtue of (3.7.20) any function $H(y)$ being harmonic in $D \supset \bar{\Omega}$ implies that

$$J(H) = 0, \tag{3.7.21}$$

where

$$J(H) = \int_{\partial\Omega} H(y)\,\mu(y)\,(\mathbf{R}_y, \mathbf{n}_y)\,ds_y - \int_{\Omega} H(y)\left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k}(y_k\,\mu)\right] dy \tag{3.7.22}$$

and $(\mathbf{R}_y, \mathbf{n}_y)$ is the scalar product of the radius-vector $\mathbf{R}_y$ on $\mathbf{n}_y$. Having in our disposal the difference of functions $\mu(y) = \mu_1(y) - \mu_2(y)$, we conclude that $\mu(y) \not\equiv \text{const}$ on $\partial\Omega$ if $\mu(y) \not\equiv 0$ in Ω. Let a function $f(y)$ be defined on the boundary $\partial\Omega$ by the relation

$$f(y) = \operatorname{sign}\mu(y) \quad \text{for} \quad y \in \partial\Omega. \tag{3.7.23}$$

Equation (3.7.21) is again extended to involve in subsequent reasonings the function $H_f(y)$ being harmonic in Ω and having the boundary values (3.7.23) almost everywhere on $\partial\Omega$. All this enables us to write down the equation

$$J(H_f) = 0, \tag{3.7.24}$$

where

$$J(H_f) = \int_{\partial\Omega} |\mu(y)|\,(\mathbf{R}_y, \mathbf{n}_y)\,ds_y - \int_{\Omega} H_f(y)\left[\sum_{k=1}^{n} \frac{\partial}{\partial y_k}(y_k\,\mu)\right] dy. \tag{3.7.25}$$

Observe that $|H_f(y)| < 1$. In light of the properties of the function $\mu(y)$ expression (3.7.25) implies $J(H_f) > 0$, violating (3.7.24). In order to carry out the proof of Theorem 3.7.2 in the remaining cases we adopt the arguments used in the proof of Theorem 3.7.1 by replacing $(\mathbf{q}, \mathbf{n}_y)$ and $\partial\mu(y)/\partial y_k$ by $(\mathbf{R}_y, \mathbf{n}_y)$ and $\sum_{k=1}^{n} \partial(\mu y_k)/\partial y_k$, respectively. ■

Finally, we proceed to the third class of functions $\mu_\alpha(y)$, $\alpha = 1, 2$, represented by

$$\mu_\alpha(y) = \gamma\,\xi(y)\,\delta_\alpha(y) + \beta\,\eta(y)\,\nu_\alpha(y)\,, \tag{3.7.26}$$

where γ and β are positive constants, $\gamma^2 + \beta^2 \neq 0$, the functions $\delta_\alpha(y)$ and $\nu_\alpha(y)$ are so chosen as to satisfy (3.7.2), the functions $\eta(y)$ and $\xi(y)$ obey the following properties:

$$\eta(y) > 0\,, \quad \frac{\partial}{\partial y_{k_1}} \eta(y) \geq 0\,, \quad \xi(y) > 0\,, \quad \frac{\partial}{\partial \rho}\,(\rho^n \xi) > 0 \quad \text{for} \quad y \in A\,.$$

Combination of appropriate expedients from the proofs of Theorems 3.7.1–3.7.2 gives the following result.

Theorem 3.7.3 *Let the functions $\mu_\alpha(y)$ admit decomposition* (3.7.26). *If the exterior potentials of a given set $A = \cup_{j=1}^{j_1} \Omega_j$ are such that*

$$u(x; A, \mu_1) = u(x; A, \mu_2) \quad \textit{for} \quad x \in \mathbf{R}^n \setminus \bar{A}\,,$$

then the function

$$\mu(y) = \mu_1(y) - \mu_2(y)$$

has constant sign on the boundaries $\partial\Omega_j$.

Our next goal is to develop an integro-differential equation of the first kind for the density of the potential assuming that the function $\mu(y) \in C^{1+h}(\bar{\Omega})$ does not vanish almost everywhere on $\bar{\Omega}$ and the boundary $\partial\Omega$ is piecewise smooth. The intention is to use the volume mass potential

$$u(x) = \int\limits_{\Omega} \mu(y)\, E(x, y)\; dy\,, \tag{3.7.27}$$

where $E(x, y)$ is the fundamental solution of the Laplace equation. As a matter of fact, the inverse problem of finding a density via the exterior potential amounts to recovering the function $\mu(x)$, $x \in \Omega$, involved in (3.7.1) from available values of the function $u(x)$ for $x \in \mathbf{R}^n \setminus \bar{\Omega}$.

Other ideas are connected with introduction of the function $w(x)$ by means of the relations

$$w(x) = \begin{cases} 2\gamma_1 u(x) - \displaystyle\sum_{k=1}^{n} (\gamma_1 x_k + \beta_1 q_k) \frac{\partial}{\partial x_k} u(x) & \text{for } n > 2, \\ 2\gamma_1 u(x) - \gamma_1 - \displaystyle\sum_{k=1}^{n} (\gamma_1 x_k + \beta_1 q_k) \frac{\partial}{\partial x_k} u(x) & \text{for } n = 2, \end{cases} \tag{3.7.28}$$

where constants β_1 and γ_1 are such that $\gamma_1^2 + \beta_1^2 \neq 0$ and $\mathbf{q} = (q_1, \dots, q_n)$ is a unit vector along the Oy_{k_1}-axis.

With the aid of the decompositions

$$\sum_{k=1}^{n} \frac{\partial}{\partial y_k} \left[(\gamma_1 y_1 + \beta_1 q_k)\, E(x,y) \right] =$$

$$\begin{cases} 2\gamma_1 E(x,y) - \sum\limits_{k=1}^{n} (\gamma_1 x_k + \beta_1 q_k) \dfrac{\partial}{\partial x_k} E(x,y)\,, & n > 2\,, \\ 2\gamma_1 E(x,y) - \gamma_1 - \sum\limits_{k=1}^{n} (\gamma_1 x_k + \beta_1 q_k) \dfrac{\partial}{\partial x_k} E(x,y)\,, & n = 2\,, \end{cases}$$

the function $w(x)$ can be rewritten as

$$w(x) = \int\limits_{\partial\Omega} \mu(y)\, \Phi(y)\, E(x,y)\; ds_y - \int\limits_{\Omega} \left[\sum_{k=1}^{n} (\gamma_1 y_k + \beta_1 q_k) \frac{\partial \mu}{\partial y_k} \right] E(x,y)\; dy\,, \tag{3.7.29}$$

where

$$\Phi(y) = (\gamma_1 \mathbf{R}_y + \beta_1 \mathbf{q}, \mathbf{n}_y)$$

is the scalar product of the vectors $\gamma_1 \mathbf{R}_y + \beta_1 \mathbf{q}$ and $\mathbf{n}_y$.

Let $\mathbf{n}_{x_0}$ be a unit external normal to the surface $\partial\Omega$ at point x_0 and let $(\widehat{\mathbf{n}_{x_0}, \mathbf{r}_{xy}})$ be the angle between $\mathbf{n}_{x_0}$ and $\mathbf{r}_{xy}$, where $\mathbf{r}_{xy}$ is the vector joining x and y provided that x lies on the normal $\mathbf{n}_{x_0}$ and $x \in \mathbf{R}^n \setminus \bar{\Omega}$. We thus have

$$\frac{\partial w(x)}{\partial n_{x_0}} = \int\limits_{\partial\Omega} \frac{\cos(\widehat{\mathbf{n}_{x_0}, \mathbf{r}_{xy}})}{(n-2)\,\omega_n\, |\, x - y\,|^{n-1}}\, \mu(y)\, \Phi(y)\; ds_y$$

$$- \int\limits_{\Omega} \frac{\cos(\widehat{\mathbf{n}_{x_0}, \mathbf{r}_{xy}})}{(n-2)\,\omega_n\, |\, x - y\,|^{n-1}}$$

$$\times \left[\sum_{k=1}^{n} (\gamma_1 y_k + \beta_1 q_k) \frac{\partial \mu}{\partial y_k} \right] dy\,.$$

Recall that the normal derivative of the simple layer potential undergoes a jump, while the first derivatives of the Newtonian potential (3.7.27) are really continuous on the entire space. With this in mind, we obtain

$$A(\mu\Phi) = f + P(\mu)\,, \qquad n > 2\,, \tag{3.7.30}$$

where

$$
(3.7.31) \qquad A(\mu\,\Phi) = \tfrac{1}{2}\,\mu\,\Phi - \int\limits_{\partial\Omega} \frac{\cos\,(\widehat{\mathbf{n}_x, \mathbf{r}_{xy}})}{(n-2)\,\omega_n\,|\,x-y\,|^{n-1}} \times \mu(y)\,\Phi(y)\;ds_y\,, \quad x \in \partial\Omega\,,
$$

$$
(3.7.32) \qquad P(\mu) = -\int\limits_{\Omega} \frac{\cos\,(\widehat{\mathbf{n}_x, \mathbf{r}_{xy}})}{(n-2)\,\omega_n\,|\,x-y\,|^{n-1}} \times \left[\sum_{k=1}^{n} (\gamma_1 y_k + \beta_1 q_k)\,\frac{\partial \mu}{\partial y_k}\right] dy\,, \quad x \in \partial\Omega\,,
$$

$$
(3.7.33) \qquad f(x) = \frac{\partial w}{\partial n_x}\bigg|^{-} = \lim_{x' \to x} \frac{\partial w(x')}{\partial n_x}\,.
$$

In formula (3.7.33) the points $x' \in \mathbf{R}^n \setminus \bar{\Omega}$ are located on the normal $\mathbf{n}_x$ taken at $x \in \partial\Omega$. Summarizing, the following statement is established.

Lemma 3.7.1 *If a potential $u(x)$ of the type* (3.7.27) *is given for all $x \in \mathbf{R}^n \setminus \bar{\Omega}$, then its density $\mu(x)$ satisfies the integro-differential equation* (3.7.30).

Lemma 3.7.1 implies a number of useful corollaries for certain types of densities. Indeed, consider one more class of functions $\mu(y)$ satisfying the condition

$$
(3.7.34) \qquad \frac{\partial \mu}{\partial \rho} = 0\,, \qquad (\rho,\,\theta) = y \in \bar{\Omega}\,.
$$

By merely setting $\gamma_1 = 0$ and $\beta_1 = 0$ in (3.7.30)–(3.7.33) it is possible to derive the equation

$$
(3.7.35) \qquad A(\mu\,\Phi) = f\,,
$$

where $\Phi(x) = (\mathbf{R}_x, \mathbf{n}_x)$ for $x \in \partial\Omega$.

In certain functional spaces both equations (3.7.30) and (3.7.35) admit alternative forms. For example, one can consider a space B, whose norm is equivalent to the C^{1+h}-norm (for more detail see Prilepko (1965a)). When the boundary $\partial\Omega$ happens to be of class C^{2+h} and the function μ satisfying (3.7.34) belongs to the class $C^{1+h}(\partial\Omega)$, all the ingredients of (3.7.35) belong to the space B in light of the properties of the potential. Due to the uniqueness of a solution of the exterior Neymann problem for the Laplace equation ($n \geq 3$) and the Banach theorem on inverse operator, for any operator A in the space B there exists the bounded inverse A^{-1}, by means of which equation (3.7.35) reduces to the equivalent one:

$$\mu\,\Phi = A^{-1}(f)\,. \tag{3.7.36}$$

Having involved the preceding equation in later discussions we arrive at the following assertions.

Corollary 3.7.1 *If potential* (3.7.27), *whose density is of the type* (3.7.34), *satisfies the condition*

$$u(x) = 0 \quad \textit{for} \quad x \in \mathbf{R}^n \setminus \bar{\Omega}\,, \qquad n \geq 3\,,$$

then $\mu(y) = 0$ *for* $y \in \Omega$.

Corollary 3.7.2 *Let* $\Omega \subset \mathbf{R}^n$, $n \geq 3$, *be a "star-shaped" domain with respect to an inner point and let* (3.7.34) *hold. Then the values of the function* $\mu(x)$ *for* $x \in \partial\Omega$ *can be expressed in terms of the exterior potential* $u(x)$ *by*

$$\mu(x) = A^{-1}(f)\,\Phi^{-1}(x)\,,$$

where $\Phi(x) = (\mathbf{R}_x, \mathbf{n}_x)$ *for all* $x \in \partial\Omega$ *and the function* $f(x)$ *is given by formulae* (3.7.33) *and* (3.7.28).

For another formulae expressing the values of μ via the corresponding potential in the cases of a "star-shaped" domain and a density of the type (3.7.34) we refer the readers to Antokhin (1966).

Of special investigation is the class of functions $\mu(y)$ satisfying the condition

$$\frac{\partial \mu(y)}{\partial y_{k_1}} = 0 \quad \text{for} \quad y \in \bar{\Omega}\,. \tag{3.7.37}$$

Upon substituting $\beta_1 = 1$ and $\gamma_1 = 0$ into (3.7.30)-(3.7.33) we derive the equation

$$A(\mu\,\Phi) = f\,, \tag{3.7.38}$$

where $\Phi(x) = (\mathbf{q}, \mathbf{n}_x)$ for $x \in \partial\Omega$. Equation (3.7.38) implies the following result.

Corollary 3.7.3 *If potential* (3.7.27) *has a density* μ *of the type* (3.7.37) *and*

$$u(x) = 0 \quad \textit{for} \quad x \in \mathbf{R}^n \setminus \bar{\Omega} , \qquad n \geq 2 ,$$

then $\mu(y) = 0$ *for* $y \in \Omega$.

In what follows we are in a new framework making it possible to determine in the domain Ω the density $\mu(y)$ of a given potential $u(x)$ for $x \in \mathbf{R}^n \setminus \bar{\Omega}$ in the class of functions $\mu(y)$ solving the equation

$$L \mu = 0 , \tag{3.7.39}$$

where L is a differential operator of the second order acting on $y \in \Omega$. Under some appropriate restrictions on the smoothness of the function $\mu(y)$ and the boundary $\partial\Omega$, the properties of the potential $u(x)$ imply that

$$\big[L \,\Delta\, u\big] = 0 \quad \text{for} \quad x \in \Omega . \tag{3.7.40}$$

The new functions $\varphi(x_0)$ and $\psi(x_0)$ are defined on the boundary $\partial\Omega$ by

$$\varphi(x_0) = \lim_{x \to x_0} u(x) , \qquad x \in \mathbf{R}^n \setminus \bar{\Omega} , \quad x_0 \in \partial\Omega , \tag{3.7.41}$$

$$\psi(x_0) = \lim_{x \to x_0} \sum_{k=1}^{n} a_k \frac{\partial u(x)}{\partial x_k} , \quad x \in \mathbf{R}^n \setminus \bar{\Omega} , \quad x_0 \in \partial\Omega , \tag{3.7.42}$$

where the known functions $a_k = a_k(x_0)$ are defined on $\partial\Omega$. One thing is worth noting here. If we have at our disposal the function $u(x)$ defined by (3.7.27) everywhere in $\mathbf{R}^n \setminus \bar{\Omega}$, it is possible to assign the values of $u(x)$ and $\frac{\partial u(x)}{\partial x_k}$ for $x \in \partial\Omega$. The problem here is to find the function $h(x)$, $x \in \Omega$, being a solution to the equation

$$\big[L \,\Delta\, h\big](x) = 0 , \qquad x \in \Omega , \tag{3.7.43}$$

supplied by the boundary conditions

$$h(x_0) = \varphi(x_0) , \qquad x_0 \in \partial\Omega , \tag{3.7.44}$$

$$\sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} h(x_0) = \psi(x_0) , \qquad x_0 \in \partial\Omega . \tag{3.7.45}$$

Having resolved (3.7.43)–(3.7.45) we need a suitably chosen form of writing, namely

$$\mu(x) = \Delta\, h(x) \quad \text{for} \quad x \in \Omega .$$

When the Laplace operator is considered in place of L, we give as one possible example the class of functions $\mu(y)$ satisfying the Laplace equation in the domain Ω:

$$\Delta \mu = 0 . \tag{3.7.46}$$

In such a setting the function $h(x)$ involved in (3.7.43)–(3.7.45) should be recovered from the set of relations

$$\Delta^2 h = 0 , \qquad x \in \Omega , \tag{3.7.47}$$

$$h(x) = u(x) , \qquad x \in \partial\Omega , \tag{3.7.48}$$

$$\frac{\partial}{\partial n_x} h(x) = \frac{\partial}{\partial n_x} u(x) , \qquad x \in \partial\Omega . \tag{3.7.49}$$

It is worth emphasizing once again that the values of $u(x)$ and $\frac{\partial u(x)}{\partial n_x}$ do exist on $\partial\Omega$, since the exterior potential was defined by (3.7.27) everywhere on $\mathbf{R}^n \setminus \bar{\Omega}$. As we have mentioned above, the solution of problem (3.7.47)–(3.7.49) is one of the ways of finding the function $\mu(x)$, $x \in \Omega$. Moreover, due to the uniqueness of this solution we obtain the following result.

Corollary 3.7.4 *If potential* (3.7.27) *has a density* μ *of the type* (3.7.46) *and the condition* $u(x) = 0$ *holds for all* $x \in \mathbf{R}^n \setminus \bar{\Omega}$, *then* $\mu(y) = 0$ *for all* $y \in \Omega$.

In addition, we formulate the relevant existence theorem which will be used in later discussions.

Theorem 3.7.3 *When the density* μ *of potential* (3.7.27) *happens to be of the type* (3.7.46), *there exists a solution of the problem of finding the density of a given body via its exterior potential, this solution is unique and*

$$\mu(x) = \Delta h(x) , \qquad x \in \Omega ,$$

where $h(x)$ *is a unique solution of the direct problem* (3.7.47)–(3.7.49).

When the function $\mu(y)$ happens to be of class (3.7.37), Corollary 3.7.3 formulated above can be proved in another way for $L \equiv \partial/\partial x_k$. In that case the density $\mu(y)$ evidently satisfies (3.7.37). Then any potential $u(x)$ of the type (3.7.27) gives a solution to the equation

$$\left(\frac{\partial}{\partial x_k} \Delta u \right)(x) = 0 \quad \text{for} \quad x \in \Omega . \tag{3.7.50}$$

By merely setting $Z(x) = \partial u(x)/\partial x_k$ we recast equation (3.7.50) as

$$(\Delta Z)(x) = 0 \quad \text{for} \quad x \in \Omega .$$

From the premises of Corollary 3.7.3 we know that the exterior potential $u(x)$ vanishes for $x \in \mathbf{R}^n \setminus \bar{\Omega}$. This provides reason enough to conclude that the boundary condition

$$Z(x) = 0 , \qquad x \in \partial\Omega ,$$

is fulfilled. We thus have $Z(x) = 0$ for all $x \in \Omega$ and, thereby, $\partial u(x)/\partial x_k = 0$ for all $x \in \Omega$. Together, the preceding and the condition

$$u(x) = 0 , \qquad x \in \partial\Omega ,$$

which is a corollary of the initial assumption, assure us of the validity of the relation $u(x) = 0$ for all $x \in \Omega$ and, in turn, $\mu(x) = 0$ for all $x \in \Omega$.

3.8 Uniqueness of the inverse problem solution for the simple layer potential

This section is devoted to the exterior inverse problem of recovering the shape of a given body from available values of the exterior potential of a simple layer. First, the uniqueness theorems will be proved for potentials of arbitrary contact bodies with variable densities having no constant signs. These results differ markedly from the preceding assertions emerging from the uniqueness theorems for magnetic potentials of contact bodies with densities having constant signs. Second, we will prove the uniqueness theorems for noncontact bodies.

Let A_1 and A_2 be open bounded sets under the agreement that either of these sets is a union of a finite number of domains; meaning, as further developments occur,

$$(3.8.1) \qquad A_1 = \bigcup_{j=1}^{m_1} \Omega_1^j , \qquad A_2 = \bigcup_{j=1}^{m_2} \Omega_2^j ,$$

$$(3.8.2) \qquad \begin{aligned} \Gamma_1^i &= \partial A_1 \cap \bar{A}_1 \cap \bar{A}_2 , & \bar{\Gamma}_1^e &= \partial A_1 \setminus \Gamma_1^i , \\ \bar{\Gamma}_2^e &= \partial A_2 \cap \bar{\Gamma}^e , & \Gamma_2^i &= \partial A_2 \setminus \bar{\Gamma}_2^e , \end{aligned}$$

where m_1 and m_2 are fixed numbers and $\bar{\Gamma}^e = \partial(\bar{A}_1 \cup \bar{A}_2)$ (for more detail we refer to (3.2.4) and (3.2.38)). Within these notations, we now consider the **simple layer potential** v defined by

$$v(x; \partial A_\alpha, \rho_\alpha) = \int_{\partial A_\alpha} E(x, y)\, \rho_\alpha(y)\; ds_y\,, \tag{3.8.3}$$

where $\rho_\alpha \neq 0$ almost everywhere on ∂A_α. Recall that the definition of externally contact set is available in Section 3.6.

Theorem 3.8.1 *Let A_α, $\alpha = 1, 2$, be externally contact sets and the simple layer potentials $v(x; \partial A_\alpha, \rho)$ with density $\rho \in C(\partial A_\alpha)$ satisfy the condition*

$$v(x; \partial A_1, \rho) \;=\; v(x; \partial A_2, \rho)\,, \qquad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2)\,. \tag{3.8.4}$$

Then $A_1 = A_2$.

Proof Observe that we imposed no restriction on the sign of the function ρ. For this reason it is necessary to examine two possible cases. If the function $\rho(y)$ has constant sign, the assertion in question is an immediate implication of Theorem 3.6.2 with $\gamma = 1$ and $\beta = 0$. If $\rho(y)$ does not have constant sign and $A_1 \neq A_2$, then Lemma 3.2.4 with $\beta = 0$ and $\gamma = 1$ implies that a solution $h(y)$ of the Laplace equation

$$(\Delta\, h)(y) = 0\,, \qquad y \in D\,, \tag{3.8.5}$$

being regular in a domain $D \subset \mathbf{R}^n$, $\bar{D} \supset D \supset \bar{B}_0$, satisfies the equality

$$\int_{(\partial B_0)^e} h(y)\, \rho(y)\; ds_y - \int_{(\partial B_0)^i} h(y)\, \rho(y)\; ds_y = 0\,. \tag{3.8.6}$$

Here the domain B_0 was taken from (3.2.39)–(3.2.42).

The function $f(y)$ is defined on $\partial B_0 = (\partial B_0)^e \cup (\partial B_0)^i$ by the relations

$$f(y) = \begin{cases} \operatorname{sign} \rho(y)\,, & y \in (\partial B_0)^e, \\ -\operatorname{sign} \rho(y)\,, & y \in (\partial B_0)^i. \end{cases}$$

As in the proof of Theorem 3.3.1 equality (3.8.6) can be extended to involve a generalized solution h_f of the Dirichlet problem for the Laplace equation with the boundary values associated with $f(y)$. In this line,

$$J(h_f) \;=\; 0\,,$$

where

$$J(h_f) = \int_{(\partial B_0)^e} h_f(y)\,\rho(y)\,ds_y - \int_{(\partial B_0)^i} h_f(y)\,\rho(y)\,ds_y\,.$$

The function h_f being a regular solution to the equation

$$(\Delta\, h_f)(y) = 0 \quad \text{for} \quad y \in B_0$$

takes the values of $f(y)$ almost everywhere on ∂B_0 and ensures the decomposition

$$J(h_f) = \int_{(\partial B_0)^e} |\,\rho(y)\,|\,ds_y + \int_{(\partial B_0)^i} |\,\rho(y)\,|\,ds_y\,,$$

yielding $J(h_f) > 0$, which disagrees with the equality $J(h_f) = 0$. The obtained contradiction proves the assertion of the theorem. ∎

It is worth bearing in mind that the sets in the above theorem were contact and this restriction cannot be relaxed. There is an example describing two different bodies with a constant density, whose exterior logarithmic potentials of a simple layer are equal to each other.

We now raise the question of the solution uniqueness for the problem of recovering the shape of a body from available values of its potential.

Theorem 3.8.2 *Let bounded sets A_α and functions $\rho_\alpha \in C(\partial A_\alpha)$, $\alpha = 1, 2$, be such that*

$$(3.8.7) \qquad \int_{\Gamma_1^i} |\,\rho_1(y)\,|\,ds_y + \int_{\Gamma_2^i} |\,\rho_2(y)\,|\,ds_y < \int_{\bar\Gamma_1^e} |\,\rho_1(y)\,|\,ds_y + \int_{\bar\Gamma_2^e} |\,\rho_2(y)\,|\,ds_y\,.$$

If $A_1 \neq A_2$, then the exterior potentials $v(x; \partial A_1, \rho_1)$ and $v(x; \partial A_2, \rho_2)$ of a simple layer are different, that is, there is a point $\hat x \in \mathbf{R}^n \setminus (\bar A_1 \cup \bar A_2)$ at which

$$(3.8.8) \qquad v(\hat x; \partial A_1, \rho_1) \neq v(\hat x; \partial A_2, \rho_2)\,.$$

Proof Let $A_1 \neq A_2$ and

$$(3.8.9) \qquad v(x; \partial A_1, \rho_1) = v(x; \partial A_2, \rho_2) \quad \text{for all} \quad x \in \mathbf{R}^n \setminus (\bar A_1 \cup \bar A_2)\,.$$

Then by Lemma 3.2.2 with $\beta = 0$ and $\gamma = 1$ the equality

$$J(h) = 0 \tag{3.8.10}$$

is valid with

$$J(h) = \int_{\partial A_1} \rho_1(y)\, h(y)\, ds_y - \int_{\partial A_2} \rho_2(y)\, h(y)\, ds_y\,, \tag{3.8.11}$$

where $h(y)$ is an arbitrary regular solution to the equation

$$(\Delta\, h)(y) = 0 \quad \text{for} \quad y \in D\,. \tag{3.8.12}$$

Here D is ordered with respect to inclusion:

$$\mathbf{R}^n \supset \bar{D} \supset D \supset (\bar{A}_1 \cup \bar{A}_2)\,. \tag{3.8.13}$$

Our subsequent arguments are based on the use of the function $f(y)$ defined on the surface $\bar{\Gamma}^e = \bar{\Gamma}_1^e \cup \bar{\Gamma}_2^e = \partial(\bar{A}_1 \cup \bar{A}_2)$ by means of the relations

$$f(y) = \begin{cases} \operatorname{sign} \rho_1(y)\,, & y \in \bar{\Gamma}_1^e\,, \\ -\operatorname{sign} \rho_2(y)\,, & y \in \bar{\Gamma}_2^e\,. \end{cases} \tag{3.8.14}$$

As in the proof of Theorem 3.3.1 equality (3.8.10) should be extended to be valid for a generalized solution $h_f(y)$ of the Dirichlet problem for the equation

$$(\Delta\, h_f)(y) = 0 \quad \text{for} \quad y \in A^e, \qquad A^e = \overline{A_1 \cup A_2} \setminus \Gamma^e, \tag{3.8.15}$$

with the boundary data (3.8.14). It follows from the foregoing that

$$J(h_f) = 0\,, \tag{3.8.16}$$

where

$$\begin{aligned} J(h_f) = & \int_{\bar{\Gamma}_1^e} |\, \rho_1(y) \,|\, ds_y + \int_{\bar{\Gamma}_2^e} |\, \rho_2(y) \,|\, ds_y \\ & + \int_{\Gamma_1^i} h_f(y)\, \rho_1(y)\, ds_y \\ & - \int_{\Gamma_2^i} h_f(y)\, \rho_2(y)\, ds_y\,. \end{aligned} \tag{3.8.17}$$

Since the function $h_f(y)$ is a solution of (3.8.15) with the boundary values (3.8.14), the **Hopf principle** yields the estimate

$$|h_f| \leq 1 \quad \text{for} \quad y \in \bar{A}^e, \tag{3.8.18}$$

where $\bar{A}^e = \bar{A}_1 \cup \bar{A}_2$. From (3.8.17)–(3.8.18) it follows that

$$\begin{aligned} J(h_f) \geq & \int\limits_{\bar{\Gamma}_1^e} |\rho_1(y)|\, ds_y + \int\limits_{\bar{\Gamma}_2^e} |\rho_2(y)|\, ds_y \\ & - \int\limits_{\Gamma_1^i} |\rho_1(y)|\, ds_y - \int\limits_{\Gamma_2^i} |\rho_2(y)|\, ds_y . \end{aligned} \tag{3.8.19}$$

Because of (3.8.7), the right-hand side of (3.8.19) is strictly positive, meaning $J(h_f) > 0$. But this contradicts (3.8.16) and thereby completes the proof of the theorem. ■

Theorem 3.8.3 *One assumes that A_α, $\alpha = 1, 2$, are open sets and either of these sets is a union of the same finite number of convex domains Ω_α^j. If the exterior potentials $v(x; A_\alpha, 1)$ of a simple layer with densities $\rho_\alpha \equiv 1$, $\alpha = 1, 2$, coincide, that is,*

$$v(x; A_1, 1) = v(x; A_2, 1) \quad \textit{for} \quad x \in \mathbf{R}^n \setminus (\bar{A}_1 \cup \bar{A}_2), \tag{3.8.20}$$

then

$$A_1 = A_2 . \tag{3.8.21}$$

Proof Assume to the contrary that $A_1 \neq A_2$. The domains Ω_α^j being convex imply that

$$\operatorname{mes} \bar{\Gamma}_1^e + \operatorname{mes} \bar{\Gamma}_2^e > \operatorname{mes} \Gamma_1^i + \operatorname{mes} \Gamma_2^i . \tag{3.8.22}$$

With the relation $\rho_\alpha \equiv 1$ in view, we deduce (3.8.7) from (3.8.22). Now Theorem 3.8.2 becomes valid, so that equality (3.8.8) holds true. But this contradicts condition (3.8.20). Thus, the theorem is completely proved. ■

As an immediate implication of the above result we quote

Theorem 3.8.4 *If for two convex domains Ω_α, $\alpha = 1, 2$, the exterior potentials $v(x; \partial\Omega_\alpha, 1)$ of a simple layer with densities $\rho_\alpha \equiv 1$, $\alpha = 1, 2$, coincide, that is,*

$$\int\limits_{\partial\Omega_1} E(x,y)\, ds_y = \int\limits_{\partial\Omega_2} E(x,y)\, ds_y \quad \textit{for} \quad x \in \mathbf{R}^n \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2),$$

then $\Omega_1 = \Omega_2$.

3.9 Stability in inverse problems for the potential of a simple layer in the space $\mathbf{R}^n$, $n \geq 3$

In this section some stability estimates will be derived for the exterior inverse problem related to the potential of a simple layer in the case of recovering the shape of a given body.

Let Ω_α be bounded domains with boundaries $\partial\Omega_\alpha$, $\alpha = 1, 2$. Within notation (3.2.38), it will be sensible to introduce

$$
\begin{aligned}
s^e &= \partial(\bar{\Omega}_1 \cup \bar{\Omega}_2)\,, \quad s_1^i = \partial\Omega_1 \cap (\bar{\Omega}_1 \cap \bar{\Omega}_2)\,, \quad s_1^e = \partial\Omega_1 \setminus s_1^i\,, \\
s_2^e &= \partial\Omega_2 \cap s^e\,, \qquad\qquad s_2^i = \partial\Omega_2 \setminus s_2^e\,.
\end{aligned}
\tag{3.9.1}
$$

In the case when $\Omega_1 = \Omega_2$, the boundaries s_α^e will be taken to be $\partial\Omega_\alpha$, $\alpha = 1, 2$. Unless otherwise is explicitly stated, Ω_α are supposed to be simply connected domains for the reader's convenience only. Then so are both sets

$$\Omega_\alpha = \Omega_1 \cap \Omega_2 \neq \varnothing$$

and

$$\mathbf{R}^n \setminus (\bar{\Omega}_1 \cup \bar{\Omega}_2)$$

for $n \geq 3$.

As we will see a little later, the function

$$v(x) = v(x; \partial\Omega_1, \rho_1) - v(x; \partial\Omega_2, \rho_2)\,, \tag{3.9.2}$$

where $v(x; \partial\Omega_\alpha, \rho_\alpha)$ is the potential of a simple layer over the boundary $\partial\Omega_\alpha$ with density $\rho_\alpha \in C(\partial\Omega_\alpha)$, finds a wide range of applications in this field (see (3.2.2)).

For $\Omega_1 \neq \Omega_2$ further numbers are defined by

$$G^e = \int\limits_{s_1^e} |\, \rho_1(y) \,|\; ds_y + \int\limits_{s_2^e} |\, \rho_2(y) \,|\; ds_y\,, \tag{3.9.3}$$

$$G^i = \int\limits_{s_1^i} |\, \rho_1(y) \,|\; ds_y + \int\limits_{s_2^i} |\, \rho_2(y) \,|\; ds_y\,. \tag{3.9.4}$$

Theorem 3.9.1 *One assumes that the simple layer potentials $v(x; \partial\Omega_\alpha, \rho_\alpha)$ can be extended from $\mathbf{R}^n \setminus \bar{\Omega}_\alpha$ onto $\mathbf{R}^n \setminus D^*$ as a regular solution to the equation*

$$\Delta v(x; \partial\Omega_\alpha, \rho_\alpha) = 0 \quad \textit{for} \quad x \in \mathbf{R}^n \setminus D^*, \quad n \geq 3, \quad \alpha = 1, 2\,, \tag{3.9.5}$$

where D^* is a simply connected domain, whose boundary ∂D^* is of class C^{2+h}, $\overline{D^*} \subset \Omega^e$ and $\Omega^e \cup \partial\Omega^e = \bar{\Omega}_1 \cup \bar{\Omega}_2$. Then the estimate

$$G^e - G^i \leq c_1 \max_{x \in \partial D^*} \left| \frac{\partial}{\partial \nu_x} v(x) \right| \tag{3.9.6}$$

is valid, where $c_1 = \text{const} > 0$ depends only on the configuration of the boundary ∂D^* and the values of $\partial v(x)/\partial \nu_x$ are taken at points $x \in \partial D^*$.

Proof Let D and D_1 be domains with boundaries ∂D and ∂D_1 of class C^{1+h} which can be ordered with respect to inclusion:

$$D \supset \bar{D}_1 \supset D_1 \supset (\bar{\Omega}_1 \cup \bar{\Omega}_2) .$$

We refer to the functional with the values

$$J(h) = \int_{\partial\Omega_1} \rho_1(y)\, h(y)\, ds_y - \int_{\partial\Omega_2} \rho_2(y)\, h(y)\, ds_y , \tag{3.9.7}$$

where h is a regular in D solution to the equation

$$(\Delta h)(y) = 0 , \qquad y \in D . \tag{3.9.8}$$

An auxiliary lemma may be useful in the sequel.

Lemma 3.9.1 *The functional $J(h)$ specified by* (3.9.7) *admits the estimate*

$$| J(h) | \leq c_2 \max_{y \in \partial D_1} | h(y) | \max_{y \in \partial D^*} \left| \frac{\partial}{\partial \nu_y} v(y) \right| . \tag{3.9.9}$$

Proof Indeed, any regular in D solution $h(y)$ to equation (3.9.8) is representable by

$$\int_{\partial D_1} M_x \big[E(x,y); h(x)\big]\, ds_x = \begin{cases} h(y) , & y \in D_1 , \\ 0 , & y \in \mathbf{R}^n \setminus \bar{D}_1 , \end{cases} \tag{3.9.10}$$

where $E(x,y)$ is the fundamental solution of the Laplace equation and the expression for $M[\,\cdot\,;\,\cdot\,]$ amounts to (3.2.9). Multiplying relation (3.9.10) by $\rho_\alpha(y)$ and integrating the resultig expression over $\partial\Omega_\alpha$, we arrive at

$$\int_{\partial\Omega_\alpha} \rho_\alpha(y)\, h(y)\, ds_y \tag{3.9.11}$$

$$= \int_{\partial D_1} M_x \left[\int_{\partial\Omega_\alpha} \rho_\alpha(y)\, E(x,y)\, ds_y ;\, h(x) \right] ds_x .$$

From (3.9.1)–(3.9.11) it follows that the functional $J(h)$ can be rewritten as

$$J(h) = \int_{\partial D_1} M_x\left[v(x);\, h(x)\right]\, ds_x\,. \tag{3.9.12}$$

Recalling that $v(x)$ is a regular solution to the equation

$$(\Delta v)(x) = 0 \quad \text{for} \quad x \in \mathbf{R}^n \setminus D^*, \tag{3.9.13}$$

we now turn to the simple layer potential

$$\tilde{v}(x) = \int_{\partial D^*} \omega(y)\, E(x,y)\, ds_y\,, \tag{3.9.14}$$

which solves the exterior Neymann problem

$$\begin{aligned} &\Delta\tilde{v}(x) = 0\,, \quad x \in \mathbf{R}^n \setminus D^*, \quad n \geq 3\,, \\ &\left[\frac{\partial}{\partial \nu_x}\tilde{v}(x)\right]^- = \frac{\partial}{\partial \nu_x}\left[v(x)\right]^-, \quad x \in \partial D^*, \\ &\tilde{v}(x) \to 0\,, \qquad |\,x\,| \to \infty\,. \end{aligned} \tag{3.9.15}$$

Here $\boldsymbol{\nu}_x$ is a normal at point $x \in \partial D^*$ and

$$\left[\frac{\partial}{\partial \nu_x}\tilde{v}(x)\right]^- = \lim_{\hat{x}\to x} \frac{\partial}{\partial \nu_x}\,\tilde{v}(\hat{x}) \tag{3.9.16}$$

$$\text{for} \quad \hat{x} \in \mathbf{R}^n \setminus D^*, \qquad x \in \partial D^*.$$

Therefore, the density $\omega(y)$ of potential (3.9.14) will satisfy an integral Fredholm equation of the second kind, whose solution does exist and is obliged to be unique.

By the uniqueness of a solution of the exterior Neymann problem $(n \geq 3)$,

$$v(x) = \tilde{v}(x) \quad \text{for} \quad \hat{x} \in \mathbf{R}^n \setminus D^*. \tag{3.9.17}$$

Relations (3.9.17) and (3.9.12) are followed by

$$
\begin{aligned}
J(h) &= \int\limits_{\partial D_1} M_x\left[\tilde{v}(x);\, h(x)\right]\, ds_x \\
&= \int\limits_{\partial D_1} M_x\left[\,\int\limits_{\partial D^*} \omega(y)\, E(x,y)\, ds_y;\, h(x)\right] ds_x \\
&= \int\limits_{\partial D^*} \omega(y) \left\{ \int\limits_{\partial D_1} M_x\left[E(x,y);\, ,h(x)\right]\, ds_x \right\} ds_y\,, \\
&= \int\limits_{\partial D_1} \omega(y)\, h(y)\, ds_y\,,
\end{aligned}
$$

so that

$$
J(h) = \int\limits_{\partial D^*} \omega(y)\, h(y)\, ds_y\,, \tag{3.9.18}
$$

which serves to motivate the estimate

$$
J(h) \leq \max_{y\in\partial D^*} |\, h(y)\,| \int\limits_{\partial D^*} |\,\omega(y)\,|\, ds_y\,, \tag{3.9.19}
$$

where the function $\omega(y)$ is a solution to the integral equation

$$
-\omega(y) + \int\limits_{\partial D^*} K(y,\xi)\, \omega(\xi)\, ds_\xi = f_1(y)\,, \qquad y \in \partial D^*\,, \tag{3.9.20}
$$

with

$$
f_1(y) = 2\, \frac{\partial}{\partial \nu_y}\, v(y)\,, \tag{3.9.21}
$$

$$
K(y,\xi) = 2 \left(\frac{\partial E(y,\xi)}{\partial \nu_y} \right)^{-}, \qquad y,\, \xi \in \partial D^*\,.
$$

Now estimate (3.9.9) is a corollary of (3.9.19)–(3.9.21) and thereby the lemma is completely proved. ■

With this result established, we return to the proof of the theorem involving the function $f(y)$, which is defined on the surface s^e by the relations

$$(3.9.22)\qquad f(y) = \begin{cases} \operatorname{sign} \rho_1(y), & y \in s_1^e, \\ -\operatorname{sign} \rho_2(y), & y \in s_2^e. \end{cases}$$

It is obvious that $|h_f| \le 1$. By exactly the same reasoning as in the derivation of (3.4.32) from (3.4.25) we should take into account the preceding inequality $|h_f| \le 1$ and appeal to (3.9.19), whose use permits us to find that

$$(3.9.23)\qquad |J(h_f)| \le \int_{\partial D^*} |\omega(y)|\, ds_y .$$

On the other hand, when treating (3.9.22) as the boundary data for the function h_f, the following expansion arises from (3.9.22):

$$(3.9.24)\qquad \int_{s_1^e} |\rho_1(y)|\, ds_y + \int_{s_2^e} |\rho_2(y)|\, ds_y + \int_{s_1^i} h_f(y)\, \rho_1(y)\, ds_y - \int_{s_2^i} h_f(y)\, \rho_2(y)\, ds_y = J(h_f).$$

To demonstrate this decomposition, we should adopt the arguments used in Theorem 3.7.2.

Recall that $|h_f| \le 1$. Then the combination of relations (3.9.2)–(3.9.4) and (3.9.23) gives

$$(3.9.25)\qquad G^e - G^i \le J(h_f).$$

Estimates (3.9.23) and (3.9.25) together yield

$$(3.9.26)\qquad G^e - G^i \le \int_{\partial D^*} |\omega(y)|\, ds_y ,$$

where $\omega(y)$ is a unique solution ($n \ge 3$) to the integral Fredholm equation (3.9.20), whose right-hand side is expressed in terms of the values of the normal derivative of the function $v(x)$ on ∂D^* with the aid of the first formula (1.9.21). Estimate (3.9.6) follows from (3.9.26) in light of the well-known properties of the function $\omega(y)$ being a solution to equation (3.9.20). This proves the assertion of the theorem. ■

Remark 3.9.1 Under the assumptions of Theorem 3.9.1 estimate (3.9.6) can be replaced by (3.9.26).

We cite now some assertions afforded by the above results.

Corollary 3.9.1 *Let a set P contain all of the singular points of the potentials $v(x; \partial A_\alpha, \rho_\alpha)$, $\alpha = 1, 2$, and let D^* will be ordered with respect to inclusion:*

$$\bar{P} \subset D^* \subset \overline{D^*} \subset \Omega^e .$$

If

$$| v(x; \partial A_1, \rho_1) - v(x; \partial A_2, \rho_2) | < \varepsilon , \qquad \forall x \in s^e , \tag{3.9.27}$$

then

$$G^e - G^i \leq c_2 \ \frac{\varepsilon}{l} , \tag{3.9.28}$$

where $c_2 = \text{const} = c_2(\partial D^*) > 0$ *and* $l = \operatorname{dist}(\partial D^*, s^e)$ *is the distance between the boundaries* ∂D^* *and* s^e.

This assertion is a corollary to Theorem 3.8.3 and the relevant properties of harmonic functions.

Corollary 3.9.2 *Let all the conditions of Theorem* 3.9.1 *or Corollary* 3.9.1 *hold. If* $\rho_\alpha \equiv 1$ *and* Ω_α, $\alpha = 1, 2$, *are convex domains, then the estimate*

$$\operatorname{mes} s^e - \operatorname{mes} s^i \leq c_4 \max_{x \in \partial D^*} \left| \frac{\partial v(x)}{\partial \nu_x} \right|$$

is valid or, what amounts to the same,

$$\operatorname{mes} s^e - \operatorname{mes} s^i \leq c_5 \ \frac{\varepsilon}{l} .$$

Chapter 4

Inverse Problems in Dynamics of Viscous Incompressible Fluid

4.1 Preliminaries

We begin our exposition with a brief survey of the facts regarding the solvability of direct problems for linearized and nonlinear Navier–Stokes equations. It is worth noting here that we quote merely the theorems which will serve in the sequel as a necessary background for special investigations of plenty of inverse problems.

We give a number of definitions for functional spaces which will be encountered throughout the entire chapter. Let Ω be a bounded domain in the space $\mathbf{R}^n$ with boundary $\partial\Omega$ of class C^2.

The space $\mathbf{L}_2(\Omega)$ consists of all vector functions $\mathbf{v}(x)$ with components $v_i(x) \in L_2(\Omega)$, $i = 1, 2, \dots, n$, and is equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_{2,\Omega} = \int_\Omega \mathbf{u}(x) \cdot \mathbf{v}(x)\, dx\,, \qquad \mathbf{u}(x) \cdot \mathbf{v}(x) = \sum_{i=1}^{n} u_i(x)\, v_i(x)\,.$$

The associated norm on that space is defined by

$$\|\mathbf{v}\|_{2,\Omega} = \sqrt{(\mathbf{v}, \mathbf{v})_{2,\Omega}}\,.$$

The space $\mathbf{W}_2^1(\Omega)$ comprises all vector functions $\mathbf{v}$ with components $v_i(x) \in W_2^1(\Omega)$, $i = 1, 2, \dots, n$, and the norm

$$\| \mathbf{v} \|_{2,\Omega}^{(1)} = \sqrt{\int_\Omega |\mathbf{v}| + |\mathbf{v}_x|^2 \, dx}\,, \qquad |\mathbf{v}_x| = \sqrt{\sum_{i,j=1}^{n} \left(\frac{\partial v_i}{\partial x_j} \right)^2}\,.$$

Let $\overset{\circ}{\mathbf{J}}(\Omega)$ be the closure in the $\mathbf{L}_2(\Omega)$-norm of the set of all smooth solenoidal and compactly supported vectors. It is well-known that $\mathbf{L}_2(\Omega) = \overset{\circ}{\mathbf{J}}(\Omega) \oplus \mathbf{G}(\Omega)$, where $\mathbf{G}(\Omega)$ is the orthogonal complement of $\overset{\circ}{\mathbf{J}}(\Omega)$ to $\mathbf{L}_2(\Omega)$. The space $\mathbf{G}(\Omega)$ consists of all vectors having the form $\nabla \psi$, where ψ is a single-valued, measurable and square summable function, whose first derivatives belong to the space $L_2(\Omega)$. The norms on the functional spaces $\mathbf{L}_q(\Omega)$, $\mathbf{W}_l^p(\Omega)$, $\mathbf{L}_{q,r}(Q_T)$ and $\mathbf{W}_p^{l_1, l_2}(Q_T)$ of vector functions are defined in the usual way. As before, it is convenient to introduce the notations

$$\| \mathbf{u} \|_{q,\Omega} = \| \, |\mathbf{u}_x| \, \|_{q,\Omega}\,,$$

$$|\mathbf{u}_{xx}| = \sqrt{\sum_{i,j=1}^{n} \left(\frac{\partial^2 u_i}{\partial x_k \, \partial x_j} \right)^2}\,,$$

$$\| \mathbf{u}_{xx} \|_{q,\Omega} = \| \, |\mathbf{u}_{xx}| \, \|_{q,\Omega}\,.$$

Let $\mathbf{W}_{2,0}^{2,1}(Q_T)$ be a subspace of Sobolev's space $\mathbf{W}_2^{2,1}(Q_T)$, which consists of all vector functions vanishing on $S_T \equiv \partial\Omega \times [0, T]$. The space $\overset{\circ}{\mathbf{J}}(Q_T)$ comprises all vectors from $\mathbf{L}_2(Q_T)$ that belong to $\overset{\circ}{\mathbf{J}}(\Omega)$ for almost all $t \in [0, T]$.

We now focus our attention on the formulations of some results concerning the solvability of stationary and nonstationary **direct problems in hydrodynamics**.

The following assertion is valid for the linear stationary direct problem of finding a pair of the functions $\{\mathbf{u}, \nabla q\}$, which satisfy in the domain Ω the system of equations

$$(4.1.1) \qquad -\nu \Delta \mathbf{u} = -\nabla q + \mathbf{h}\,, \qquad \operatorname{div} \mathbf{u} = 0\,,$$

and the boundary condition

$$(4.1.2) \qquad \mathbf{u}(x) = 0\,, \qquad x \in \partial\Omega\,.$$

Theorem 4.1.1 (Temam (1979), p. 37) *For* $\mathbf{h} \in \mathbf{L}_m(\Omega)$, $m > 1$, *problem* (4.1.1)-(4.1.2) *has a solution* $\mathbf{u} \in \mathbf{W}^2_m(\Omega)$, $\nabla q \in \mathbf{L}_m(\Omega)$, *this solution is unique in the indicated class of functions and the estimate*

$$\|\mathbf{u}\|^{(2)}_{m,\Omega} + \|\nabla q\|_{m,\Omega} \le c\,\|\mathbf{h}\|_{m,\Omega}$$

is valid with constant c depending only on m *and* Ω.

Consider now the linear nonstationary direct problem of recovering a pair of the functions $\{\mathbf{v}, \nabla p\}$, which satisfy in Q_T the system of **linearized Navier–Stokes equations** along with the **incompressibility equation**

$$\text{(4.1.3)} \quad \mathbf{v}_t - \nu\Delta\mathbf{v} = -\nabla p + \mathbf{F}_1, \quad \operatorname{div}\mathbf{v} = 0, \quad (x,t) \in Q_T \equiv \Omega \times (0,T),$$

the initial condition

$$\text{(4.1.4)} \qquad \mathbf{v}(x,0) = \mathbf{a}_1(x), \qquad x \in \Omega,$$

and the boundary condition

$$\text{(4.1.5)} \qquad \mathbf{v}(x,t) = 0, \qquad (x,t) \in S_T \equiv \partial\Omega \times [0,T],$$

where the vector functions $\mathbf{F}_1$, $\mathbf{a}_1$ and the coefficient ν are given.

Theorem 4.1.2 (Ladyzhenskaya (1970), p. 109) *For* $\mathbf{F}_1 \in \mathbf{L}_2(Q_T)$ *and* $\mathbf{a}_1 \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$ *a solution* $\mathbf{v} \in \mathbf{W}^{2,1}_{2,0}(Q_T) \cap \overset{\circ}{\mathbf{J}}(Q_T)$, $\nabla p \in \mathbf{L}_2(Q_T)$ *of the direct problem* (4.1.3)-(4.1.5) *exists and is unique and the following estimate is true:*

$$\text{(4.1.6)} \qquad \|\mathbf{v}\|^{(2,1)}_{2,Q_T} + \|\nabla p\|_{2,Q_T} \le \tilde{C}\Big(\|\mathbf{F}_1\|_{2,Q_T} + \|\mathbf{a}_1\|^{(1)}_{2,\Omega}\Big).$$

Theorem 4.1.3 (Ladyzhenskaya (1970), p. 112) *For* $\mathbf{a}_1 \in \overset{\circ}{\mathbf{J}}(\Omega)$ *and* $\mathbf{F}_1 \in \mathbf{L}_{2,1}(Q_T)$ *there always exists a solution* $\mathbf{v}$ *of problem* (4.1.3)-(4.1.5) *that belongs to* $\overset{\circ}{\mathbf{V}}{}^{1,0}_2(Q_T)$. *This solution is unique in the indicated class of functions and the estimate*

$$\text{(4.1.7)} \quad \|\mathbf{v}(\cdot,t)\|^2_{2,\Omega} + 2\nu\int_0^t \|\mathbf{v}_x(\cdot,\tau)\|^2_{2,\Omega}\,d\tau$$

$$\le 2\|\mathbf{a}_1\|^2_{2,\Omega} + 3\Big(\int_0^t \|\mathbf{F}_1(\cdot,\tau)\|_{2,\Omega}\,d\tau\Big)^2, \qquad 0 \le t \le T,$$

is valid.

Here $\overset{\circ}{\mathbf{V}}{}_2^{1,0}(Q_T)$ is a Banach space which is the closure in the norm

$$\uparrow \mathbf{v} \uparrow_{\overset{\circ}{\mathbf{V}}{}_2^{1,0}(Q_T)} = \max_{t\in[0,T]} \| \mathbf{v}(\,\cdot\,,t)\|_{2,\Omega} + \|\mathbf{v}_x\|_{2,Q_T}$$

of the set of all smooth vectors vanishing in a neighborhood of S_T.

Of special interest is the direct problem of finding a pair of the functions $\{\mathbf{v}, \nabla p\}$, which satisfy in Q_T the **nonlinear time-dependent Navier–Stokes system**

$$(4.1.8)\quad \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v},\nabla)\mathbf{v} = -\nabla p + \mathbf{F}_2, \quad \operatorname{div}\mathbf{v} = 0, \quad (x,t)\in Q_T,$$

the initial condition

$$(4.1.9)\qquad \mathbf{v}(x,0) = \mathbf{a}_2(x), \qquad x\in\Omega,$$

and the boundary condition

$$(4.1.10)\qquad \mathbf{v}(x,t) = 0, \qquad (x,t)\in S_T,$$

where the functions $\mathbf{F}_2$, $\mathbf{a}_2$ and the coefficient ν are known in advance and

$$(\mathbf{v},\nabla)\mathbf{v} = \sum_{i=1}^{n} v_i \frac{\partial\mathbf{v}}{\partial x_i}.$$

We begin by considering the case of a **three-dimensional flow**, that is, the case when $\Omega\subset\mathbf{R}^3$.

Theorem 4.1.4 (Ladyzhenskaya (1970), p. 202) *Let* $\Omega\subset\mathbf{R}^3$, $\mathbf{F}_2\in\mathbf{L}_2(Q_T)$, $(\mathbf{F}_2)_t\in\mathbf{L}_2(Q_T)$, $\mathbf{a}_2\in\mathbf{W}_2^2(\Omega)\cap\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)\cap\overset{\circ}{\mathbf{J}}(\Omega)$ *and let the inequality*

$$(4.1.11)\qquad M_4 = \nu - \left(\tfrac{4}{3}\right)^{3/4}\Big[c_1(\Omega)\,\nu^{-1}M_1(M_2+M_3)\Big]^{1/2} > 0$$

be valid with

$$M_1 = \|\mathbf{a}_2\|_{2,\Omega} + \int_0^T \|\mathbf{F}_2(\,\cdot\,,t)\|_{2,\Omega}\,dt,$$

$$M_2 = \| P_{\mathbf{J}}\left[\nu\Delta\mathbf{a}_2 - (\mathbf{a}_2,\nabla)\mathbf{a}_2 + \mathbf{F}_2(\,\cdot\,,0)\right]\|_{2,\Omega} + \int_0^T \|(\mathbf{F}_2)_t(\,\cdot\,,t)\|_{2,\Omega}\,dt,$$

$P_{\mathbf{J}}$ being the orthogonal projector of $\mathbf{L}_2(\Omega)$ onto $\overset{\circ}{\mathbf{J}}(\Omega)$ and $c_1(\Omega)$ appearing in the Poincaré–Friedrichs inequality (see (4.2.21) below). Then problem (4.1.8)-(4.1.10) has in Q_T a generalized solution $\mathbf{v}$ such that $|\mathbf{v}|^2$, $\mathbf{v}_x$, $\mathbf{v}_t$, $\mathbf{v}_{tx} \in \mathbf{L}_2(Q_T)$ and $\mathbf{v}_t(\,\cdot\,, t)$ is an element of $\overset{\circ}{\mathbf{J}}(\Omega)$ continuously depending on $t \in [0, T]$ in the weak topology of $\overset{\circ}{\mathbf{J}}(\Omega)$. Moreover, this solution is unique in the indicated class of functions and the function $\mathbf{v}$ satisfies for each $t \in [0, T]$ the estimates

$$(4.1.12)\quad \|\mathbf{v}_x(\,\cdot\,, t)\|_{2,\Omega} \le \left[\nu^{-1} M_1 (M_2 + M_3)\right]^{1/2},$$

$$(4.1.13)\quad \|\mathbf{v}_t(\,\cdot\,, t)\|_{2,\Omega} \le M_2,$$

$$(4.1.14)\quad 2\, M_4 \int\limits_0^t \|\mathbf{v}_{tx}(\,\cdot\,, \tau)\|_{2,\Omega}^2\, d\tau \le \left\| P_{\mathbf{J}}\left[\nu\, \Delta\, \mathbf{a}_2 - (\mathbf{a}_2, \nabla)\, \mathbf{a}_2 + \mathbf{F}_2(\,\cdot\,, 0)\right] \right\|_{2,\Omega}^2 + 2\, M_2^2 .$$

Theorem 4.1.5 (Ladyzhenskaya (1970), p. 209) *Under the conditions of Theorem 4.1.4 the function $\mathbf{v}(\,\cdot\,, t)$ is continuous with respect to $t \in [0, T]$ in the $\mathbf{W}_2^2(\Omega)$-norm, the function $\mathbf{v}_t(\,\cdot\,, t)$ is continuous with respect to $t \in [0, T]$ in the $\mathbf{L}_2(\Omega)$-norm and the following estimate holds:*

$$(4.1.15)\qquad \|\mathbf{v}(\,\cdot\,, t)\|_{2,\Omega}^{(2)} \le M_5, \qquad 0 \le t \le T,$$

where

$$M_5 = c_1^* \Bigg\{ \Big[\sup_{t\in[0,T]} \|\mathbf{F}_2(\,\cdot\,, t)\|_{2,\Omega} + M_2 \Big] \times \Big\{ 1 + c_2^* \Big[\nu^{-1} M_1 (M_2 + M_3)\Big]^{1/2} \Big\} + c_3^* \Big[M_1\, \nu^{-1} (M_2 + M_3) \Big]^{3/2} \Bigg\}$$

and c_1^*, c_2^*, c_3^* are constants depending only on Ω, ν and T.

It is worth noting here that M_5 does not depend on t.

In the case of a **two-dimensional flow**, that is, the case when $\Omega \subset \mathbf{R}^2$ the direct problem for the nonstationary nonlinear system (4.1.8) is, in general, uniquely solvable. This profound result is revealed in the following theorem.

Theorem 4.1.6 (Ladyzhenskaya (1970), p. 198) *Let* $\Omega \subset \mathbf{R}^2$, $\mathbf{F}_2 \in \mathbf{L}_2(Q_T)$, $(\mathbf{F}_2)_t \in \mathbf{L}_2(Q_T)$ *and* $\mathbf{a}_2 \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$. *Then problem* (4.1.8)–(4.1.10) *has in* Q_T *a generalized solution* $\mathbf{v}$ *such that* $|\mathbf{v}|^2, \mathbf{v}_x, \mathbf{v}_t$, $\mathbf{v}_{tx} \in \mathbf{L}_2(Q_T)$ *and* $\mathbf{v}_t(\,\cdot\,, t)$ *is an element of* $\overset{\circ}{\mathbf{J}}(\Omega)$ *continuously depending on* $t \in [0, T]$ *in the weak topology of* $\overset{\circ}{\mathbf{J}}(\Omega)$. *This solution is unique in the indicated class of functions and the function* $\mathbf{v}$ *satisfies for* $0 \le t \le T$ *the estimates*

$$\|\mathbf{v}(\,\cdot\,, t)\|_{2,\Omega} \le \|\mathbf{a}_2\|_{2,\Omega} + \int_0^t \|\mathbf{F}_2(\,\cdot\,, t)\|_{2,\Omega}\, dt\,, \quad 0 \le t \le T, \tag{4.1.16}$$

$$\nu\, \|\mathbf{v}_x\|_{2,Q_T}^2 \le \|\mathbf{a}_2\|_{2,\Omega}^2 + \frac{3}{2}\Big(\int_0^T \|\mathbf{F}_2(\,\cdot\,, t)\|_{2,\Omega}\, dt\Big)^2, \tag{4.1.17}$$

$$\|\mathbf{v}_t(\,\cdot\,, t)\|_{2,\Omega} \le M_2\, \exp\{M_6\, \nu^{-2}\}, \quad 0 \le t \le T, \tag{4.1.18}$$

$$\nu \int_0^t \|\mathbf{v}_{tx}(\,\cdot\,, \tau)\|_{2,\Omega}^2\, d\tau \le M_2^2 \times \Big[1 + 2\exp\{M_6\, \nu^{-2}\} + 2\, M_6\, \nu^{-2} \exp\{2\, M_6\, \nu^{-2}\}\Big], \tag{4.1.19}$$

where

$$M_2 = \|\, P_{\mathbf{J}}\, [\nu\, \Delta\, \mathbf{a}_2 - (\mathbf{a}_2, \nabla)\, \mathbf{a}_2 + \mathbf{F}_2(\,\cdot\,, 0)]\, \|_{2,\Omega} + \int_0^T \|(\mathbf{F}_2)_t(\,\cdot\,, t)\|_{2,\Omega}\, dt,$$

$$M_6 = \tfrac{1}{2}\|\mathbf{a}_2\|_{2,\Omega}^2 + \Big(\|\mathbf{a}_2\|_{2,\Omega} + \int_0^T \|\mathbf{F}_2(\,\cdot\,, t)\|_{2,\Omega}\, dt\Big) \int_0^T \|\mathbf{F}_2(\,\cdot\,, t)\|_{2,\Omega}\, dt.$$

Theorem 4.1.7 *Under the conditions of Theorem* 4.1.6 *the function* $\mathbf{v}(\,\cdot\,, t)$ *is continuous with respect to* $t \in [0, T]$ *in the* $\mathbf{W}_2^2(\Omega)$*-norm, the function* $\mathbf{v}_t(\,\cdot\,, t)$ *is continuous with respect to* $t \in [0, T]$ *in the* $\overset{\circ}{\mathbf{J}}(\Omega)$*-norm* *and the following estimates hold:*

$$\text{(4.1.20)}\qquad \nu \, \| \mathbf{v}_x(\,\cdot\,,t) \|^2_{2,\Omega} \le \Big(\| \mathbf{a}_2 \|_{2,\Omega} + \| \mathbf{F}_2 \|_{2,1,Q_T} \Big) \times \Big(M_2 \exp\{ M_6 \, \nu^{-2} \} + \| \mathbf{F}_2(\,\cdot\,,t) \|_{2,\Omega} \Big), \quad 0 \le t \le T,$$

$$\text{(4.1.21)}\qquad \| \mathbf{v}(\,\cdot\,,t) \|^{(2)}_{2,\Omega} \le M_7, \qquad 0 \le t \le T,$$

where

$$\begin{aligned} M_7 = c_1^* \Big[\sup_{t\in[0,T]} \| \mathbf{F}_2(\,\cdot\,,t) \|_{2,\Omega} + M_2 \exp\{ M_6 \, \nu^{-2} \} \Big] \\ \times \Big\{ 1 + c_2^* \Big(\| \mathbf{a}_2 \|_{2,\Omega} + \| \mathbf{F}_2 \|_{2,1,Q_T} \Big)^{1/2} \\ \times \Big(M_2 \exp\{ M_6 \, \nu^{-2} \} + \sup_{t\in[0,T]} \| \mathbf{F}_2(\,\cdot\,,t) \|_{2,\Omega} \Big)^{1/2} \\ \times \Big[1 + \Big(\| \mathbf{a}_2 \|_{2,\Omega} + \| \mathbf{F}_2 \|_{2,1,Q_T} \Big)^{1/2} \Big] \Big\} \end{aligned}$$

and c_1^*, c_2^* are constants depending only on Ω, ν and T.

The proof of this assertion is similar to that carried out in Theorem 4.1.5. One can see that M_7 does not depend on t.

4.2 Nonstationary linearized system of Navier–Stokes equations: the final overdetermination

The main goal of our studies is to consider the inverse problem with the final overdetermination for nonstationary linearized Navier–Stokes equations. All the methods we develop throughout this section apply equally well to systems which can arise in various approaches to the linearization of nonlinear Navier–Stokes equations. For more a detailed outline of the results obtained we offer the linearization type presented by the system (4.1.3).

Before considering details, we are interested in a common setting of the inverse problem. For this, suppose that an unknown external force function $\mathbf{F}_1(x,t)$ is sought on the basis of indirect measurements. Our approach is connected with a suitably chosen statement of the inverse problem and some restriction on the input data making it possible to treat an initial problem as a well-posed one for which

(i) a solution exists and is unique;
(ii) the solution is stable in the norm of the corresponding space of functions.

Without loss of generality we may assume that the function $\mathbf{v}$ satisfies the homogeneous initial condition (4.1.2) almost everywhere in Ω, that is, we accept $\mathbf{a}_1 = 0$.

Additional information is available here on a solution of the system (4.1.3)–(4.1.5) as the condition of final overdetermination in which the traces of the velocity $\mathbf{v}$ and the pressure gradient ∇p are prescribed at the final moment $t = T$ of the segment $[0, T]$ under consideration.

The vector $\mathbf{F}_1$ is taken to be

$$\mathbf{F}_1 = \mathbf{f}(x)\, g(x, t)\,, \tag{4.2.1}$$

where the vector $\mathbf{f}(x)$ is unknown and the scalar $g(x, t)$ is given.

With these ingredients, we are led to the inverse problem of finding a collection of the vector functions $\{\mathbf{v},\, \nabla p,\, \mathbf{f}\}$, which satisfy the system

$$\begin{aligned} &\mathbf{v}_t - \nu\,\Delta\,\mathbf{v} = -\nabla p + \mathbf{f}(x)\, g(x, t)\,, \\ &\operatorname{div}\, \mathbf{v} = 0\,, \qquad (x, t) \in Q_T\,, \end{aligned} \tag{4.2.2}$$

the initial condition

$$\mathbf{v}(x, 0) = 0\,, \qquad x \in \Omega\,, \tag{4.2.3}$$

the boundary condition

$$\mathbf{v}\,(x, t) = 0\,, \qquad (x, t) \in S_T\,, \tag{4.2.4}$$

and the conditions of **final overdetermination**

$$\mathbf{v}(x, T) = \varphi(x)\,, \qquad \nabla p(x, T) = \nabla\,\psi(x)\,, \qquad x \in \Omega\,, \tag{4.2.5}$$

provided that the functions g, φ, $\nabla\,\psi$ and the coefficient ν are given.

We list the basic trends of further development. As a first step towards the solution of this problem, we are going to derive an operator equation of the second kind for $\mathbf{f}$. If we succeed in showing that the resulting equation is equivalent to the inverse problem posed above, the question of the solvability of the inverse problem and the uniqueness of its solution will reduce to the study of an operator equation of the second kind.

In this regard, it is necessary to give a rigorous definition for a solution of the inverse problem (4.2.2)–(4.2.5).

Definition 4.2.1 *A collection of functions* $\{\mathbf{v}, \nabla p, \mathbf{f}\}$ *is called a generalized solution of the inverse problem* (4.2.2)–(4.2.5) *if* (4.2.2)–(4.2.5) *are fulfilled,* $\mathbf{v} \in \mathbf{W}_{2,0}^{2,1}(Q_T) \cap \overset{\circ}{\mathbf{J}}(Q_T)$, $\mathbf{f} \in \mathbf{L}_2(\Omega)$, $\nabla p(\cdot, t) \in \mathbf{G}(\Omega)$ *for each* t *from the segment* $[0, T]$ *and continuously depends on* t *in the* $\mathbf{L}_2(\Omega)$*-norm on* $[0, T]$.

Furthermore, assume that $g, g_t \in C(\bar{Q}_T)$ and $|g(x, T)| \geq g_T > 0$ for each $x \in \bar{\Omega}$. By relating $\mathbf{f} \in \mathbf{L}_2(\Omega)$ to be fixed we might treat (4.2.2)–(4.2.4) as the system corresponding to the direct problem of finding a pair of the vector functions $\{\mathbf{v}, \nabla p\}$. Theorem 4.1.2 implies that $\{\mathbf{v}, \nabla p\}$ with this property exists and is unique. Observe that the function $\mathbf{v}_t(\cdot, t)$ is continuous in the $\mathbf{L}_2(\Omega)$-norm on the segment $[0, T]$, since

$$(\mathbf{F}_1)_t = \mathbf{f}\, g_t \in \mathbf{L}_2(Q_T)\,.$$

This provides reason enough to appeal in subsequent studies to the linear operator

$$A\colon\ \mathbf{L}_2(\Omega) \mapsto \mathbf{L}_2(\Omega)$$

with the values

$$(A\,\mathbf{f})(x) = \frac{1}{g(x, T)}\, \mathbf{v}_t(x, T)\,, \qquad x \in \Omega\,, \tag{4.2.6}$$

where $\mathbf{v}$ has been found as the function involved in the solution $\{\mathbf{v}, \nabla p\}$ of the system (4.2.2)–(4.2.4) in the sense indicated above and associated with the function $\mathbf{f}$.

We now consider in $\mathbf{L}_2(\Omega)$ a linear operator equation of the second kind for such a function $\mathbf{f}$:

$$\mathbf{f} = A\,\mathbf{f} + \chi\,, \tag{4.2.7}$$

where χ from the space $\mathbf{L}_2(\Omega)$ is given.

The first theorem provides a framework in which the solvability of equation (4.2.7) implies that of the inverse problem (4.2.2)–(4.2.5) and vice versa.

Theorem 4.2.1 *Let*

$$g,\, g_t \in C(\bar{Q}_T)\,, \quad |g(x, T)| \geq g_T > 0 \quad \textit{for} \quad x \in \bar{\Omega}\,,$$

$$\varphi \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \qquad \nabla \psi \in \mathbf{G}(\Omega)\,,$$

and

$$\chi = \frac{1}{g(x,\,T)}\;(-\nu\,\Delta\,\varphi + \nabla\,\psi)\,. \tag{4.2.8}$$

Then the following assertions are valid:

(a) *if the linear equation* (4.2.7) *is uniquely solvable, then so is the inverse problem* (4.2.2)-(4.2.5);

(b) *if there exists a solution of the inverse problem* (4.2.2)-(4.2.5) *and this solution is unique, then the linear equation has a unique solution.*

Proof To prove item a) we assume that (4.2.8) holds and (4.2.7) has a unique solution, say $\mathbf{f}$. Upon substituting $\mathbf{f}$ into (4.2.2) we make use of the system (4.2.2)–(4.2.4) for finding a pair of the functions $\{\mathbf{v},\,\nabla\,p\}$ as the solution of the direct problem corresponding to the external force function $\mathbf{F}_1 = \mathbf{f}(x)\,g(x,\,t)$.

Theorem 4.1.2 yields the existence of a unique pair $\{\mathbf{v},\,\nabla\,p\}$ solving the direct problem at hand. By the same token, $\mathbf{v} \in \mathbf{W}_{2,\,0}^{2,\,1}(Q_T) \cap \overset{\circ}{\mathbf{J}}(Q_T)$ and $\nabla\,p \in \mathbf{L}_2(Q_T)$. As $\mathbf{F}_1$, $(\mathbf{F}_1)_t \in \mathbf{L}_2(Q_T)$, the functions $\mathbf{v}$ and $\nabla\,p$ obey some nice properties such as $\mathbf{v}_t(\,\cdot\,,\,t)$, $\Delta\,\mathbf{v}(\,\cdot\,,\,t) \in \mathbf{L}_2(\Omega)$ and $\nabla\,p(\,\cdot\,,\,t) \in \mathbf{G}(\Omega)$, which fit our purposes. Moreover, $\mathbf{v}_t(\,\cdot\,,\,t)$, $\Delta\,\mathbf{v}(\,\cdot\,,\,t)$ and $\nabla\,p(\,\cdot\,,\,t)$ are continuous in the $\mathbf{L}_2(\Omega)$-norm as the functions of the argument t on the segment $[0,\,T]$.

We claim that $\mathbf{v}$ and $\nabla\,p$ so constructed are subject to the conditions of the final overdetermination (4.2.5). Indeed,

$$\varphi_1 \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \qquad \nabla\,\psi_1 \in \mathbf{L}_2(\Omega)$$

and

$$\mathbf{v}_t(x,\,T) - \nu\,\Delta\,\varphi_1 \;=\; -\nabla\,\psi_1 + \mathbf{f}(x)\;g(x,\,T) \tag{4.2.9}$$

if we agree to consider

$$\mathbf{v}(x,\,T) = \varphi_1(x)$$

and

$$\nabla\,p(x,\,T) = \nabla\,\psi_1(x)\,.$$

On the other hand, (4.2.7)–(4.2.8) together imply that

$$g(x,\,T)\;A\;\mathbf{f} - \nu\,\Delta\,\varphi \;=\; -\nabla\,\psi + \mathbf{f}(x)\;g(x,\,T)\,. \tag{4.2.10}$$

By successively applying (4.2.6), (4.2.9) and (4.2.10) it is not difficult to verify that the functions $\varphi - \varphi_1$ and $\nabla(\psi - \psi_1)$ satisfy the system of stationary linear equations

$$(4.2.11)\quad -\nu\,\Delta\,(\varphi - \varphi_1) = -\nabla\,(\psi - \psi_1), \quad \operatorname{div}\,(\varphi - \varphi_1) = 0\,, \quad x \in \Omega\,,$$

supplied by the boundary condition

$$(4.2.12)\qquad (\varphi - \varphi_1)\,(x) = 0\,, \qquad x \in \partial\Omega\,.$$

Taking the scalar product of both sides of the first equation (4.2.11) and $\varphi - \varphi_1$ in $\mathbf{L}_2(\Omega)$, we obtain $\nu\,\|\,(\varphi - \varphi_1)_x\,\|_{2,\Omega}^2 = 0$, what means that $\varphi = \varphi_1$ and $\nabla\,\psi = \nabla\,\psi_1$ almost everywhere in Ω. This provides support for decision-making that the inverse problem (4.2.2)–(4.2.5) is solvable.

Assuming that the solution $\{\mathbf{v},\ \nabla\,p,\ \mathbf{f}\}$ of the inverse problem (4.2.2)–(4.2.5) is nonunique, it is plain to show the nonuniqueness of the solution of (4.2.7) when χ happens to be of the form (4.2.8). But this contradicts the initial assumption.

We proceed to prove item b). Let the inverse problem (4.2.2)–(4.2.5) have a unique solution, say $\{\mathbf{v},\ \nabla\,p,\ \mathbf{f}\}$. Since $\mathbf{F}_1 = \mathbf{f}\,g$ and $(\mathbf{F}_1)_t = \mathbf{f}\,g_t$ belong to the space $\mathbf{L}_2(Q_T)$, we shall need as yet the smoothness properties of $\mathbf{v}_t(\,\cdot\,,\,t)$, $\Delta\,\mathbf{v}(\,\cdot\,,\,t)$ and $\nabla\,p(\,\cdot\,,\,t)$ arguing as in the proof of item a).

When system (4.2.2) is considered at $t = T$, we have

$$\mathbf{v}_t(x,\,T) - \nu\,\Delta\,\mathbf{v}(x,\,T) = -\nabla\,p(x,\,T) + \mathbf{f}(x)\;g(x,\,T)$$

and, because of (4.2.5),

$$\mathbf{v}_t(x,\,T) - \nu\,\Delta\,\varphi = -\nabla\,\psi + \mathbf{f}(x)\;g(x,\,T)\,.$$

From (4.2.6) it follows that $A\,\mathbf{f} = \mathbf{f} + \chi$, where

$$\chi = \frac{1}{g(x,\,T)}\;\big(\nu\,\Delta\,\varphi + \nabla\,\psi\big)\,.$$

With this relation established, we can deduce that there exists a solution of (4.2.7). On the contrary, let (4.2.7) will have more than one solution. In just the same way as we did in item a) there is no difficulty to find that problem (4.2.2)–(4.2.5) can have more than one solution. But this contradicts the initial assumption and thereby proves the assertion of the theorem. ■

In the following theorem we impose rather mild restrictions on the input data and prove on their basis the existence and uniqueness of the solution of problem (4.2.2)–(4.2.5).

Theorem 4.2.2 *Let*

$$g,\, g_t \in C(\bar{Q}_T)\,, \quad |\, g(x,\, T)\,| \geq g_T > 0 \quad for \quad x \in \bar{\Omega}\,,$$

$$\varphi \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \qquad \nabla \psi \in \mathbf{G}(\Omega)$$

and let

$$m_1 < 1\,, \tag{4.2.13}$$

where

$$m_1 = \frac{1}{g_T} \Big[\sup_{x\in\Omega} \; |\, g(x,\, 0)\,| \, \exp\{ -\nu\, T / c_1(\Omega) \}$$

$$+ \int_0^T - \exp\{ -\nu\, (T-t)\, / c_1(\Omega) \} \sup_{x\in\Omega} \; |\, g_t(x,\, t)\,|\; dt \Big]$$

and $c_1(\Omega)$ *is the constant from the Poincaré–Friedrichs inequality (see* (4.2.21) *below). Then there exists a solution of the inverse problem* (4.2.2)–(4.2.5), *this solution is unique in the indicated class of functions and the estimates*

$$\| \mathbf{f} \|_{2,\,\Omega} \;\leq\; \frac{1}{(1-m_1)\, g_T} \; \| \; -\nu\, \Delta\, \varphi + \nabla\, \psi \, \|_{2,\,\Omega}\,, \tag{4.2.14}$$

$$\| \mathbf{v} \|_{2,\,Q_T}^{(2,\,1)} + \| \nabla\, p \, \|_{2,\,Q_T} \;\leq\; \frac{\tilde{c}}{(1-m_1)\, g_T} \; \| -\nu\, \Delta\, \varphi + \nabla\, \psi \, \|_{2,\,\Omega} \times \Big(\int_0^T \sup_{x\in\Omega} \; |\, g(x,\, t)\, |^2 \; dt \Big)^{1/2} \tag{4.2.15}$$

are valid with constant $\tilde{c}$ *from* (4.1.6).

Proof Before we undertake the proof, it will be useful to reveal some remarkable properties of the direct problem (4.1.3)–(4.1.5), due to which an important a priori estimate for the operator A is obtained.

If the conditions of Theorem 4.1.2 are supplied by

$$(\mathbf{F}_1)_t \in \mathbf{L}_{2,\,1}(Q_T)\,, \qquad \mathbf{a}_1 \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,,$$

then $\mathbf{v}$ will possess extra differential properties. When this is the case, we know from Ladyzhenskaya (1966) that the derivative $\mathbf{v}_t \in \overset{\circ}{\mathbf{V}}{}_2^{1,\,0}(Q_T)$ gives a solution of the system

$$(4.2.16)\qquad \mathbf{w}_t - \nu\,\Delta\,\mathbf{w} = \nabla\, q + \mathbf{F}^*\,, \quad \operatorname{div}\,\mathbf{w} = 0\,, \quad (x,\,t) \in Q_T\,,$$

$$(4.2.17)\qquad \mathbf{w}(x,0) = \mathbf{a}^*(x)\,, \qquad x \in \Omega\,,$$

$$(4.2.18)\qquad \mathbf{w}(x,t) = 0\,, \qquad x \in S_T\,,$$

which is satisfied by $\mathbf{v}_t$ in the sense of the integral identity:

$$\int_0^t \int_\Omega (-\mathbf{w}\,\cdot\,\Phi_\tau + \nu\,\mathbf{w}_x\,\cdot\,\Phi_x)\;dx\;d\tau$$

$$+ \int_\Omega \mathbf{w}(x,t)\,\cdot\,\Phi(x,t)\;dx - \int_\Omega \mathbf{a}^*\,\cdot\,\Phi(x,0)\;dx$$

$$= \int_0^t \int_\Omega \mathbf{F}^*\,\cdot\,\Phi\;dx\;d\tau\,, \qquad 0 \le t \le T\,,$$

$$\forall\,\Phi\colon\ \Phi \in \mathbf{W}_2^{1,\,1}(Q_T) \bigcap \overset{\circ}{\mathbf{J}}(Q_T)\,, \qquad \Phi(x,\,t) = 0\,, \qquad (x,\,t) \in S_T\,.$$

Here we accepted

$$\mathbf{F}^* = (\mathbf{F}_1)_t\,, \qquad \mathbf{a}^* = P_{\mathbf{J}}(\nu\,\Delta\,\mathbf{a}_1 + \mathbf{F}_1(x,0))$$

and $P_{\mathbf{J}}$ is the orthogonal projector of $\mathbf{L}_2(\Omega)$ onto $\overset{\circ}{\mathbf{J}}(\Omega)$.

Moreover, any solution $\mathbf{w}$ of problem (4.2.16)–(4.2.18) from $\overset{\circ}{\mathbf{V}}{}_2^{1,\,0}(Q_T)$ possesses the generalized derivatives $\mathbf{w}_t$ and $\mathbf{w}_{x_i x_j}$, which are square summable on $\Omega \times [\varepsilon,\,T]$ for any ε from the interval $(0,\,T)$ and the derivatives $\mathbf{w}_{x_i}(\,\cdot\,,\,t)$ are elements of $\mathbf{L}_2(\Omega)$ continuously depending on t in the $\mathbf{L}_2(\Omega)$-norm for all $t \in [\varepsilon,\,T]$.

Let ε be an arbitrary fixed number from $(0,\,T)$. In the light of differential properties of solutions to (4.2.16)–(4.2.18) we establish with the aid of the preceding integral identity the **energy relation**

$$(4.2.19)\qquad \frac{1}{2}\,\frac{d}{dt}\;\|\,\mathbf{w}(\,\cdot\,,\,t)\,\|_{2,\,\Omega}^2 + \nu\,\|\,\mathbf{w}_x(\,\cdot\,,\,t)\,\|_{2,\,\Omega}^2$$

$$= \int_\Omega g_t(x,t)\;\mathbf{f}(x)\,\cdot\,\mathbf{w}(x,t)\;dx\,, \qquad 0 < \varepsilon \le t \le T\,,$$

from which it follows for $0 < \varepsilon \le t \le T$ that

$$(4.2.20)\quad \frac{d}{dt} \| \mathbf{w}(\cdot, t) \|_{2,\Omega} + \frac{\nu}{c_1(\Omega)} \| \mathbf{w}(\cdot, t) \|_{2,\Omega} \le \| \mathbf{f}(\cdot)\, g_t(\cdot, t) \|_{2,\Omega} .$$

Here we used also the Poincaré–Friedrichs inequality

$$(4.2.21)\qquad \| \mathbf{u} \|_{2,\Omega}^2 \le c_1(\Omega)\, \| \mathbf{u}_x \|_{2,\Omega}^2 ,$$

which is valid for any $\mathbf{u} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ (see Temam (1979), Chapter 1, (1.9)) and a constant $c_1(\Omega)$ depending solely on Ω and bounded by the value $4\,(\operatorname{diam}\Omega)^2$.

Multiplying both sides of (4.2.20) by $\exp\{-\nu\,(T-t)/c_1(\Omega)$, integrating the resulting expressions with respect to t from ε to T and letting $\varepsilon \to 0$, we deduce that any solution of the system (4.2.16)–(4.2.18) satisfies the inequality

$$(4.2.22)\quad \| \mathbf{w}(\cdot, T) \|_{2,\Omega} \le \| \mathbf{a}^* \|_{2,\Omega} \exp\{-\nu T / c_1(\Omega)\} + \int_0^T \| \mathbf{f}(\cdot) g_t(\cdot, t) \|_{2,\Omega} \exp\{-\nu\,(T-t)/c_1(\Omega)\}\, dt .$$

Since $v_t \in \overset{\circ}{\mathbf{V}}{}_2^{1,0}(Q_T)$ solves problem (4.2.16)–(4.2.18), we thus have

$$\mathbf{w}(x, T) = \mathbf{v}_t(x, T) , \qquad \mathbf{a}^* = P_{\mathrm{J}}(\mathbf{f}(x)\, g(x, 0)) .$$

Therefore, (4.2.22) implies that the linear operator A specified by (4.2.6) is bounded and admits the estimate

$$(4.2.23)\qquad \| A\, \mathbf{f} \|_{2,\Omega} \le m_1 \| \mathbf{f} \|_{2,\Omega} , \qquad \mathbf{f} \in \mathbf{L}_2(\Omega) ,$$

where m_1 is taken from (4.2.13).

As $m_1 < 1$, the linear operator equation (4.2.7) has a unique solution for any χ from the space $\mathbf{L}_2(\Omega)$ and, in particular, we might agree with

$$\chi = (-\nu\, \Delta\, \varphi + \nabla\, \psi) / g(x, T) ,$$

for which (4.2.14) holds true and Theorem 4.2.1 implies the existence and uniqueness of the solution $\{\mathbf{v},\, \nabla p,\, \mathbf{f}\}$ of the inverse problem (4.2.2)–(4.2.5). Estimate (4.2.15) immediately follows from (4.1.6) and (4.2.14). This completes the proof of the theorem. ■

To illustrate the results obtained it is worth noting three things.

Remark 4.2.1 Suppose that under the conditions of Theorem 4.2.2 the function g depends only on t under the constraints

$$g,\, g' \in C([0,T])\,, \qquad g(t) \geq 0\,, \qquad g'(t) \geq 0\,, \qquad g(T) \neq 0\,.$$

In this case $m_1 < 1$ for any $T > 0$ and Theorem 4.2.2 turns out to be of global character. That is to say, the inverse problem (4.2.2)–(4.2.5) with these input data is uniquely solvable for any T, $0 < T < \infty$.

Remark 4.2.2 If

$$g = g(t)\,, \qquad g\,,\, g' \in C([0,T])\,, \qquad g(T) \neq 0\,,$$

then the inverse problem (4.2.2)–(4.2.5) can be investigated by the method of separating variables by means of which it is plain to expand its solution in the Fourier series with respect to the eigenfunctions of the **Stokes operator** $\nu\, P_{\mathbf{J}}\, \Delta$. If this happens, the function $\mathbf{f}$ is sought in the space $\mathbf{L}_2(\Omega)$ assuming that both components $P_{\mathbf{J}}\,\mathbf{f}$ and $P_{\mathbf{G}}\,\mathbf{f}$ in the subspaces $\overset{\circ}{\mathbf{J}}(\Omega)$ and $\mathbf{G}(\Omega)$, respectively, are, generally speaking, nontrivial (recall that $\mathbf{L}_2(\Omega) = \overset{\circ}{\mathbf{J}}(\Omega) \oplus \mathbf{G}(\Omega)$).

Remark 4.2.3 Once the function g depends only on t, it will be possible to impose the condition of the final overdetermination (4.2.5) omitting the information about the final value of the pressure gradient $\nabla\, p(x,T)$. In other words, the final overdetermination contains $\mathbf{v}(x,\, T) = \varphi(x)$ and no more. However, in this case we look for the function $\mathbf{f}$ in the space $\overset{\circ}{\mathbf{J}}(\Omega)$.

In conclusion an example is given as one possible application.

Example 4.2.1 Consider the inverse problem of determining a collection of functions $\{\mathbf{v},\, \nabla\, p,\, \mathbf{f}\}$, which satisfy the set of relations

$$\mathbf{v}_t - \nu\,\Delta\,\mathbf{v} = -\nabla\, p + \mathbf{f}(x)\,, \quad \operatorname{div}\,\mathbf{v} = 0\,, \qquad (x,\, t) \in Q_T\,, \tag{4.2.24}$$

$$\mathbf{v}(x,0) = 0\,, \qquad x \in \Omega\,, \tag{4.2.25}$$

$$\mathbf{v}\,(x,\, t) = 0\,, \qquad (x,\, t) \in S_T\,, \tag{4.2.26}$$

$$\mathbf{v}(x,\, T) = \varphi(x)\,, \qquad x \in \Omega\,. \tag{4.2.27}$$

In the case where $g(x, t) \equiv 1$ one can see that $m_1 < 1$. Therefore, Theorem 4.2.2 implies that there exists a unique solution $\{\mathbf{v}, \nabla p, \mathbf{f}\}$, which is of global character. The condition of the final overdetermination (4.2.27) contains only the value of $\mathbf{v}$ at the final moment $t = T$. Therefore, according to Remark 4.2.3, the function $\mathbf{f}$ should be sought in $\overset{\circ}{\mathbf{J}}(\Omega)$. We employ the method of separating variables that provides a powerful tool for finding a unique solution of the inverse problem (4.2.24)–(4.2.27) in an explicit form. It is well-known that all the eigenvalues of the Stokes operator are nonnegative, have finite multiplicity and tend to $-\infty$. The eigenfunctions of the Stokes operator $\{\mathbf{X}_k(x)\}_{k=1}^{\infty}$ constitute a complete and orthogonal system in the metrics of the spaces $\overset{\circ}{\mathbf{J}}(\Omega)$ and $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$. In the framework of the method of separating variables for the system (4.2.24)–(4.2.26) we thus have

$$\mathbf{v}(x,t) = \sum_{k=1}^{\infty} \mathbf{X}_k(x) \int_0^t f_k \exp\{-\lambda_k (t-\tau)\}\, d\tau \tag{4.2.28}$$

with

$$f_k = \int_\Omega \mathbf{f}(x) \cdot \mathbf{X}_k(x)\, dx .$$

With the aid of (4.2.27) and (4.2.28) we derive the expansion

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \Big[1 - \exp\{-\lambda_k T\} \Big] f_k\, \mathbf{X}_k(x), \tag{4.2.29}$$

which implies that

$$f_k = \lambda_k\, \varphi_k\, (1 - \exp\{-\lambda_k T\})^{-1}, \tag{4.2.30}$$

where

$$\varphi_k = \int_\Omega \varphi(x) \cdot \mathbf{X}_k(x)\, dx .$$

Consequently, using (4.2.30) we find the formula

$$\mathbf{f}(x) = \sum_{k=1}^{\infty} \frac{\lambda_k}{1 - \exp\{-\lambda_k T\}}\, \varphi_k\, \mathbf{X}_k(x),$$

thereby representing the function $\mathbf{f}$ only in terms of the input data of the inverse problem (4.2.24)–(4.2.27). The function $\mathbf{v}$ admits for now expansion (4.2.28), so that the pressure gradient ∇p can be found from (4.2.24) by merely inserting $\mathbf{v}$ and $\mathbf{f}$ both.

More a detailed exploration of the properties of the operator A allows us to establish the following result.

Theorem 4.2.3 *Let g, $g_t \in C(\bar{Q}_T)$ and $|g(x, T)| \geq g_T > 0$ for $x \in \bar{\Omega}$. Then the operator A is completely continuous on the space $\mathbf{L}_2(\Omega)$.*

Proof We again appeal to the differential properties of the solution $\mathbf{w}$ of problem (4.2.16)–(4.2.18) arguing as in the proof of Theorem 4.2.2. As usual, an arbitrary number ε from $(0, T)$ is considered to be fixed. As far as $\| \mathbf{w}_x(\cdot, t) \|_{2,\Omega}$ is continuous on the segment $[\varepsilon, T]$, there is $\tau^* \in [\varepsilon, T]$ such that

$$
(4.2.31) \qquad \int_{\varepsilon}^{T} \| \mathbf{w}_x(\cdot, t) \|_{2,\Omega}^2 \, dt = (T - \varepsilon) \, \| \mathbf{w}_x(\cdot, \tau^*) \|_{2,\Omega}^2 \,.
$$

By means of the element $\mathbf{F}^* = \mathbf{f} \, g_t$ from the space $\mathbf{L}_2(Q_T)$ it is straightforward to verify that the system (4.2.16) implies that

$$
(4.2.32) \qquad \int_{\tau^*}^{T} \int_{\Omega} \Big| \mathbf{w}_t - \nu \, P_{\mathrm{J}} \, \Delta \, \mathbf{w} \Big|^2 \, dx \, dt = \int_{\tau^*}^{T} \int_{\Omega} \Big| P_{\mathrm{J}} \, \mathbf{F}^* \Big|^2 \, dx \, dt \,.
$$

With the aid of the equality

$$
\int_{\tau^*}^{T} \int_{\Omega} \mathbf{w}_t \cdot \nu \, P_{\mathrm{J}} \Delta \mathbf{w} \, dx \, dt = -\frac{\nu}{2} \int_{\tau^*}^{T} \frac{d}{dt} \, \| \mathbf{w}_x(\cdot, t) \|_{2,\Omega}^2 \, dt
$$

it is reasonable to recast (4.2.32) as

$$
(4.2.33) \quad \nu \, \| \mathbf{w}_x(\cdot, T) \|_{2,\Omega}^2 + \int_{\tau^*}^{T} \int_{\Omega} \Big(| \mathbf{w}_t |^2 + | \nu \, P_{\mathrm{J}} \, \Delta \, \mathbf{w} |^2 \Big) \, dx \, dt
$$

$$
= \int_{\tau^*}^{T} \int_{\Omega} | P_{\mathrm{J}} \, \mathbf{F}^* |^2 \, dx \, dt + \nu \, \| \mathbf{w}_x(\cdot, \tau^*) \|_{2,\Omega}^2 \,.
$$

From (4.2.31) and (4.2.33) we derive the inequality

$$
(4.2.34) \qquad \nu \, \| \mathbf{w}_x(\cdot, T) \|_{2,\Omega}^2 \leq \| \mathbf{F}^* \|_{2, Q_T}^2 + \frac{\nu}{T - \varepsilon} \, \| \mathbf{w}_x \|_{2, Q_T}^2 \,.
$$

As stated in Ladyzhenskaya (1970, p. 113]), (4.2.19) implies the estimate

$$
(4.2.35) \qquad \nu \, \| \mathbf{w}_x \|_{2, Q_T}^2 \leq \| \mathbf{a}^* \|_{2,\Omega}^2 + \frac{3}{2} \left(\int_{0}^{T} \| \mathbf{F}^*(\cdot, t) \|_{2,\Omega}^2 \, dt \right)^2
$$

with $\mathbf{F}^* = \mathbf{f}(x)\, g_t(x,t)$, $\mathbf{a}^* = P_J(\mathbf{f}(x)\, g(x,0))$ and $\mathbf{w}(\,\cdot\,,T) = \mathbf{v}_t(\,\cdot\,,T)$. Using (4.2.34)–(4.2.35) behind we establish the relation

$$\| \mathbf{v}_{tx}(\,\cdot\,, T) \|_{2,\Omega}^2 \le m_2 \| \mathbf{f} \|_{2,\Omega}^2 , \tag{4.2.36}$$

yielding

$$\| (A\mathbf{f})_x \|_{2,\Omega}^2 \le m_3 \| \mathbf{f} \|_{2,\Omega} , \tag{4.2.37}$$

where the constants m_2 and m_3 are independent of $\mathbf{f}$. On the other hand, the operator A acts, in fact, from $\mathbf{L}_2(\Omega)$ into $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$. In view of this, Rellich's theorem on compactness of the imbedding $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \subset \mathbf{L}_2(\Omega)$ yields that the linear operator A is completely continuous on $\mathbf{L}_2(\Omega)$, thereby proving the assertion of the theorem. ■

Corollary 4.2.1 *Under the conditions of Theorem* 4.2.3 *the Fredholm alternative is true for equation* (4.2.7).

It is to be hoped that this result adds interest and aim in understanding one specific property of the inverse problem (4.2.2)–(4.2.5). It turns out that under certain restrictions on the input data the uniqueness of the solution of the inverse problem at hand implies its existence and stability. This property is known as the **Fredholm-type property** of the inverse problem and is much involved in subsequent reasoning.

Theorem 4.2.4 *Let*

$$g,\, g_t \in C(\bar{Q}_T)\,, \quad | g(x, T) | \ge g_T > 0 \quad \textit{for} \quad x \in \bar{\Omega}\,,$$

$$\varphi \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)\,, \qquad \nabla \psi \in \mathbf{G}(\Omega)\,.$$

Then the following assertions are valid:

(a) *if the linear homogeneous equation $A\mathbf{f} = \mathbf{f}$ admits a trivial solution only, then the inverse problem* (4.2.2)–(4.2.5) *has a solution and this solution is unique in the indicated class of functions;*

(b) *if the uniqueness theorem for the inverse problem* (4.2.2)–(4.2.5) *holds, then there exists a solution of the inverse problem* (4.2.2)–(4.2.5) *and this solution is unique.*

The proof of Theorem 4.2.4 follows immediately from Theorem 4.2.1 and Corollary 4.2.1.

4.3 Nonstationary linearized system of Navier–Stokes equations: the integral overdetermination

We are much interested in the inverse problem involving the nonstationary linearized system of Navier–Stokes equations when the overdetermination condition is given in a certain integral form.

As in Subsection 4.2 an unknown vector function of the external force $\mathbf{F}_1$ is sought via the measurement of its indirect indications: the velocity of the flow $\mathbf{v}$ and (or) the pressure gradient ∇p. In addition, $\mathbf{F}_1$ is taken to be

$$\mathbf{F}_1 = f(t)\ \mathrm{g}(x, t)\,, \tag{4.3.1}$$

where $f(t)$ is unknown, while $\mathrm{g}(x, t)$ is given.

With these assumptions, we may set up the inverse problem in which it is required to find a triple of the functions $\{\mathbf{v},\ \nabla p,\ f\}$, which satisfy the system

$$\mathbf{v}_t - \nu\,\Delta\,\mathbf{v} = -\nabla p + f(t)\ \mathrm{g}(x, t), \qquad \operatorname{div}\mathbf{v} = 0\,, \qquad (x, t) \in Q_T\,, \tag{4.3.2}$$

the initial condition

$$\mathbf{v}(x, 0) = 0\,, \qquad x \in \Omega\,, \tag{4.3.3}$$

the boundary condition

$$\mathbf{v}\,(x, t) = 0\,, \qquad (x, t) \in S_T\,, \tag{4.3.4}$$

and the **integral overdetermination condition**

$$\int\limits_\Omega \mathbf{v}(x, t)\cdot \boldsymbol{\omega}(x)\ dx = \varphi(t)\,, \qquad 0 \le t \le T\,, \tag{4.3.5}$$

provided that the functions g, $\boldsymbol{\omega}$, φ and the coefficient ν are known in advance. We look for a solution of the inverse problem (4.3.2)–(4.3.5) in the class of functions

$$\mathbf{v} \in \mathbf{W}^{2,1}_{2,0}(Q_T) \bigcap \overset{\circ}{\mathbf{J}}(Q_T)\,, \qquad \nabla p \in \mathbf{G}(Q_T)\,, \qquad f \in C([0, T])\,.$$

The norm of the space $C([0, T])$ is defined by

$$\|\, f\,\|_C = \sup_{t\in[0, T]} |\exp\{-\gamma\, t\}\ f(t)|\,, \tag{4.3.6}$$

where γ is a positive number which will be specified below.

In tackling problem (4.3.2)–(4.3.5) one might reasonably try to adapt the method ascribed to Prilepko and Vasin (1989a). Their methodology guides the derivation of a linear operator equation of the second kind for f and necessitates imposing the extra restrictions

$$\mathbf{g} \in \mathbf{C}\Big([0, T], \mathbf{L}_2(\Omega)\Big), \qquad \omega \in \mathbf{W}_2^2(\Omega), \qquad \varphi, \varphi' \in C([0, T]),$$

$$\Big| \int_\Omega \mathbf{g}(x, t) \cdot \omega(x) \, dx \Big| \geq g_0 > 0 \quad \text{for} \quad 0 \leq t \leq T \quad (g_0 \equiv \text{const}).$$

When an arbitrary function f from $C([0, T])$ is held fixed, the system (4.3.2)–(4.3.4) serves as a basis for recovering a pair of the functions $\{\mathbf{v}, \nabla p\}$ as a solution of the direct problem with the prescribed function $\mathbf{g}$ and coefficient ν. In agreement with Theorem 4.1.2 there exists a unique solution $\{\mathbf{v}, \nabla p\}$ of problem (4.3.2)–(4.3.4)

$$\mathbf{v} \in \mathbf{W}_{2,\,0}^{2,\,1}(Q_T) \bigcap \overset{\circ}{\mathbf{J}}(Q_T), \qquad \nabla p \in \mathbf{G}(Q_T),$$

it being understood that any function f from the space $C([0, T])$ is associated with the unique function $\mathbf{v}$ thus obtained. The traditional way of covering this is to introduce the linear operator

$$A_1 \colon C([0, T]) \mapsto C([0, T])$$

acting in accordance with the rule

$$(A_1 f)(t) = -\frac{\nu}{g_1(t)} \int_\Omega \mathbf{v} \cdot \Delta\omega \, dx, \tag{4.3.7}$$

where

$$g_1(t) = \int_\Omega \mathbf{g}(x, t) \cdot \omega(x) \, dx$$

and $\mathbf{v}$ is the function involved in the solution $\{\mathbf{v}, \nabla p\}$ of the system (4.3.2)–(4.3.4), the meaning of which we have discussed above.

Our further step is to consider the linear operator equation of the second kind over the space $C([0, T])$:

$$f = A_1 f + h_1, \quad \text{where} \quad h_1 = \frac{\varphi'(t)}{g_1(t)}. \tag{4.3.8}$$

Theorem 4.3.1 *Let*

$$\mathbf{g} \in \mathbf{C}\Big([0,\, T],\, \mathbf{L}_2(\Omega)\Big), \qquad \boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega),$$

$$\varphi,\, \varphi' \in C([0,\, T]),\ \varphi(0) = 0, \qquad \left| \int\limits_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}(x)\, dx \right| \geq g_0 > 0 \ \textit{for}\ t \in [0,\, T].$$

Then for a solution of the inverse problem (4.3.2)–(4.3.5) *to exist it is necessary and sufficient that equation* (4.3.8) *is solvable.*

Proof We first prove the necessity. Let the inverse problem (4.3.2)–(4.3.5) possess a solution, say $\{\mathbf{v},\, \nabla p,\, f\}$. Taking the scalar product in $\mathbf{L}_2(\Omega)$ between $\boldsymbol{\omega}$ and both sides of the first equation (4.3.2) we arrive at

$$\int\limits_\Omega \mathbf{v}_t \cdot \boldsymbol{\omega}\ dx - \nu \int\limits_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}\ dx = f(t) \int\limits_\Omega \mathbf{g} \cdot \boldsymbol{\omega}\ dx\,, \tag{4.3.9}$$

recalling that

$$\int\limits_\Omega \nabla\, p(x,\, t) \cdot \boldsymbol{\omega}(x)\ dx = 0 \quad \text{for} \quad \boldsymbol{\omega} \in \overset{\circ}{\mathbf{J}}(\Omega)\,.$$

From (4.3.9), (4.3.5) and (4.3.7) we conclude that f solves the equation

$$f = A_1\, f + h_1\,.$$

But this means that (4.3.8) is solvable.

Proceeding to the proof of the sufficiency we suppose that equation (4.3.8) possesses a solution, say $f \in C([0,\, T])$. By Theorem 4.1.2 on the unique solvability of the direct problem we are able to recover $\{\mathbf{v},\, \nabla\, p\}$ as the solution of (4.3.2)–(4.3.4) associated with f, so that it remains to show that the function $\mathbf{v}$ satisfies the overdetermination condition (4.3.5). Under the agreement

$$\int\limits_\Omega \mathbf{v}(x,\, t)\ \cdot\, \boldsymbol{\omega}(x)\ dx = \varphi_1(t)\,, \qquad t \in [0,\, T]\,,$$

it is not difficult to check that the initial condition (4.3.3) gives $\varphi_1(0) = 0$. By exactly the same reasoning as in the derivation of (4.3.9) we find that

$$f = A_1\, f + \varphi_1'(t)/g_1(t)\,. \tag{4.3.10}$$

On the other hand,

$$f = A_1 f + \varphi'(t)/g_1(t)\,, \tag{4.3.11}$$

which is consistent with the initial assumption.

From such reasoning it seems clear that the combination of (4.3.10) and (4.3.11) gives

$$\varphi_1'(t) = \varphi'(t)$$

for $t \in [0, T]$. By the same token,

$$\varphi_1(0) = \varphi(0) = 0\,,$$

yielding

$$\varphi_1(t) = \varphi(t)$$

for $t \in [0, T]$. This provides support for the view that $\{\mathbf{v}, \nabla p, f\}$ is just a solution of the inverse problem (4.3.2)–(4.3.5). ■

In the next theorem we establish sufficient conditions under which the inverse problem solution can be shown to exist and to be unique.

Theorem 4.3.2 *Let*

$$\mathbf{g} \in \mathbf{C}\Big([0, T], \mathbf{L}_2(\Omega)\Big)\,, \qquad \boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,,$$

$$\varphi, \varphi' \in C([0, T]), \varphi(0) = 0, \qquad \left| \int_\Omega \mathbf{g}(x, t) \cdot \boldsymbol{\omega}(x)\, dx \right| \geq g_0 > 0 \textit{ for } t \in [0, T].$$

Then there exists a solution of the inverse problem (4.3.2)–(4.3.5), *this solution is unique in the indicated class of functions and the estimates hold:*

$$\| f \|_C \leq m_1 \| \varphi' \|_C\,, \tag{4.3.12}$$

$$\| \mathbf{v} \|_{2, Q_T}^{(2,1)} + \| \nabla p \|_{2, Q_T} \leq m_2 \| \varphi' \|_C\,, \tag{4.3.13}$$

where the constants m_1 and m_2 depend only on the input data of the inverse problem under consideration.

Proof It follows from relation (4.3.7) specifying the operator A_1 that

$$(4.3.14)\quad \| A_1 f \|_C \leq \frac{1}{g_0}\, \nu \, \| \Delta \omega \|_{2,\Omega} \sup_{t\in[0,T]} \Big(\exp\{-\gamma t\} \, \| \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega} \Big) .$$

Taking the scalar product in the space $\mathbf{L}_2(\Omega)$ between $\mathbf{v}$ and the first equation (4.3.2) we establish the relation known as the **energy identity**:

$$\frac{1}{2}\,\frac{d}{dt}\, \| \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega}^2 + \nu \, \| \mathbf{v}_x(\,\cdot\,, t) \|_{2,\Omega}^2 = f(t) \int_\Omega \mathbf{g} \cdot \mathbf{v} \, dx ,$$

which implies the inequality

$$\frac{d}{dt}\, \| \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega} \leq \| f(t)\, \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} , \qquad 0 \leq t \leq T .$$

Integrating over τ from 0 to t the resulting expression yields

$$(4.3.15)\quad \| \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega}$$
$$\leq \| \mathbf{v}(\,\cdot\,, 0) \|_{2,\Omega} + \sup_{t\in[0,T]} \| \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} \int_0^t |f(\tau)| \, d\tau .$$

Putting these together with the initial condition (4.3.3) we obtain the estimate

$$(4.3.16)\qquad \| A_1 f \|_C \leq m_3 \, \| f \|_C ,$$

where

$$m_3 = \frac{\nu}{\gamma \, g_0} \, \| \Delta \mathbf{w} \|_{2,\Omega} \sup_{t\in[0,T]} \| \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} .$$

For the moment, we choose γ so that

$$(4.3.17)\qquad m_3 < 1 .$$

From (4.3.16) and (4.3.17) we deduce that there exists a unique solution to equation (4.3.8) in $C([0, T])$ and (4.3.12) holds true. Relations (4.1.6) and (4.3.12) imply estimate (4.3.13). Due to Theorem 4.3.1 the inverse problem (4.3.2)–(4.3.5) has a solution $\{\mathbf{v}, \nabla p, f\}$. In order to prove its uniqueness, assume to the contrary that there were two distinct collections $\{\mathbf{v}_1, \nabla p_1, f_1\}$ and $\{\mathbf{v}_2, \nabla p_2, f_2\}$ both solving the inverse problem (4.3.2)–(4.3.5). We claim that f_1 does not coincide with f_2, since otherwise $\mathbf{v}_1$ and ∇p_1 would be equal to $\mathbf{v}_2$ and ∇p_2, respectively, due to the uniqueness theorem for the direct problem related to the system (4.3.2)–(4.3.4). Repeating the same arguments adopted in the development of (4.3.9) for these specific cases we see that both functions f_1 and f_2 satisfy (4.3.8). But this contradicts the solution uniqueness established above for the governing equation, thereby completing the proof of the theorem. ■

In practical applications of advanced theory one could require to measure the pressure gradient ∇p instead of the flow velocity $\mathbf{v}$. In that case the statement of the inverse problem is somewhat different (see Vasin (1992b)) and necessitates seeking a collection of the functions $\{\mathbf{v}, \nabla p, f\}$, which satisfy in Q_T the system

$$\mathbf{v}_t - \nu \Delta \mathbf{v} = -\nabla p + f(t)\, \mathbf{g}(x, t), \quad \operatorname{div} \mathbf{v} = 0, \quad (x, t) \in Q_T, \tag{4.3.18}$$

the initial condition

$$\mathbf{v}(x,0) = 0\,, \qquad x \in \Omega\,, \tag{4.3.19}$$

the boundary condition

$$\mathbf{v}\,(x, t) = 0\,, \qquad (x, t) \in S_T\,, \tag{4.3.20}$$

and the condition of **integral overdetermination**

$$\int\limits_{\Omega} \nabla p(x, t) \cdot \chi(x)\, dx = \psi(t) \tag{4.3.21}$$

provided that the functions g, χ, ψ and the coefficient ν are given.

In a common setting we look for a solution of (4.3.18)–(4.3.21) in the class of functions

$$\mathbf{v} \in \mathbf{W}_{2,0}^{2,1}(Q_T) \bigcap \overset{\circ}{\mathbf{J}}(Q_T)\,, \qquad f \in C([0, T])\,, \qquad \nabla p \in \mathbf{G}(Q_T)\,,$$

under the agreement that the integral

$$\int\limits_{\Omega} \nabla p(x,t)\, \Phi(x)\, dx$$

is continuous in $t \in [0, T]$ for any $\Phi \in \mathbf{G}(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$. In addition, one supposes that

$$\mathbf{g} \in \mathbf{C}\Big([0, T], \mathbf{L}_2(\Omega)\Big)\,, \qquad \chi \in \mathbf{W}_2^2(\Omega)\,, \qquad \psi \in C([0, T])\,,$$

$$\left| \int\limits_{\Omega} \mathbf{g}(x, t) \cdot \chi(x)\, dx \right| \geq g_0 > 0\,, \qquad 0 \leq t \leq T \qquad (g_0 \equiv \text{const})\,.$$

On the same grounds as in the derivation of (4.3.8), we involve in the further development the linear operator

$$A_2 \colon C([0, T]) \mapsto C([0, T])$$

acting in accordance with the rule

$$(A_2 f)(t) = -\frac{\nu}{g_2(t)} \int_\Omega \mathbf{v}(x, t) \cdot \Delta\chi(x) \, dx , \tag{4.3.22}$$

where

$$g_2(t) = \int_\Omega \mathbf{g}(x, t) \cdot \chi(x) \, dx .$$

In the theorem below we establish the conditions under which the linear operator equation of the second kind

$$f = A_2 f + h_2 \tag{4.3.23}$$

with $h_2 = -\psi(t)/g_2(t)$ is equivalent to the inverse problem at hand.

Theorem 4.3.3 *Let*

$$\mathbf{g} \in \mathbf{C}\Big([0, T], \mathbf{L}_2(\Omega)\Big) , \quad \chi \in \mathbf{G}(\Omega) \bigcap \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) ,$$

$$\psi \in C([0, T]) , \quad \left| \int_\Omega \mathbf{g}(x, t) \cdot \chi(x) \, dx \right| \geq g_0 > 0 , \quad 0 \leq t \leq T .$$

Then for a solution of the inverse problem (4.3.18)–(4.3.21) *to exist it is necessary and sufficient that there exists a solution to equation* (4.3.23).

Proof We proceed to prove the necessity. Let the inverse problem (4.3.18)–(4.3.21) admit a solution, say $\{\mathbf{v}, \nabla p, f\}$. In dealing with $\mathbf{v}_t \in \overset{\circ}{\mathbf{J}}(Q_T)$ and $\chi \in \mathbf{G}(\Omega)$ we can show that the system (4.3.18) implies that

$$\int_\Omega \nabla p(x, t) \cdot \chi(x) \, dx = \nu \int_\Omega \mathbf{v}(x, t) \cdot \Delta \chi(x) \, dx + f(t) \int_\Omega \mathbf{g}(x, t) \cdot \chi(x) \, dx , \tag{4.3.24}$$

since

$$\int_\Omega \mathbf{v}_t(x, t) \cdot \chi(x)\, dx = 0 .$$

Observe that the function on the left-hand side of (4.3.24) is continuous on the segment $[0, T]$. This is due to the fact that

$$\Big| \int_\Omega \Big(\nabla p(x, t+\Delta t) - \nabla p(x, t)\Big) \cdot \chi(x)\, dx \Big|$$

$$\leq \nu \| \Delta \chi \|_{2,\Omega} \cdot \| \mathbf{v}(\,\cdot\,, t+\Delta t) - \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega}$$

$$+ |f(t+\Delta t)|\, \| \chi \|_{2,\Omega} \cdot \| \mathbf{g}(\,\cdot\,, t+\Delta t) - \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega}$$

$$+ \| \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} \cdot \| \chi \|_{2,\Omega}\, |f(t+\Delta t) - f(t)| ,$$

where $f \in C([0, T])$ and $\mathbf{v} \in \mathbf{W}_{2,0}^{2,1}(Q_T)$.

With relation (4.3.22) in view, which specifies the operator A_2, we deduce that (4.3.24) yields $f = A_2 f + h_2$. This proves the necessity.

The sufficiency can be established in a similar way as in the proof of Theorem 4.3.1 and the details are omitted here. Thus, the theorem is completely proved. ■

Theorem 4.3.4 *Let*

$$\mathbf{g} \in \mathbf{C}\Big([0, T], \mathbf{L}_2(\Omega)\Big), \qquad \chi \in \mathbf{G}(\Omega) \bigcap \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) ,$$

$$\psi \in C([0, T]), \qquad \Big| \int_\Omega \mathbf{g}(x, t) \cdot \chi(x)\, dx \Big| \geq g_0 > 0 , \quad 0 \leq t \leq T .$$

Then there exists a solution of the inverse problem (4.3.18)–(4.3.21), *this solution is unique in the indicated class of functions and the estimates hold:*

$$\| f \|_C \leq m_4 \| \psi \|_C , \tag{4.3.25}$$

$$\| \mathbf{v} \|_{2, Q_T}^{(2,1)} + \| \nabla p \|_{2, Q_T} \leq m_5 \| \psi \|_C , \tag{4.3.26}$$

where m_4 and m_5 are the constants depending only on the input data of the inverse problem under consideration.

Proof We rely on the estimate for the operator A_2 specified by (4.3.22):

$$\left|(A_2 f)(t)\right| \le \frac{\nu}{g_0} \, \| \Delta \boldsymbol{\chi} \|_{2,\Omega} \cdot \| \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega} \,. \tag{4.3.27}$$

Relations (4.3.16) and (4.3.27) together imply the inequality

$$\| A_2 f \|_C \le m_6 \, \| f \|_C \,, \tag{4.3.28}$$

where

$$m_6 = \frac{\nu}{\gamma \, g_0} \, \| \Delta \boldsymbol{\chi} \|_{2,\Omega} \sup_{t \in [0,T]} \| \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} \,.$$

Arguing as in the proof of Theorem 4.3.2 we conclude that the inverse problem (4.3.18)–(4.3.21) has a solution and this solution is unique. Estimate (4.3.26) follows from (4.1.4) and (4.3.25). ■

It is worth noting here that the results set forth in this section can easily be generalized for the linearized Navier–Stokes equations of rather general form where the system (4.3.2) will be replaced by

$$\begin{aligned} & \mathbf{v}_t - \nu \Delta \mathbf{v} + \sum_{k=1}^{n} B_k(x,t)\, \mathbf{v}_{x_k} + A(x,t)\, \mathbf{v} = -\nabla p + f(t)\, \mathbf{g}(x,t) \,, \\ & \operatorname{div} \mathbf{v} = 0 \,, \qquad (x,\, t) \in Q_T \,, \end{aligned} \tag{4.3.29}$$

where B_k and A are given $(n \times n)$-matrices with entries b_k^{ij} and a^{ij}, respectively. Once again, the initial and boundary conditions as well as the overdetermination condition are prescribed by (4.3.3)–(4.3.5). As an axample we cite here the theorem proved by Vasin (1992a).

Theorem 4.3.5 *Let* $b_k^{ij},\ a^{ij} \in C\big([0,T], L_4(\Omega)\big)$, $\mathbf{b}^{ij} \in \mathbf{J}(Q_T)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\varphi \in C^1([0,T])$, $\varphi(0) = 0$, $\mathbf{g} \in \mathbf{C}\big([0,T]; \mathbf{L}_2(\Omega)\big)$ *and let the inequality*

$$\left| \int_\Omega \mathbf{g}(x,t)\, \boldsymbol{\omega}(x)\, dx \right| \ge g_0 > 0 \qquad (g_0 = \text{const}) \,, \quad 0 \le t \le T \,,$$

hold. Then a solution

$$\mathbf{v} \in \mathbf{W}_{2,0}^{2,1}(Q_T) \bigcap \overset{\circ}{\mathbf{J}}(Q_T) \,, \qquad \nabla p \in \mathbf{G}(Q_T) \,, \qquad f \in C([0,\, T])$$

of the inverse problem (4.3.29), (4.3.3)–(4.3.5) *exists, is unique and has estimates* (4.3.12)–(4.3.13).

A similar result is still valid in the case where the overdetermination condition is of the form (4.3.21). Here $\mathbf{b}^{ij}$ refers to the vector with components b_k^{ij} ($k = 1, 2$ or $k = 1, 2, 3$). The symbol $\overset{\circ}{\mathbf{J}}(Q_T)$ stands for the subspace of $\mathbf{L}_2(Q_T)$ consisting of all vector functions belonging to $\mathbf{J}(\Omega)$ for almost all $t \in [0, T]$. The space $\mathbf{J}(\Omega)$ contains the $\mathbf{L}_2(\Omega)$-vectors being orthogonal to all vectors like $\nabla \psi$ for $\psi \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$.

4.4 Nonstationary nonlinear system of Navier–Stokes equations: three-dimensional flow

We begin by placing the statement of the inverse problem for the system (4.1.8) assuming that the vector function $\mathbf{F}_2$ involved in (4.1.8) is representable by

$$\mathbf{F}_2 = \mathbf{f}(x)\, g(x, t),$$

where $g(x, t)$ is a given scalar function and $\mathbf{f}(x)$ is an unknown vector function.

The inverse problem here is to find a triple of the vector functions $\{\mathbf{v}, \nabla p, \mathbf{f}\}$, which satisfy in Q_T the **nonstationary nonlinear Navier–Stokes system**

$$\begin{aligned} &\mathbf{v}_t - \nu\, \Delta\, \mathbf{v} + (\mathbf{v}, \nabla)\, \mathbf{v} = -\nabla p + \mathbf{f}(x)\, g(x, t), \\ &\operatorname{div} \mathbf{v} = 0, \qquad (x, t) \in Q_T, \end{aligned} \tag{4.4.1}$$

the initial condition

$$\mathbf{v}(x, 0) = \mathbf{a}(x), \qquad x \in \Omega, \tag{4.4.2}$$

the boundary condition

$$\mathbf{v}(x, t) = 0, \qquad (x, t) \in S_T, \tag{4.4.3}$$

and the **final overdetermination** conditions

$$\mathbf{v}(x, T) = \varphi(x), \qquad \nabla p(x, T) = \nabla \psi(x), \qquad x \in \Omega, \tag{4.4.4}$$

provided that the functions $\mathbf{a}$, φ, $\nabla \psi$, g and the coefficient ν are given. By a solution of the inverse problem (4.4.1)–(4.4.4) we mean a triple $\{\mathbf{v}, \nabla p, \mathbf{f}\}$ such that

$$\mathbf{v} \in \mathbf{W}_{2,0}^{2,1}(Q_T) \bigcap \overset{\circ}{\mathbf{J}}(Q_T), \qquad \mathbf{f} \in \mathbf{L}_2(\Omega), \qquad \nabla p(\,\cdot\,, t) \in \mathbf{G}(\Omega)$$

for any $t \in [0, T]$ and it continuously depends on t in the $\mathbf{L}_2(\Omega)$-norm on the segment $[0, T]$ and, in addition, all of the relations (4.4.1)–(4.4.4) occur.

Throughout Section 4.4, we will assume that the dimension of the domain of spatial variables Ω is equal to 3, that is, $\Omega \subset \mathbf{R}^3$. First, we are going to derive some stability estimates for the solutions of the system (4.1.8)–(4.1.10). Let $\mathbf{v}' = \mathbf{v}'(x, t)$ and $\mathbf{v}'' = \mathbf{v}''(x, t)$ be the solutions of the system (4.1.8)–(4.1.10) corresponding to the initial velocities $\mathbf{a}_2'$, $\mathbf{a}_2''$ and the external forces $\mathbf{F}_2'$, $\mathbf{F}_2''$, respectively. The function $\mathbf{v}'$ is subject to the integral identity

$$\int_\Omega \mathbf{v}_t{}' \cdot \Phi \, dx + \nu \int_\Omega \mathbf{v}_x{}' \cdot \Phi_x \, dx + \int_\Omega (\mathbf{v}', \nabla)\, \mathbf{v}' \cdot \Phi \, dx = \int_\Omega \mathbf{F}_2' \cdot \Phi \, dx \,,$$

where Φ is an arbitrary function from the space $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$.

A similar relation remains valid with regard to the function $\mathbf{v}''$. Therefore, by subtracting the first identity from the second we deduce that

$$\begin{aligned}(4.4.5)\qquad & \int_\Omega (\mathbf{v}' - \mathbf{v}'')_t \cdot \Phi \, dx + \nu \int_\Omega (\mathbf{v}' - \mathbf{v}'')_x \cdot \Phi_x \, dx \\ & + \int_\Omega (\mathbf{v}', \nabla)\,(\mathbf{v}' - \mathbf{v}'') \cdot \Phi \, dx \\ & + \int_\Omega (\mathbf{v}' - \mathbf{v}'', \nabla)\, \mathbf{v}'' \cdot \Phi \, dx \\ & = \int_\Omega (\mathbf{F}_2' - \mathbf{F}_2'') \cdot \Phi \, dx \,, \quad 0 \le t \le T .\end{aligned}$$

Since $\mathbf{v}'(\,\cdot\,, t)$ and $\mathbf{v}''(\,\cdot\,, t)$ both are located in $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, it is meaningful to substitute $\Phi = \mathbf{v}' - \mathbf{v}''$ into (4.4.5). The outcome of this is

$$\begin{aligned}(4.4.6)\qquad & \frac{1}{2}\,\frac{d}{dt} \| (\mathbf{v}' - \mathbf{v}'')(\,\cdot\,, t) \|_{2,\Omega}^2 + \nu \| (\mathbf{v}' - \mathbf{v}'')_x(\,\cdot\,, t) \|_{2,\Omega}^2 \\ & = - \int_\Omega (\mathbf{v}' - \mathbf{v}'', \nabla)\, \mathbf{v}'' \cdot (\mathbf{v}' - \mathbf{v}'') \, dx + \int_\Omega (\mathbf{F}_2' - \mathbf{F}_2'') \cdot (\mathbf{v}' - \mathbf{v}'') \, dx \,.\end{aligned}$$

With the aid of the relation

$$\int_\Omega (\mathbf{v}', \nabla)\,(\mathbf{v}' - \mathbf{v}'') \cdot (\mathbf{v}' - \mathbf{v}'') \, dx = 0$$

we estimate the first term on the right-hand side of (4.4.6) by

$$(4.4.7)\quad \left| \int_\Omega (\mathbf{v}' - \mathbf{v}'', \nabla)\, \mathbf{v}'' \cdot (\mathbf{v}' - \mathbf{v}'')\, dx \right| \le \| \mathbf{v}_x''(\,\cdot\,, t) \|_{2,\Omega} \cdot \| (\mathbf{v}' - \mathbf{v}'')(\,\cdot\,, t) \|_{4,\Omega}^2 .$$

Recall that, being an element of the space $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ (see Ladyzhenskaya (1970, p. 20)), any function $\mathbf{u}$ (if $\Omega \in \mathbf{R}^3$) is subject to the following relation:

$$(4.4.8)\qquad \| \mathbf{u} \|_{4,\Omega}^4 \le (4/3)^{3/2} \| \mathbf{u} \|_{2,\Omega} \cdot \| \mathbf{u}_x \|_{2,\Omega}^3 .$$

This approach applies equally well to the second factor on the right-hand side of (4.4.7). As a final result we get

$$(4.4.9)\qquad \left| \int_\Omega (\mathbf{v}' - \mathbf{v}'', \nabla)\, \mathbf{v}'' \cdot (\mathbf{v}' - \mathbf{v}'')\, dx \right|$$

$$\le \left(\frac{4}{3}\right)^{3/2} \| \mathbf{v}_x''(\,\cdot\,, t) \|_{2,\Omega} \cdot \| (\mathbf{v}' - \mathbf{v}'')(\,\cdot\,, t) \|_{2,\Omega}^{1/2} \times \| (\mathbf{v}' - \mathbf{v}'')_x(\,\cdot\,, t) \|_{2,\Omega}^{3/2}$$

$$\le \frac{3}{4}\, \varepsilon^{4/3}\, \| (\mathbf{v}' - \mathbf{v}'')_x(\,\cdot\,, t) \|_{2,\Omega}^2 + \frac{1}{4\,\varepsilon^4} \left(\frac{4}{3}\right)^3 \| \mathbf{v}_x''(\,\cdot\,, t) \|_{2,\Omega}^4 \times \| (\mathbf{v}' - \mathbf{v}'')(\,\cdot\,, t) \|_{2,\Omega}^2 ,$$

where ε is free to be chosen among positive constants.

In the derivation of the last estimate in (4.4.9) we used Young's inequality

$$a\, b \le m^{-1}\, \varepsilon^m\, a^m + (m-1)\, m^{-1}\, \varepsilon^{-m/(m-1)}\, b^{m/(m-1)} ,$$

which is valid for any positive a, b, ε and $m > 1$. Upon substituting (4.4.9) into (4.4.6) with $\varepsilon = \left(\frac{\nu}{2}\cdot\frac{4}{3}\right)^{3/4}$ we arrive at the chain of relations

$$(4.4.10)\quad \frac{1}{2}\,\frac{d}{dt}\,\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|^2_{2,\Omega} + \nu\,\|(\mathbf{v}'-\mathbf{v}'')_x(\,\cdot\,,t)\|^2_{2,\Omega}$$
$$\le \frac{\nu}{2}\,\|(\mathbf{v}'-\mathbf{v}'')_x(\,\cdot\,,t)\|^2_{2,\Omega}$$
$$+ \frac{2}{\nu^3}\,\|\mathbf{v}_x{}''(\,\cdot\,,t)\|^4_{2,\Omega}\cdot\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|^2_{2,\Omega}$$
$$+ \|(\mathbf{F}_2'-\mathbf{F}_2'')(\,\cdot\,,t)\|_{2,\Omega}\cdot\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|_{2,\Omega}\,,$$

which imply that

$$(4.4.11)\quad \frac{1}{2}\,\frac{d}{dt}\,\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|^2_{2,\Omega} + \frac{\nu}{2}\,\|(\mathbf{v}'-\mathbf{v}'')_x(\,\cdot\,,t)\|^2_{2,\Omega}$$
$$\le \frac{2}{\nu^3}\,\|\mathbf{v}_x{}''(\,\cdot\,,t)\|^4_{2,\Omega}\cdot\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|^2_{2,\Omega}$$
$$+ \|(\mathbf{F}_2'-\mathbf{F}_2'')(\,\cdot\,,t)\|_{2,\Omega}\cdot\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|_{2,\Omega}\,.$$

By appeal to Gronwall's lemma we obtain from (4.4.11) the first stability estimate

$$(4.4.12)\quad \|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|_{2,\Omega} \le \exp\left\{\frac{2}{\nu^3}\int_0^t \|\mathbf{v}_x{}''(\,\cdot\,,\tau)\|^4_{2,\Omega}\,d\tau\right\}$$
$$\times\left(\|\mathbf{a}_2'-\mathbf{a}_2''\|_{2,\Omega} + \int_0^t \|(\mathbf{F}_2'-\mathbf{F}_2'')(\,\cdot\,,\tau)\|_{2,\Omega}\,d\tau\right).$$

On the other hand, the integration of (4.4.10) over t from 0 to T allows to construct the second stability estimate

$$(4.4.13)\quad \frac{\nu}{2}\,\|(\mathbf{v}'-\mathbf{v}'')_x\|^2_{2,Q_T} \le \frac{2}{\nu^3}\int_0^T \|\mathbf{v}_x{}''(\,\cdot\,,t)\|^4_{2,\Omega}$$
$$\times\,\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|^2_{2,\Omega}\,dt$$
$$+\int_0^T \|(\mathbf{F}_2'-\mathbf{F}_2'')(\,\cdot\,,t)\|_{2,\Omega}$$
$$\times\,\|(\mathbf{v}'-\mathbf{v}'')(\,\cdot\,,t)\|_{2,\Omega}\,dt$$
$$+\tfrac{1}{2}\,\|\mathbf{a}_2'-\mathbf{a}_2''\|^2_{2,\Omega}\,.$$

All this enables us to explore the inverse problem (4.4.1)–(4.4.4) more deeply by means of a subset $D \subset \mathbf{L}_2(\Omega)$ such that

$$D = \left\{ \mathbf{f} \in \mathbf{L}_2(\Omega) \colon \ \| \mathbf{f} \|_{2,\,\Omega} \le 1 \right\} .$$

In the sequel we will always assume that $\mathbf{a}$ and φ belong to $\mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$ and

$$\nabla \, , \psi \in \mathbf{G}(\Omega) \, , \qquad g \, , \, g_t \in C(\bar{Q}_T) \, ,$$

$$|g(x,\, T)| \ge g_T > 0 \quad \text{for} \quad x \in \bar{\Omega} \quad (g_T \equiv \text{const}\,) \, ,$$

and

$$m_1 \, m_2 \, < \, \nu^3 \, (4 \, \beta^2)^{-1} , \tag{4.4.14}$$

where

$$\beta = \, (4/3)^{3/4} \, (c_1(\Omega))^{1/4} \, ,$$

$$m_1 = \left[\, \| \mathbf{a}_2 \|_{2,\,\Omega} + \int_0^T \sup_{x \in \Omega} \; |g(x,\, t)| \; dt \, \right] ,$$

$$m_2 = \| \nu \, \Delta \, \mathbf{a} - (\mathbf{a}, \, \nabla) \, \mathbf{a} \|_{2,\,\Omega} + \sup_{(x,\, t) \in Q_T} |g(x,\, t)|$$

$$+ \sup_{x \in \Omega} \; |g(x,\, 0)| + \int_0^T \sup_{x \in \Omega} \; |g_t(x,\, t)| \; dt$$

and $c_1(\Omega)$ is the constant from the Poincaré–Friedrichs inequality (4.2.21).

The system (4.4.1)–(4.4.3) with $\mathbf{f} \in D$ can be viewed as a system of the type (4.1.8)–(4.1.10). It is clear that for any $\mathbf{f}$ lying within D either of the functions $\mathbf{F}_2 = \mathbf{f}\, g$ and $(\mathbf{F}_2)_t = \mathbf{f}\, g_t$ belongs to the space $\mathbf{L}_2(\Omega_T)$. Granted (4.4.14), the function $\mathbf{F}_2 = \mathbf{f}\, g$ satisfies condition (4.1.11) for any $\mathbf{f}$ from D. Consequently, by Theorem 4.1.4 one can find a unique pair of the functions $\{\mathbf{v}, \, \nabla p\}$ as a solution of the direct problem (4.4.1)–(4.4.3) corresponding to a suitably chosen function $\mathbf{f} \in D$. In the framework of Theorems 4.1.4–4.1.5 we refer to the nonlinear operator

$$A\colon D \mapsto \mathbf{L}_2(\Omega)$$

acting in accordance with the rule

$$(A\mathbf{f})(x) = \mathbf{v}_t(x, T), \qquad x \in \Omega,$$

where $\mathbf{v}$ is the function entering the set $\{\mathbf{v}, \nabla p\}$ and solving the direct problem (4.4.1)–(4.4.3).

One more nonlinear operator

$$B\colon D \mapsto \mathbf{L}_2(\Omega)$$

with the values

$$(B\,\mathbf{f})(x) = \frac{1}{g(x, T)}\,(A\mathbf{f})(x), \qquad x \in \Omega, \tag{4.4.15}$$

complements careful analysis of the nonlinear operator equation of the second kind for $\mathbf{f}$:

$$\mathbf{f} = B\mathbf{f} + \chi, \tag{4.4.16}$$

where

$$\chi = \frac{1}{g(x, T)}\,\Big[-\nu\,\Delta\,\varphi + (\varphi, \nabla)\,\varphi + \nabla\,\psi\Big].$$

Theorem 4.4.1 *Let*

$$\Omega \subset \mathbf{R}^3, \quad g, g_t \in C(\bar{Q}_T), \qquad |g(x, T)| \geq g_T > 0 \quad \textit{for} \quad x \in \bar{\Omega},$$

$$\mathbf{a} \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega).$$

If (4.4.14) *holds, then the operator B is completely continuous on D.*

Proof The theorem will be proved if we succeed in showing that the operator A is completely continuous on D. Choosing $\mathbf{f}$ in D arbitrarily we consider a sequence $\{\mathbf{f}_k\}_{k=1}^{\infty}$ of elements $\mathbf{f}_k \in D$ such that

$$\|\mathbf{f} - \mathbf{f}_k\|_{2,\Omega} \longrightarrow 0, \qquad k \to \infty.$$

Let $\{\mathbf{v}_k, \nabla p_k\}$ be the solution of the direct problem (4.4.1)–(4.4.3) corresponding to the external force function $\mathbf{f}_k\, g$ and the initial velocity $\mathbf{a}$

and let $\{\mathbf{v},\ \nabla p\}$ be the solution of the same problem corresponding to the external force function $\mathbf{f}\, g$ and the initial velocity $\mathbf{a}$. Then the functions $\mathbf{v}_k - \mathbf{v}$ and $\nabla(p_k - p)$ satisfy the system

$$(4.4.17) \qquad (\mathbf{v}_k - \mathbf{v})_t - \nu\, \Delta(\mathbf{v}_k - \mathbf{v}) = -\nabla(p_k - p) + \mathbf{F}_k\,, \qquad \operatorname{div}(\mathbf{v}_k - \mathbf{v}) = 0\,,$$

$$(4.4.18) \qquad (\mathbf{v}_k - \mathbf{v})(x,\, 0) = 0\,, \qquad x \in \Omega\,,$$

$$(4.4.19) \qquad (\mathbf{v}_k - \mathbf{v})(x,\, t) = 0\,, \qquad (x,\, t) \in S_T\,,$$

where

$$\mathbf{F}_k = (\mathbf{f}_k - \mathbf{f})\, g + (\mathbf{v}_k,\, \nabla)(\mathbf{v} - \mathbf{v}_k) + (\mathbf{v} - \mathbf{v}_k,\, \nabla)\, \mathbf{v}\,.$$

We note in passing that the norm of $\mathbf{F}_k$ can be estimated as follows:

$$(4.4.20) \qquad \|\, \mathbf{F}_k \,\|_{2,\, Q_T} \leq \|\, (\mathbf{f}_k - \mathbf{f})\, g \,\|_{2,\, Q_T} + \|\, (\mathbf{v}_k,\, \nabla)(\mathbf{v} - \mathbf{v}_k) \,\|_{2,\, Q_T} + \|\, (\mathbf{v} - \mathbf{v}_k,\, \nabla)\, \mathbf{v} \,\|_{2,\, Q_T}\,.$$

The Cauchy–Schwarz inequality is involved in the estimation of the second and third terms on the right-hand side of (4.4.20). The same procedure works with a great success in establishing the relations

$$(4.4.21) \qquad \|\, (\mathbf{v}_k,\, \nabla)(\mathbf{v} - \mathbf{v}_k) \,\|_{2,\, Q_T} \leq \left[\, 3 \int_0^T \int_\Omega |\mathbf{v}_k|^2\, |(\mathbf{v}_k - \mathbf{v})_x|^2 \; dx\, dt \,\right]^{1/2} \leq \left\{ \sup_{t\in[0,\,T]} \left[\, \sup_{x\in\Omega} |\mathbf{v}_k(x,\, t)| \,\right] \right\} \times \sqrt{3}\, \|\, (\mathbf{v}_k - \mathbf{v})_x \,\|_{2,\, Q_T}$$

and

$$(4.4.22) \qquad \|\, (\mathbf{v} - \mathbf{v}_k,\, \nabla)\, \mathbf{v} \,\|_{2,\, Q_T} \leq \left[\, 3 \int_0^T \|\, (\mathbf{v}_k - \mathbf{v})(\,\cdot\,,\, t) \,\|_{4,\, \Omega}^2 \cdot \|\, \mathbf{v}_x(\,\cdot\,,\, t) \,\|_{4,\, \Omega}^2 \; dt \,\right]^{1/2}.$$

The estimation of the first factor on the right-hand side of (4.4.21) can be done using (4.1.15) and the theorem of embedding $\mathbf{W}_2^2(\Omega)$ into $C^{1/2}(\bar{\Omega})$ (see Ladyzhenskaya et al. (1968, p. 78)), due to which we thus have

$$\| (\mathbf{v}_k, \nabla)(\mathbf{v} - \mathbf{v}_k) \|_{2, Q_T} \le c^* \widetilde{M}_5(\mathbf{f}_k) \| (\mathbf{v} - \mathbf{v}_k)_x \|_{2, Q_T} , \tag{4.4.23}$$

where

$$\widetilde{M}_5(\mathbf{f}_k) = \Big\{ \Big[\sup_{(x,t)\in Q_T} |g(x, t)| \ \| \mathbf{f}_k \|_{2,\Omega} + \widetilde{M}_2(\mathbf{f}_k) \Big]$$

$$\times \Big\{ 1 + c^* \Big[\widetilde{M}_1(\mathbf{f}_k) (\widetilde{M}_2(\mathbf{f}_k) + \widetilde{M}_3(\mathbf{f}_k)) \Big]^{1/2} \Big\}$$

$$+ c^* \Big[\widetilde{M}_1(\mathbf{f}_k) (\widetilde{M}_2(\mathbf{f}_k) + \widetilde{M}_3(\mathbf{f}_k)) \Big]^{3/2} \Big\} ,$$

$$\widetilde{M}_1(\mathbf{f}_k) = \| \mathbf{a} \|_{2,\Omega} + \| \mathbf{f}_k \|_{2,\Omega} \int_0^T \sup_{x\in\Omega} | g(x, t) | \, dt ,$$

$$\widetilde{M}_2(\mathbf{f}_k) = \| \nu \Delta \mathbf{a} - (\mathbf{a}, \nabla) \mathbf{a} \|_{2,\Omega} + \| \mathbf{f}_k \|_{2,\Omega} \sup_{x\in\Omega} |g(x, 0)|$$

$$+ \| \mathbf{f}_k \|_{2,\Omega} \int_0^T \sup_{x\in\Omega} | g_t(x, t) | \, dt ,$$

$$\widetilde{M}_3(\mathbf{f}_k) = \| \mathbf{f}_k \|_{2,\Omega} \sup_{(x,t)\in Q_T} |g(x, t)| .$$

The symbol c^* denotes the constant depending on T, Ω and ν. In what follows the same symbol c^* will stand for different constants. It is to be hoped that this should cause no confusion. It is straightforward to verify that the values of $\widetilde{M}_5(\mathbf{f}_k)$, $\widetilde{M}_1(\mathbf{f}_k)$, $\widetilde{M}_2(\mathbf{f}_k)$ and $\widetilde{M}_3(\mathbf{f}_k)$ are bounded as $k \to \infty$, since the sequence $\{\mathbf{f}_k\}_{k=1}^{\infty}$ converges to $\mathbf{f}$ in the $\mathbf{L}_2(\Omega)$-norm.

By (4.1.15), (4.2.21) and the theorem of embedding from $\mathbf{W}_2^1(\Omega)$ into $\mathbf{L}_4(\Omega)$ ($\Omega \subset \mathbf{R}^3$), we see that (4.4.22) implies the estimate

(4.4.24) $\| (\mathbf{v} - \mathbf{v}_k, \nabla) \mathbf{v} \|_{2, Q_T}$

$$\leq c^* \left\{ \int_0^T \left[\| (\mathbf{v} - \mathbf{v}_k)(\cdot, t) \|_{2,\Omega}^{(1)} \right]^2 \left[\| \mathbf{v}(\cdot, t) \|_{2,\Omega}^{(2)} \right]^2 dt \right\}^{1/2}$$

$$\leq c^* \left\{ \int_0^T \| (\mathbf{v} - \mathbf{v}_k)_x (\cdot, t) \|_{2,\Omega}^2 \left[\| \mathbf{v}(\cdot, t) \|_{2,\Omega}^{(2)} \right]^2 dt \right\}^{1/2}$$

$$\leq c^* M_5(\mathbf{f}) \| (\mathbf{v} - \mathbf{v}_k)_x \|_{2, Q_T} ,$$

where

$$M_5(\mathbf{f}) = \left\{ \Big[\sup_{(x,t) \in Q_T} | g(x, t) | \, \| \mathbf{f} \|_{2,\Omega} + M_2(\mathbf{f}) \Big] \right.$$

$$\times \left\{ 1 + c^* \left[M_1(\mathbf{f}) \, (M_2(\mathbf{f}) + M_3(\mathbf{f})) \right]^{1/2} \right\}$$

$$\left. + c^* \left[M_1(\mathbf{f}) (M_2(\mathbf{f}) + M_3(\mathbf{f})) \right]^{3/2} \right\} ,$$

$$M_1(\mathbf{f}) = \| \mathbf{a} \|_{2,\Omega} + \| \mathbf{f} \|_{2,\Omega} \int_0^T \sup_{x \in \Omega} | g(x, t) | \, dt ,$$

$$M_2(\mathbf{f}) = \| \nu \Delta \mathbf{a} - (\mathbf{a}, \nabla) \mathbf{a} \|_{2,\Omega} + \| \mathbf{f} \|_{2,\Omega} \sup_{x \in \Omega} | g(x, 0) |$$

$$+ \| \mathbf{f} \|_{2,\Omega} \int_0^T \sup_{x \in \Omega} | g_t(x, t) | \, dt ,$$

$$M_3(\mathbf{f}) = \| \mathbf{f} \|_{2,\Omega} \sup_{(x,t) \in Q_T} | g(x, t) | .$$

Substituting (4.4.23) and (4.4.24) into (4.4.20) yields

$$\text{(4.4.25)} \quad \| \mathbf{F}_k \|_{2, Q_T} \leq \Big[\int_0^T \sup_{x \in \Omega} |g(x, t)|^2 \, dt \Big]^{1/2} \| \mathbf{f}_k - \mathbf{f} \|_{2,\Omega}$$

$$+ c^* \left(\widetilde{M}_5(\mathbf{f}_k) + M_5(\mathbf{f}) \right) \| (\mathbf{v} - \mathbf{v}_k)_x \|_{2, Q_T} .$$

We now apply the stability estimates (4.4.12)–(4.4.13) with $\mathbf{F}_k = \mathbf{f}_k\, g$ and $\mathbf{F} = \mathbf{f}\, g$ standing in place of the functions $\mathbf{F}_2'$ and $\mathbf{F}_1'$, respectively, thus causing $\mathbf{v}' = \mathbf{v}_k$, $\mathbf{v}'' = \mathbf{v}$ and $\mathbf{a}_2' = \mathbf{a}_2'' = \mathbf{a}$. With these relations in view, (4.4.12) implies that

$$\text{(4.4.26)} \quad \|(\mathbf{v}_k - \mathbf{v})(\,\cdot\,, t)\|_{2,\Omega} \le \exp\left\{ \frac{2}{\nu^3} \int\limits_0^T \|\mathbf{v}_x(\,\cdot\,, \tau)\|_{2,\Omega}^4 \, d\tau \right\} \times \left[\int\limits_0^T \sup_{x\in\Omega} |g(x,\tau)|\, d\tau \right] \|\mathbf{f}_k - \mathbf{f}\|_{2,\Omega}$$

and (4.4.13) is followed by

$$\text{(4.4.27)} \quad \|(\mathbf{v}_k - \mathbf{v})_x\|_{2,Q_T}^2 \le \frac{4}{\nu^3} \left[\sup_{t\in[0,T]} \|\mathbf{v}_x(\,\cdot\,, t)\|_{2,\Omega} \right]^4 \times \|\mathbf{v}_k - \mathbf{v}\|_{2,Q_T}^2 + \sup_{(x,t)\in Q_T} |g(x,t)| \times \|\mathbf{f}_k - \mathbf{f}\|_{2,\Omega}\, T^{1/2}\, \|\mathbf{v}_k - \mathbf{v}\|_{2,Q_T} .$$

If one squares both sides of (4.4.26) and integrates the resulting expressions over t from 0 to T, then

$$\text{(4.4.28)} \quad \|\mathbf{v}_k - \mathbf{v}\|_{2,Q_T} \le T \exp\left\{ \frac{4}{\nu^3} \int\limits_0^T \|\mathbf{v}_x(\,\cdot\,, \tau)\|_{2,\Omega}^4 \, d\tau \right\} \times \left[\int\limits_0^T \sup_{x\in\Omega} |g(x,\tau)|\, d\tau \right]^2 \|\mathbf{f}_k - \mathbf{f}\|_{2,\Omega} .$$

Since the sequence $\{\mathbf{f}_k\}_{k=1}^\infty$ converges to $\mathbf{f}$ in the $\mathbf{L}_2(\Omega)$-norm, inequality (4.4.28) implies the convergence

$$\text{(4.4.29)} \qquad \|\mathbf{v}_k - \mathbf{v}\|_{2,Q_T} \longrightarrow 0, \qquad k \to \infty,$$

which in combination with (4.4.27) gives

$$\text{(4.4.30)} \qquad \|(\mathbf{v}_k - \mathbf{v})_x\|_{2,Q_T} \longrightarrow 0, \qquad k \to \infty.$$

By successively applying (4.4.28), (4.4.30) and (4.4.25) we arrive at the limit relation

$$\text{(4.4.31)} \qquad \|\mathbf{F}_k\|_{2,Q_T} \longrightarrow 0, \qquad k \to \infty.$$

As $\mathbf{F}_k \in \mathbf{L}_2(Q_T)$, the system (4.4.17)–(4.4.19) can be viewed as a system of the type (4.1.3)–(4.1.5), for which estimate (4.1.6) is certainly true and admits an alternative form of writing

$$\| \mathbf{v}_k - \mathbf{v} \|_{2,Q_T}^{(2,1)} \le c^* \; \| \mathbf{F}_k \|_{2,Q_T} .$$

Whence, due to convergence (4.4.31), it follows that

$$\| (\mathbf{v}_k - \mathbf{v})_t \|_{2,Q_T} \longrightarrow 0 , \qquad k \to \infty . \tag{4.4.32}$$

We proceed to the estimation of $(\mathbf{F}_k)_t$ in the $\mathbf{L}_2(Q_T)$-norm. The preliminary expressions may be of help in preparation for this:

$$\begin{aligned} &\| (\mathbf{v}_t, \nabla) \mathbf{v} \|_{2,Q_T} + \| (\mathbf{v}, \nabla) \mathbf{v}_t \|_{2,Q_T} \\ &\le \Big(3 \int_0^T \| \mathbf{v}_t(\cdot, t) \|_{4,\Omega}^2 \cdot \| \mathbf{v}_x(\cdot, t) \|_{4,\Omega}^2 \, dt \Big)^{1/2} \\ &\quad + \left(3 \int_0^T \int_\Omega |\mathbf{v}|^2 \, |\mathbf{v}_{tx}|^2 \, dx \, dt \right)^{1/2} \\ &\le c_1^* \left\{ 3 \int_0^T \Big[\| \mathbf{v}_t(\cdot, t) \|_{2,\Omega}^{(1)} \Big]^2 \Big[\| \mathbf{v}(\cdot, t) \|_{2,\Omega}^{(2)} \Big]^2 \, dt \right\}^{1/2} \\ &\quad + \sqrt{3} \; c_2^* \, M_5(\mathbf{f}) \, \| \mathbf{v}_{tx} \|_{2,Q_T} \le c^* \, M_5(\mathbf{f}) \; \| \mathbf{v}_{tx} \|_{2,Q_T} . \end{aligned} \tag{4.4.33}$$

It is worth recalling here that

$$(\mathbf{F}_k)_t = \frac{\partial}{\partial t} \Big(\mathbf{f}_k \, g - \mathbf{f} \, g \Big) - \frac{\partial}{\partial t} \Big[(\mathbf{v}_k, \nabla) \, \mathbf{v}_k - (\mathbf{v}, \nabla) \, \mathbf{v} \Big] .$$

In just the same way as in the derivation of (4.4.33) we majorize the norm of $(\mathbf{F}_k)_t$ as follows:

$$\begin{aligned} \| (\mathbf{F}_k)_t \|_{2,Q_T} &\le \| \mathbf{f}_k \, g_t \|_{2,Q_T} + \| \mathbf{f} \, g_t \|_{2,Q_T} \\ &\quad + c^* \, \widetilde{M}_5(\mathbf{f}_k) \, \| (\mathbf{v}_k)_{tx} \|_{2,Q_T} \\ &\quad + c^* \, M_5(\mathbf{f}) \, \| \mathbf{v}_{tx} \|_{2,Q_T} , \end{aligned} \tag{4.4.34}$$

where the explicit formulae for $\widetilde{M}_5(\mathbf{f}_k)$ and $M_5(\mathbf{f})$ are given by (4.4.23) and (4.4.24), respectively.

In the case where $\mathbf{f}_k \in D$, the norm of $(\mathbf{v}_k)_{tx}$ can be estimated with the aid of (4.1.14) as follows:

$$
\begin{aligned}
(4.4.35)\quad & \left[\nu - \beta\,(\nu^{-1}\, m_1\, m_2)^{1/2}\right] \,\|\,(\mathbf{v}_k)_{tx}\,\|^2_{2,\,Q_T} \\
& \leq \|\,\nu\,\Delta\mathbf{a} - (\mathbf{a},\,\nabla)\,\mathbf{a}\,\|^2_{2,\,\Omega} + \|\,\mathbf{f}_k\,\|^2_{2,\,\Omega}\left[\sup_{x\in\Omega}\ |g(x,\,0)|\right]^2 \\
& \quad + \Big[\|\,\nu\,\Delta\mathbf{a} - (\mathbf{a},\,\nabla)\,\mathbf{a}\,\|_{2,\,\Omega} + \|\,\mathbf{f}_k\,\|_{2,\,\Omega}\ \sup_{x\in\Omega}\ |\,g(x,\,0)\,| \\
& \quad + \|\,\mathbf{f}_k\,\|_{2,\,\Omega}\int_0^T \sup_{x\in\Omega}\ |\,g_t\,(x,\,t)\,|\ dt\Big]^2 ,
\end{aligned}
$$

where m_1, m_2 and β have been introduced in (4.4.14). From such manipulations it seems clear that the first factor on the left-hand side of (4.4.35) does not depend on k and is strictly positive by virtue of (4.4.14). Moreover, estimate (4.4.35) implies that the norm $\|\,(\mathbf{v}_k)_{tx}\,\|^2_{2,\,Q_T}$ is bounded as $k \to 0$. With this in mind, we conclude on the basis of estimate (4.4.34) that the norm $\|\,(\mathbf{F}_k)_t\,\|_{2,\,Q_T}$ is also bounded as $k \to 0$.

Being concerned with $\mathbf{F}_k$ and $(\mathbf{F}_k)_t$ from the space $\mathbf{L}_2(Q_T)$ we can treat the system (4.4.17)–(4.4.19) as a system of the type (4.1.3)–(4.1.5). Just for this reason all the assertions of Theorem 4.2.2 remain valid when expression (4.2.19) is considered in terms of the solutions of (4.4.17)–(4.4.19):

$$
\begin{aligned}
(4.4.36)\quad & \int_\varepsilon^T \left(\frac{1}{2}\,\frac{d}{dt}\,\|\,(\,\mathbf{v}_k - \mathbf{v})_t\,(\,\cdot\,,\,t)\,\|^2_{2,\,\Omega} + \nu\,\|\,(\,\mathbf{v}_k - \mathbf{v})_{tx}\,(\,\cdot\,,\,t)\,\|^2_{2,\,\Omega}\right) dt \\
& = \int_\varepsilon^T \int_\Omega (\mathbf{F}_k)_t\cdot(\,\mathbf{v}_k - \mathbf{v})_t\ dx\ dt\,, \qquad 0 < \varepsilon \leq T\,.
\end{aligned}
$$

Relation (4.4.36) implies that

$$
\begin{aligned}
& \int_\varepsilon^T \left(\frac{1}{2}\,\frac{d}{dt}\,\|\,(\,\mathbf{v}_k - \mathbf{v})_t\,(\,\cdot\,,\,t)\,\|^2_{2,\,\Omega} + \frac{\nu}{c_1(\Omega)}\,\|\,(\,\mathbf{v}_k - \mathbf{v})_x\,(\,\cdot\,,\,t)\,\|^2_{2,\,\Omega}\right)\,dt \\
& \leq \int_\varepsilon^T \|\,(\mathbf{F}_k)_t(\,\cdot\,,\,t)\,\|^2_{2,\,\Omega}\cdot\|\,(\,\mathbf{v}_k - \mathbf{v})_t(\,\cdot\,,\,t)\,\|_{2,\,\Omega}\ dt\,, \qquad 0 < \varepsilon \leq T\,,
\end{aligned}
$$

yielding

$$\frac{1}{2}\,\|\,(\mathbf{v}_k-\mathbf{v})_t(\,\cdot\,,T)\,\|^2_{2,\Omega}\le\|\,(\mathbf{F}_k)_t\,\|_{2,Q_T}\cdot\|\,(\mathbf{v}_k-\mathbf{v})_t\,\|_{2,Q_T}$$

$$+\frac{1}{2}\,\|\,(\mathbf{v}_k-\mathbf{v})_t(\,\cdot\,,\varepsilon)\,\|^2_{2,\Omega}\,.$$

Because $\|\,(\mathbf{v}_k-\mathbf{v})_t(\,\cdot\,,t)\,\|_{2,\Omega}$ is continuous on the segment $[0,T]$, we are able to pass in the preceding inequality to the limit as $\varepsilon\to 0$ and, as a final result, get the estimate

$$\|\,(\mathbf{v}_k-\mathbf{v})_t(\,\cdot\,,T)\,\|^2_{2,\Omega}\le\|\,(\mathbf{F}_k)_t\,\|_{2,Q_T}\cdot\|\,(\mathbf{v}_k-\mathbf{v})_t\,\|_{2,Q_T}\,,\tag{4.4.37}$$

assuming that the boundary condition (4.4.18) is homogeneous.

As stated above, the norm $\|\,(\mathbf{F}_k)_t\,\|_{2,Q_T}$ is bounded as $k\to\infty$. Applying estimate (4.4.37), relation (4.4.32) and the definition of the operator A we establish

$$\|\,A\,\mathbf{f}_k-A\,\mathbf{f}\,\|_{2,\Omega}=\|\,(\mathbf{v}_k-\mathbf{v})_t(\,\cdot\,,T)\,\|_{2,\Omega}\longrightarrow 0\,,\qquad k\to\infty\,.$$

This provides support for the view that the nonlinear operator A is continuous on D.

Let us show that any bounded subset of the set D is carried by the operator A into a set being compact in the space $\mathbf{L}_2(\Omega)$. To that end, we rewrire the system (4.4.1)–(4.4.3) as

$$\mathbf{v}_t-\nu\,\Delta\,\mathbf{v}=-\nabla\,p+\mathbf{F}_1\,,\quad \operatorname{div}\mathbf{v}=0\,,\quad (x,\,t)\in Q_T\,,\tag{4.4.38}$$

$$\mathbf{v}\,(x,\,0)=\mathbf{a}(x)\,,\quad x\in\Omega\,,\quad \mathbf{v}\,(x,\,t)=0\,,\quad (x,\,t)\in S_T\,,\tag{4.4.39}$$

where

$$\mathbf{F}_1=\mathbf{f}(x)\;g(x,\,t)-(\mathbf{v},\,\nabla)\,\mathbf{v}\,.$$

As in the derivation of (4.4.25) and (4.4.34) we make use of the estimates

$$\begin{aligned}\|\,\mathbf{F}_1\,\|_{2,Q_T}&\le\left[\int_0^T\sup_{x\in\Omega}\,|\,g(x,\,t)\,|^2\,dt\right]^{1/2}\|\,\mathbf{f}\,\|_{2,\Omega}\\&\quad+c^*\,M_5(\mathbf{f})\,\|\,\mathbf{v}_x\,\|_{2,Q_T}\,,\\ \|\,(\mathbf{F}_1)_t\,\|_{2,Q_T}&\le\left[\int_0^T\sup_{x\in\Omega}\,|\,g_t(x,\,t)\,|^2\,dt\right]^{1/2}\|\,\mathbf{f}\,\|_{2,\Omega}\\&\quad+c^*\,M_5(\mathbf{f})\,\|\,\mathbf{v}_{tx}\,\|_{2,Q_T}\,,\end{aligned}\tag{4.4.40}$$

where $M_5(\mathbf{f})$ has been introduced in (4.4.24).

Further treatment of (4.4.38)–(4.4.39) as a system of the type (4.1.3)–(4.1.4) with reference to the proof of Theorem 4.2.3 permits us to deduce, keeping $\mathbf{v}_t(\,\cdot\,, T) \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$ and holding ε from the interval $(0, T)$ fixed, that (4.2.34) yields

$$
\text{(4.4.41)} \quad \| \mathbf{v}_{tx}(\,\cdot\,, T) \|_{2,\Omega}^2 \le \frac{1}{\nu} \, \| (\mathbf{F}_1)_t \|_{2, Q_T}^2 + \frac{1}{\nu (T - \varepsilon)} \times \left[\| \nu \Delta \mathbf{a} - (\mathbf{a}, \nabla) \mathbf{a} + \mathbf{F}(\,\cdot\,, 0) \|_{2,\Omega}^2 + \frac{3 T^2}{2} \, \| (\mathbf{F}_1)_t \|_{2, Q_T}^2 \right].
$$

Finally, we get from (4.1.14), (4.4.40) and (4.4.41) the estimate for the nonlinear operator A

$$
\begin{aligned}
\text{(4.4.42)} \qquad \| (A\mathbf{f})_x \|_{2,\Omega}^2 \le {} & 2 \left[\frac{1}{\nu} + \frac{3 T^2}{2 \nu (T - \varepsilon)} \right] \\
& \times \left[1 + \frac{3 c^* M_5^2(\mathbf{f}) \, \nu^2}{\nu^3 - \beta \sqrt{m_1 m_2}} \right] . \\
& \times \| \mathbf{f} \|_{2,\Omega}^2 \int_0^T \sup_{x \in \Omega} \, | g_t(x, t) |^2 \, dt \\
& + \left\{ 8 \left[\frac{1}{\nu} + \frac{3 T^2}{2 \nu (T - \varepsilon)} \right] \right. \\
& \left. \times \frac{c^* M_5^2(\mathbf{f}) \, \nu^2}{\nu^3 - \beta \sqrt{m_1 m_2}} + \frac{2}{\nu (T - \varepsilon)} \right\} \\
& \times \Big(\| \nu \Delta \mathbf{a} - (\mathbf{a}, \nabla) \mathbf{a} \|_{2,\Omega}^2 \\
& + \| \mathbf{f} \|_{2,\Omega}^2 \sup_{x \in \Omega} \, | g(x, 0) |^2 \Big) .
\end{aligned}
$$

Let an arbitrary bounded subset D_1 of the set D be given. Recall that

$$
D = \{ \mathbf{f} \in \mathbf{L}_2(\Omega) \colon \ \| \mathbf{f} \|_{2,\Omega} \le 1 \}
$$

is referred to as the range of the operator A.

Estimate (4.4.42) implies that the nonlinear operator A acts from $D \subset \mathbf{L}_2(\Omega)$ into $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ and maps D_1 into a certain set $\tilde{D}_1$ being bounded in the space $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$.

On account of Rellich's theorem, the set $\widetilde{D}_1$ is compact in the space $\mathbf{L}_2(\Omega)$ and, therefore, the operator A is continuous on D and carries any bounded subset of D into a set being compact in the space $\mathbf{L}_2(\Omega)$. From such reasoning it seems clear that A is completely continuous on D and the operator B specified by (4.4.15) is completely continuous as the composition of a nonlinear completely continuous operator and a linear bounded one. This proves the desired assertion. ■

In the following theorem we try to establish an interconnection between the nonlinear equation (4.4.16) and the inverse problem (4.4.1)–(4.4.4).

Theorem 4.4.2 *Let*

$$\Omega \subset \mathbf{R}^3, \quad g,\, g_t \in C(\bar{Q}_T), \quad |g(x, T)| \geq g_T > 0 \quad \textit{for} \quad x \in \bar{\Omega},$$

$$\mathbf{a},\, \varphi \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega), \qquad \nabla\, \psi \in \mathbf{G}(\Omega).$$

One assumes, in addition, that inequality (4.4.14) *is valid and*

$$\sqrt{\beta}\, \|\, \varphi \,\|_{4,\,\Omega} < \nu, \tag{4.4.43}$$

where $\beta = (4/3)^{3/4}\, c_1^{1/4}(\Omega)$ *and* $c_1(\Omega)$ *is the constant from* (4.2.21). *If equation* (4.4.16) *has a solution lying within D, then there exists a solution of the inverse problem* (4.4.1)–(4.4.4).

Proof By assumption, the nonlinear equation (4.4.16) has a solution lying within D, say $\mathbf{f}$. As stated above, inequality (4.4.16) implies that the function $\mathbf{F} = \mathbf{f}\, g$ admits estimate (4.1.11). When treating (4.4.1)–(4.4.3) as a system of the type (4.1.8)–(4.1.10), we look for a set $\{\mathbf{v},\, \nabla\, p\}$ as the unique solution of the direct problem associated with the external force function $\mathbf{F} = \mathbf{f}\, g$ and the initial velocity $\mathbf{a}$ in complete agreement with Theorem 4.1.4.

In order to prove that $\mathbf{v}$ and $\nabla\, p$ satisfy the overdetermination condition (4.4.4) we agree to consider $\mathbf{v}(x, T) = \varphi_1$ and $\nabla\, p(x, T) = \nabla\, \psi_1$. It is evident that the new functions $\varphi^* = \varphi - \varphi_1$ and $\nabla\, \psi^* = \nabla(\psi - \psi_1)$ satisfy the system of equations

$$\begin{aligned} -\nu\, \Delta\varphi^* + (\varphi^*,\, \nabla)\, \varphi + (\varphi,\, \nabla)\, \varphi^* &= -\nabla\, \psi^*, \\ \operatorname{div} \varphi^* = 0\,, \qquad & x \in \Omega\,, \end{aligned} \tag{4.4.44}$$

supplied by the boundary condition

$$\varphi^* = 0\,, \qquad x \in \partial\Omega\,. \tag{4.4.45}$$

Furthermore, we multiply both sides of the first equation by φ^* scalarly in $\mathbf{L}_2(\Omega)$. Integrating by parts yields

$$\nu \, \| \varphi_x^* \|_{2,\Omega}^2 = - \int_\Omega (\varphi^*, \nabla)\, \varphi \cdot \varphi^* \, dx\,, \tag{4.4.46}$$

since

$$\int_\Omega (\varphi_1, \nabla)\, \varphi^* \cdot \varphi^* \, dx = 0\,.$$

In view of (4.4.8), relation (4.4.46) assures us of the validity of the estimate

$$\nu \, \| \varphi_x^* \|_{2,\Omega}^2 \le \sqrt{\beta}\, \| \varphi \|_{4,\Omega} \cdot \| \varphi_x^* \|_{2,\Omega}^2 \tag{4.4.47}$$

with $\beta = (4/3)^{3/4}\, c_1^{1/4}(\Omega)$. Here we used also the inequality

$$\left| \int_\Omega (\varphi^*, \nabla)\, \varphi \cdot \varphi^* \, dx \right| \le \| \varphi^* \|_{4,\Omega} \cdot \| \varphi \|_{4,\Omega} \cdot \| \varphi_x^* \|_{2,\Omega} \,.$$

When φ is recovered from (4.4.43), $\varphi_1 = \varphi$ and $\nabla \psi_1 = \nabla \psi$, valid almost everywhere in Ω, are ensured by estimate (4.4.47). Eventually, we deduce that the collection of the functions $\{\mathbf{v},\, \nabla p,\, f\}$ satisfies (4.4.1)–(4.4.4), that is, there exists a solution of the inverse problem (4.4.1)–(4.4.4), thereby justifying the assertion of the theorem. ■

In the next theorem we establish sufficient conditions under which the inverse problem under consideration will be solvable. It is worth noting here that all of the restrictions imposed above are formulated in terms of the input data.

Theorem 4.4.3 *Let*

$$\Omega \subset \mathbf{R}^3, \quad g\,,\, g_t \in C(\bar{Q}_T)\,, \quad |g(x, T)| \ge g_T > 0 \quad \textit{for} \quad x \in \bar{\Omega}\,,$$

$$\mathbf{a},\, \varphi \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \qquad \nabla \psi \in \mathbf{G}(\Omega)$$

and relations (4.4.14) *and* (4.4.43) *occur. One assumes, in addition, that the estimate*

$$m_4 = m_3 + \| \chi \|_{2,\Omega} < 1 \tag{4.4.48}$$

is valid with

$$m_3 = \Big[\inf_{x \in \Omega} | g(x, T) | \Big]^{-1} \Big\{ \Big[\| \nu \Delta \mathbf{a} - (\mathbf{a}, \nabla) \mathbf{a} \|_{2,\Omega}$$

$$+ \sup_{x \in \Omega} | g(x, 0) | \exp \Big[- \frac{\nu T}{2 c_1(\Omega)} \Big] \Big]$$

$$+ \int_0^T \sup_{x \in \Omega} | g_t(x, t) | \exp \Big[- \frac{\nu (T - t)}{2 c_1(\Omega)} \Big] \, dt \Big\}$$

and

$$\chi = \frac{1}{g(x, T)} \Big[-\nu \Delta \varphi + (\varphi, \nabla) \varphi + \nabla \psi \Big] .$$

Then there exists a solution of the inverse problem (4.4.1)–(4.4.4).

Proof We proceed to derive an a priori estimate for the nonlinear operator B on D. The energy identity (see Ladyzhenskaya (1970), p. 184) suits us perfectly after rewriting it in terms of (4.4.1):

$$\frac{1}{2} \frac{d}{dt} \| \mathbf{v}_t(\cdot, t) \|_{2,\Omega}^2 + \nu \| \mathbf{v}_{tx}(\cdot, t) \|_{2,\Omega}^2 \tag{4.4.49}$$

$$= - \int_\Omega (\mathbf{v}_t, \nabla) \mathbf{v} \cdot \mathbf{v}_t \, dx + \int_\Omega (\mathbf{f} \, g_t) \cdot \mathbf{v}_t \, dx .$$

We are led by evaluating the first term on the right-hand side of (4.4.49) with the aid of (4.2.21) and (4.4.8) to

$$\Big| \int_\Omega (\mathbf{v}_t, \nabla) \mathbf{v} \cdot \mathbf{v}_t \, dx \Big| \leq \| \mathbf{v}_t(\cdot, t) \|_{4,\Omega}^2 \cdot \| \mathbf{v}_x(\cdot, t) \|_{2,\Omega} \tag{4.4.50}$$

$$\leq (4/3)^{3/4} c_1^{1/2}(\Omega) \| \mathbf{v}_{tx}(\cdot, t) \|_{2,\Omega}^2$$

$$\times \| \mathbf{v}_x(\cdot, t) \|_{2,\Omega} .$$

Upon substituting (4.4.49) into (4.4.50) we can estimate the second term on the right by the Cauchy–Schwarz inequality and then apply (4.1.12). As a final result we get

$$(4.4.51) \qquad \frac{1}{2} \frac{d}{dt} \| \mathbf{v}_t(\cdot, t) \|^2_{2,\Omega} + \nu \| \mathbf{v}_{tx}(\cdot, t) \|^2_{2,\Omega}$$

$$\leq \| \mathbf{f}(\cdot) g_t(\cdot, t) \|_{2,\Omega} \cdot \| \mathbf{v}_t(\cdot, t) \|_{2,\Omega}$$

$$+ \beta (\nu^{-1} m_1 m_2)^{1/2} \| \mathbf{v}_{tx}(\cdot, t) \|^2_{2,\Omega}$$

with $\mathbf{f} \in D$ incorporated.

Provided that (4.4.14) holds and the norm $\| \mathbf{v}_{tx}(\cdot, t) \|_{2,\Omega}$ has been bounded by (4.2.21), expression (4.4.51) implies that

$$\frac{d}{dt} \| \mathbf{v}_t(\cdot, t) \|_{2,\Omega} - \frac{\nu}{2 c_1(\Omega)} \| \mathbf{v}_t(\cdot, t) \|_{2,\Omega} \leq \| \mathbf{f}(\cdot) g_t(\cdot, t) \|_{2,\Omega}, \quad \mathbf{f} \in D.$$

Whence by Gronwall's lemma it follows that

$$\| \mathbf{v}_t(\cdot, T) \|_{2,\Omega} \leq \| \mathbf{v}_t(\cdot, 0) \|_{2,\Omega} \, \exp \left[- \frac{\nu T}{2 c_1(\Omega)} \right]$$

$$+ \int_0^T \| \mathbf{f}(\cdot) g_t(\cdot, t) \|_{2,\Omega} \, \exp \left[- \frac{\nu (T - t)}{2 c_1(\Omega)} \right] dt, \quad \mathbf{f} \in D,$$

With this relation established, it is plain to show that the nonlinear operator B admits the estimate

$$\| B \, \mathbf{f} \|_{2,\Omega} \leq m_3 \quad \text{for any} \quad \mathbf{f} \in D.$$

Further, we refer to a nonlinear operator B_1 acting in D in accordance with the rule

$$B_1 \mathbf{f} = B \mathbf{f} + \chi,$$

where

$$\chi = \frac{1}{g(x, T)} \left[-\nu \Delta \varphi + (\varphi, \nabla) \varphi + \nabla \psi \right].$$

It is clear that (4.4.16) takes now the form

$$(4.4.52) \qquad \mathbf{f} = B_1 \mathbf{f}, \qquad \mathbf{f} \in D.$$

Because of (4.4.48), the operator B_1 admits the estimate

$$\| B_1 \mathbf{f} \|_{2,\Omega} \leq m_3 + \| \chi \|_{2,\Omega} = m_4 < 1$$

and, consequently, maps the set D onto itself.

Theorem 4.4.1 implies that B_1 is completely continuous on $D \subset \mathbf{L}_2(\Omega)$. In conformity with Schauder's fixed-point principle, valid for nonlinear operators, equation (4.4.52) has a solution lying within D. Hence there exists a solution to equation (4.4.16) and this solution also belongs to D. Finally, by Theorem 4.4.2 we deduce that there exists a solution of the inverse problem (4.4.1)–(4.4.4), thereby justifying the desired assertion. ■

As possible illustrations we give several remarks.

Remark 4.4.1 Let in the conditions of Theorem 4.4.3 $\mathbf{a} = 0$ and the function g will be independent of x. If $g \in C^1([0, T])$, $g(t) \geq 0$, $g'(t) \geq 0$ and $g(T) \neq 0$, then inequality (4.4.14) is valid as long as

$$2\, g^2(T) < \nu^3 (4\, \beta^2)^{-1}.$$

Note that $m_3 < 1$ for any T, $0 < T < \infty$. We choose the functions φ and $\nabla \psi$ so as to satisfy (4.4.43) and provide the validity of the estimate

$$\| -\nu \Delta \varphi + (\varphi, \nabla)\, \varphi + \nabla \psi \|_{2,\Omega} \leq (1 - m_3)\; g(T)\,.$$

Then there exists a solution of the inverse problem (4.4.1)–(4.4.4) with these input data and the final moment $t = T$ with the bounds

$$0 < T < \nu^3 \left\{ 8 \left[\beta\, g(T) \right]^2 \right\}^{-1}.$$

Remark 4.4.2 In the study of the inverse problem (4.4.1)–(4.4.4) we do follow proper guidelines of Remark 4.2.1. The relevant results are available in Prilepko and Vasin (1991).

4.5 Nonstaionary nonlinear system of Navier–Stokes equations: two-dimensional flow

We now focus the reader's attention on the inverse problem (4.4.1)–(4.4.4) when the fluid flow is plane-parallel, that is, $\Omega \subset \mathbf{R}^2$. The main feature of the two-dimensional case is that Theorem 4.1.6 on solvability of the corresponding nonlinear direct problem (4.1.8)–(4.1.10) is of global character in contrast to Theorem 4.1.4 with regard to $\Omega \subset \mathbf{R}^3$. The basic trends and tools of research rely essentially on the arguments similar to those adopted earlier in the three-dimensional case.

Let us derive a nonlinear operator equation associated with the function $\mathbf{f}$. Holding $\mathbf{f}$ from the space $\mathbf{L}_2(\Omega)$ fixed and arguing as in Section 4.4, we refer to the nonlinear operator

$$B\colon\ \mathbf{L}_2(\Omega) \mapsto \mathbf{L}_2(\Omega)$$

with the values

$$(B\ \mathbf{f})\,(x) = \frac{1}{g(x,\,T)}\ \mathbf{v}_t(x,\,T)\,, \tag{4.5.1}$$

where the function $\mathbf{v}$ is determined as a unique generalized solution of the direct problem (4.4.1)–(4.4.3), which exists and possesses the required smoothness in accordance with Theorems 4.1.6 and 4.1.7, respectively.

Unlike the three-dimensional case the nonlinear operator B is usually defined on the entire space $\mathbf{L}_2(\Omega)$.

Consider the nonlinear equation of the second kind for the function $\mathbf{f}$ over the space $\mathbf{L}_2(\Omega)$:

$$\mathbf{f} \;=\; B\,\mathbf{f} + \chi\,, \tag{4.5.2}$$

where

$$\chi \;=\; \frac{1}{g(x,\,T)}\ \Big[-\nu\,\Delta\,\varphi + (\varphi,\,\nabla)\,\varphi + \nabla\,\psi\Big]\,.$$

Theorem 4.5.1 *Let*

$$\Omega \in \mathbf{R}^2, \quad g,\,g_t \subset C(\bar{Q}_T)\,, \quad |\,g(x,\,T)\,| \geq g_T > 0 \quad \textit{for} \quad x \in \bar{\Omega}\,,$$

$$\mathbf{a} \in \mathbf{W}_2^2(\Omega) \bigcap \mathring{\mathbf{W}}_2^1(\Omega) \bigcap \mathring{\mathbf{J}}\,(\Omega)\,.$$

Then the nonlinear operator B is completely continuous on the space $\mathbf{L}_2(\Omega)$.

Arguing as in the proof of Theorem 4.4.1 we can arrive at the assertion of the theorem. More specifically, in this case the norms $\|\,\mathbf{v}_t(\,\cdot\,,\,t)\,\|_{2,\,\Omega}$ and $\|\,\mathbf{v}_x(\,\cdot\,,\,t)\,\|_{2,\,\Omega}$ should be majorized with the aid of (4.1.18) and (4.1.17) rather than of (4.1.13) and (4.1.12) (for more detail see Prilepko and Vasin (1989b)).

Theorem 4.5.2 *Let*

$$\Omega \subset \mathbf{R}^2, \quad g\,,\, g_t \in C(\bar{Q}_T)\,, \quad |\, g(x,\,T)| \geq g_T > 0 \quad for \quad x \in \bar{\Omega}\,,$$

$$\mathbf{a},\ \varphi \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \qquad \nabla\,\psi \in \mathbf{G}(\Omega)$$

and let

$$\|\,\varphi\,\|_{4,\,\Omega} \left[2\,c_1(\Omega)\right]^{1/4} < \nu\,, \tag{4.5.3}$$

where $c_1(\Omega)$ *is the constant from* (4.2.21). *If equation* (4.5.2) *is solvable, then so is the inverse problem* (4.4.1)–(4.4.4).

Proof In establishing this result we approve the scheme of proving Theorem 4.4.2 with minor changes. In particular, (4.4.46) should be estimated as follows:

$$\begin{aligned}
\nu\,\|\,\varphi_x^*\,\|_{2,\,\Omega}^2 &\leq \left| \int\limits_{\Omega} (\varphi^*,\,\nabla)\,\varphi\,\cdot\,\varphi^*\ dx \right| \\
&\leq \|\,\varphi^*\,\|_{4,\,\Omega}\,\cdot\,\|\,\varphi\,\|_{4,\,\Omega}\,\cdot\,\|\,\varphi_x^*\,\|_{2,\,\Omega} \\
&\leq \left[2\,c_1(\Omega)\right]^{1/4} \|\,\varphi\,\|_{4,\,\Omega}\,\cdot\,\|\,\varphi_x^*\,\|_{2,\,\Omega}^2\,.
\end{aligned}$$

This is due to the fact that any function $\mathbf{u} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$, $\Omega \subset \mathbf{R}^2$, is subject to the relation

$$\|\,\mathbf{u}\,\|_{4,\,\Omega}^4 \leq 2\,\|\,\mathbf{u}\,\|_{2,\,\Omega}^2\,\cdot\,\|\,\mathbf{u}_x\,\|_{2,\,\Omega}^2\,. \tag{4.5.4}$$

Theorem 4.5.3 *Let*

$$\Omega \subset \mathbf{R}^2, \quad g\,,\, g_t \in C(\bar{Q}_T)\,, \quad |\, g(x,\,T)| \geq g_T > 0 \quad for \quad x \in \bar{\Omega}\,,$$

$$\mathbf{a},\ \varphi \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \qquad \nabla\,\psi \in \mathbf{G}(\Omega)$$

and let estimate (4.5.3) *be true. If*

$$m_6 = m_5 + \|\,\chi\,\|_{2,\,\Omega} < 1\,, \tag{4.5.5}$$

where

$$m_5 = \Big[\inf_{x\in\Omega} |g(x,T)| \Big]^{-1} \Big\{ \Big[\| \nu\,\Delta\,\mathbf{a} - (\mathbf{a},\nabla)\,\mathbf{a} \|_{2,\Omega}$$

$$+ \sup_{x\in\Omega} |g(x,0)| \exp\Big[-\frac{\nu T}{2\,c_1(\Omega)} \Big] \Big]$$

$$+ \int_0^T \exp\Big[-\frac{\nu (T-t)}{2\,c_1(\Omega)} \Big] |g_t(x,t)|\, dt \Big\}$$

$$\times \exp\Big\{ \frac{1}{\nu^2} \Big[\| \mathbf{a} \|_{2,\Omega}^2 + \frac{3}{2} \Big(\int_0^T \sup_{x\in\Omega} |g(x,T)|\, dt \Big)^2 \Big] \Big\},$$

$c_1(\Omega)$ is the same constant as in (4.2.21) *and*

$$\chi = \frac{1}{g(x,T)} \Big[-\nu\,\Delta\,\varphi + (\varphi,\nabla)\,\varphi + \nabla\,\psi \Big],$$

then there exists a solution of the inverse problem (4.4.1)–(4.4.4).

Proof For the same reason as in proving Theorem 4.4.3 we employ identity (4.4.49). With the aid of (4.5.3) the first term on the right-hand side of (4.4.49) can be estimated as follows:

$$\text{(4.5.6)} \qquad \Big| \int_\Omega (\mathbf{v}_t,\nabla)\,\mathbf{v}\cdot\mathbf{v}_t\, dx \Big| \le \| \mathbf{v}_t(\cdot,t) \|_{4,\Omega}^2 \cdot \| \mathbf{v}_x(\cdot,t) \|_{2,\Omega}$$

$$\le \sqrt{2}\, \| \mathbf{v}_{tx}(\cdot,t) \|_{2,\Omega}$$

$$\times \| \mathbf{v}_t(\cdot,t) \|_{2,\Omega} \cdot \| \mathbf{v}_x(\cdot,t) \|_{2,\Omega}.$$

Substituting (4.5.6) into (4.4.49) yields

$$\text{(4.5.7)} \quad \frac{1}{2}\,\frac{d}{dt} \| \mathbf{v}_t(\cdot,t) \|_{2,\Omega}^2 + \nu \| \mathbf{v}_{tx}(\cdot,t) \|_{2,\Omega}^2$$

$$\le \| \mathbf{f}(\cdot)\, g_t(\cdot,t) \|_{2,\Omega} \cdot \| \mathbf{v}_t(\cdot,t) \|_{2,\Omega}$$

$$+ \sqrt{2}\, \| \mathbf{v}_{tx}(\cdot,t) \|_{2,\Omega} \cdot \| \mathbf{v}_t(\cdot,t) \|_{2,\Omega} \cdot \| \mathbf{v}_x(\cdot,t) \|_{2,\Omega}.$$

In rearranging the right-hand side of (4.5.7) we need the inequality

$$2\,a\,b < \delta\, a^2 + \delta^{-1}\, b^2,$$

which is valid for arbitrary $\delta > 0$ and yields

$$(4.5.8)\quad \| \mathbf{v}_{tx}(\,\cdot\,,t)\|_{2,\Omega}\cdot\| \mathbf{v}_t(\,\cdot\,,t)\|_{2,\Omega}\cdot\| \mathbf{v}_x(\,\cdot\,,t)\|_{2,\Omega}$$
$$\le \tfrac{1}{2}\,\delta\,\| \mathbf{v}_{tx}(\,\cdot\,,t)\|^2_{2,\Omega} + \tfrac{1}{2}\,\delta^{-1}\,\| \mathbf{v}_t(\,\cdot\,,t)\|^2_{2,\Omega}\cdot\| \mathbf{v}_x(\,\cdot\,,t)\|^2_{2,\Omega}\,.$$

We substitute (4.5.8) with $\delta = \nu/\sqrt{2}$ into (4.5.7) and take into account (4.2.21). As a final result we get

$$(4.5.9)\quad \frac{d}{dt}\,\| \mathbf{v}_t(\,\cdot\,,t)\|_{2,\Omega} + \frac{\nu}{2\,c_1\,(\Omega)}\,\| \mathbf{v}_t(\,\cdot\,,t)\|_{2,\Omega}$$
$$\le\ \frac{1}{\nu}\,\| \mathbf{v}_x(\,\cdot\,,t)\|^2_{2,\Omega}\cdot\| \mathbf{v}_t(\,\cdot\,,t)\|_{2,\Omega} + \| \mathbf{f}(\,\cdot\,)\,g_t\,(\,\cdot\,,t)\|_{2,\Omega}\,.$$

Multiplying both sides of (4.5.9) by $\exp\left[-\frac{\nu(T-t)}{2\,c_1(\Omega)}\right]$ and introducing the new functions

$$y(t) = \exp\left[-\frac{\nu\,(T-t)}{2\,c_1(\Omega)}\right]\| \mathbf{v}_t(\,\cdot\,,t)\|_{2,\Omega}\,,$$

$$\alpha_1(t) = \frac{1}{\nu}\,\| \mathbf{v}_x(\,\cdot\,,t)\|^2_{2,\Omega}\,,$$

$$\alpha_2(t) = \exp\left[-\frac{\nu\,(T-t)}{2\,c_1(\Omega)}\right]\| \mathbf{f}(\,\cdot\,)\,g_t\,(\,\cdot\,,t)\|_{2,\Omega}\,,$$

one can rewrite (4.5.9) as

$$(4.5.10)\qquad \frac{d\,y(t)}{dt}\ \le\ \alpha_1(t)\,y(t) + \alpha_2(t)\,.$$

It is easily seen that (4.5.10) satisfies Gronwall's lemma. In the preceding notations,

$$(4.5.11)\quad \| \mathbf{v}_t(\,\cdot\,,t)\|_{2,\Omega} \le \exp\left[\frac{1}{\nu}\,\| \mathbf{v}_x\|^2_{2,Q_T}\right]$$
$$\times\left\{\| \mathbf{v}_t(\,\cdot\,,0)\|_{2,\Omega}\exp\left[-\frac{\nu\,t}{2\,c_1(\Omega)}\right]\right.$$
$$\left. + \int\limits_0^t \| \mathbf{f}(\,\cdot\,)\,g_\tau(\,\cdot\,,\tau)\|_{2,\Omega}\exp\left[-\frac{\nu(T-\tau)}{2\,c_1(\Omega)}\right]d\tau\right\}.$$

Because of (4.1.17), inequality (4.5.11) assures us of the validity of the estimate

$$(4.5.12)\quad \| \mathbf{v}_t(\cdot, T) \|_{2,\Omega} \leq \exp \left\{ \frac{1}{\nu^2} \left[\| \mathbf{a} \|_{2,\Omega}^2 + \frac{3}{2} \left(\int_0^T \sup_{x\in\Omega} |g(x,t)|\, dt \right)^2 \| \mathbf{f} \|_{2,\Omega}^2 \right] \right\}$$

$$\times \left\{ \left[\| \nu \Delta \mathbf{a} - (\mathbf{a}, \nabla) \mathbf{a} \|_{2,\Omega} + \sup_{x\in\Omega} |g(x,0)| \, \| \mathbf{f} \|_{2,\Omega} \right] \exp \left[-\frac{\nu T}{2 c_1(\Omega)} \right] + \int_0^T \sup_{x\in\Omega} |g_t(x,t)| \exp \left[-\frac{\nu (T-t)}{2 c_1(\Omega)} \right] dt \| \mathbf{f} \|_{2,\Omega} \right\}.$$

Relation (4.5.12) serves to motivate that the nonlinear operator B specified by (4.5.1) admits the estimate

$$(4.5.13)\qquad \| B\, \mathbf{f} \|_{2,\Omega} \leq m_5, \qquad \mathbf{f} \in D,$$

where

$$D = \left\{ \mathbf{f} \in \mathbf{L}_2(\Omega) \colon \| \mathbf{f} \|_{2,\Omega} \leq 1 \right\}.$$

We refer to the nonlinear operator

$$B_1 \colon \mathbf{L}_2(\Omega) \mapsto \mathbf{L}_2(\Omega)$$

with the values

$$(4.5.14)\qquad B_1 \mathbf{f} = B\, \mathbf{f} + \chi,$$

where

$$\chi = \frac{1}{g(x,T)} \left[-\nu \Delta \varphi + (\varphi, \nabla) \varphi + \nabla \psi \right].$$

Theorem 4.5.1 guarantees that the operator B is completely continuous on $\mathbf{L}_2(\Omega)$. Then so is the operator B_1 on $\mathbf{L}_2(\Omega)$. From (4.5.5) and (4.5.13) it follows that

$$(4.5.15)\qquad \| B_1 \mathbf{f} \|_{2,\Omega} \leq m_5 + \| \chi \|_{2,\Omega} = m_6 < 1$$

for arbitrary $\mathbf{f} \in D$.

This provides enough reason to conclude that the nonlinear operator B_1 is completely continuous and carries the closed bounded set D into itself. Schauder's fixed-point principle implies that B_1 has a fixed point lying within D. In the language of operator equations, this means that the equation

$$\mathbf{f} = B_1 \mathbf{f} \tag{4.5.16}$$

possesses a solution and this solution belongs to D. True, it is to be shown that the same remains valid for equation (4.5.2). Then Theorem 4.5.2 yields that there exists a solution of the inverse problem (4.4.1)–(4.4.4), thereby completing the proof of the theorem. ■

Remark 4.5.1 Of special interest is one possible application to $\mathbf{a} = 0$ and $g(x, t) \equiv 1$. In that case $m_1 < 1$ for any T from the interval $\big(0, \nu^3/[3\, c_1(\Omega)]\big)$. The functions φ and $\nabla \psi$ can be so chosen as to satisfy (4.5.3) and the inequality

$$\| - \nu \Delta \varphi + (\varphi, \nabla)\, \varphi + \nabla \psi \|_{2,\Omega} < 1 - m_1 , \tag{4.5.17}$$

thereby providing the validity of estimate (4.5.5). Therefore, Theorem 4.5.3 yields that there exists a solution of the inverse problem (4.4.1)–(4.4.4) with these input data once we take the final moment $t = T$ from the interval $\big(0, \nu^3/[3\, c_1(\Omega)]\big)$.

4.6 Nonstationary nonlinear system of Navier–Stokes equations: the integral overdetermination

Given a bounded domain Ω in the plane $\mathbf{R}^2$ with boundary $\partial\Omega$ of class C^2, the system consisting of the **nonstationary nonlinear Navier–Stokes equations** and the **incompressibility equation**

$$\begin{aligned} \mathbf{v}_t - \nu \Delta \mathbf{v} + (\mathbf{v}, \Delta)\, \mathbf{v} &= -\nabla p + \mathbf{F}(x,t) , \\ \operatorname{div} \mathbf{v} &= 0 , \end{aligned} \tag{4.6.1}$$

will be of special investigations for $(x, t) \in Q_T \equiv \Omega \times (0, T)$, $0 < T < \infty$, provided that the vector external force function $\mathbf{F}$ is representable by

$$\mathbf{F} = f(t)\; \mathrm{g}(x,t) , \tag{4.6.2}$$

where the vector g is known in advance, while the unknown scalar coefficient f is sought.

The nonlinear inverse problem consists of finding a set of the functions $\{\mathbf{v}, \nabla p, f\}$, which satisfy the system (4.6.1)–(4.6.2) and the function $\mathbf{v}$ involved complies also with the initial condition

$$\mathbf{v}(x,0) = \mathbf{a}(x)\,, \qquad x \in \Omega\,, \tag{4.6.3}$$

the boundary condition

$$\mathbf{v}(x,t) = 0\,, \qquad (x,t) \in S_T = \partial\Omega \times [0,T]\,, \tag{4.6.4}$$

and the condition of **integral overdetermination**

$$\int\limits_\Omega \mathbf{v}(x,t)\cdot\boldsymbol{\omega}(x)\,dx = \varphi(t)\,, \qquad 0 \le t \le T\,, \tag{4.6.5}$$

when operating with the functions $\mathbf{g}$, $\boldsymbol{\omega}$, $\mathbf{a}$, φ and the coefficient ν.

Definition 4.6.1 *A pair of the functions $\{\mathbf{v}, f\}$ is said to be a weak generalized solution of the nonlinear inverse problem concerned if*

$$\mathbf{v} \in \mathbf{C}\big([0,T]; \overset{\circ}{\mathbf{J}}(\Omega)\big) \bigcap \mathbf{L}_2\big([0,T]; \overset{\circ}{\mathbf{W}}{}^1_2(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\big)\,;$$

$$\int\limits_\Omega \mathbf{v}(x,t)\cdot\boldsymbol{\Phi}(x)\,dx \in \mathbf{W}^1_2(0,T)\,,\ \forall \boldsymbol{\Phi} \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega)\bigcap\overset{\circ}{\mathbf{J}}(\Omega)\,;\ f \in L_2(0,T)\,,$$

and they satisfy for any $\boldsymbol{\Phi} \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$ the integral identity

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega & \mathbf{v}(x,t)\cdot\boldsymbol{\Phi}(x)\,dx \\ &+ \nu\int\limits_\Omega \mathbf{v}_x(x,t)\cdot\boldsymbol{\Phi}_x(x)\,dx \\ &+ \int\limits_\Omega \big(\mathbf{v}(x,t),\nabla\big)\mathbf{v}(x,t)\cdot\boldsymbol{\Phi}(x)\,dx \\ &= \int\limits_\Omega f(t)\,\mathbf{g}(x,t)\cdot\boldsymbol{\Phi}(x)\,dx\end{aligned} \tag{4.6.6}$$

along with the initial condition (4.6.3) *and the overdetermination condition* (4.6.5).

It is worth emphasizing here the incompressibility of the fluid and the boundary condition (4.6.4) are taken into account in Definition 4.6.1 in the sense that the function $\mathbf{v}(\,\cdot\,,t)$ belongs to the space $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$ for almost all $t \in [0, T]$.

Later discussions of the inverse problem at hand are based on the paper by Vasin (1993). In the sequel we will assume that the domain Ω of space variables is plane and the input data functions meet the requirements

$$\mathbf{a} \in \overset{\circ}{\mathbf{J}}(\Omega), \qquad \mathbf{g} \in \mathbf{C}\big([0, T], \mathbf{L}_2(\Omega)\big), \qquad \boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega),$$

$$\varphi \in \mathbf{W}_2^1(0, T), \quad \left| \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}(x)\, dx \right| \geq g_0 > 0\,, \quad 0 \leq t \leq T \quad (g_0 \equiv \text{const}\,).$$

With these assumptions, we proceed to derive an operator equation of the second kind for the scalar function f. This amounts to holding an arbitrary function f from the space $\mathbf{L}_2(0, T)$ fixed and substituting it into the system (4.6.1) and identity (4.6.6) both. Combination of identity (4.6.6) written above and the initial condition (4.6.3) constitutes what is called a weak statement of the direct problem (4.6.1)–(4.6.4) of finding a function $\mathbf{v}$.

Since $\mathbf{F} = f\,\mathbf{g} \in \mathbf{L}_2(Q_T)$, $\mathbf{a} \in \overset{\circ}{\mathbf{J}}(\Omega)$ and $\Omega \subset \mathbf{R}^2$, there exists a unique function $\mathbf{v}$ satisfying identity (4.6.6) with the coefficient f fixed and the initial condition (4.6.3). In every such case the function $\mathbf{v}$ should belong to the desired spaces of functions (see Temam (1979)) and any function f from the space $L_2(0, T)$ can uniquely be put in correspondence with the vector function

$$\mathbf{v} \in \mathbf{C}\big([0, T]; \overset{\circ}{\mathbf{J}}(\Omega)\big) \bigcap \mathbf{L}_2\big([0, T]; \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)\big)\,.$$

The traditional way of covering this is to refer to the nonlinear operator

$$A\colon\ L_2(0, T) \mapsto L_2(0, T)$$

acting in accordance with the rule

$$(4.6.7) \qquad (A\,f)(t) = \left\{ \int_\Omega \Big[\nu\, \mathbf{v}_x \cdot \boldsymbol{\omega}_x + (\mathbf{v}, \nabla)\, \mathbf{v} \cdot \boldsymbol{\omega} \Big]\, dx + \varphi'(t) \right\} \Big/\, g_1(t)\,,$$

where $\mathbf{v}$ has been already found as a weak solution of the system (4.6.1)–(4.6.4) and

$$g_1(t) = \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}(x)\, dx\,.$$

We proceed to study the operator equation of the second kind over the space $L_2([0, T])$:

$$f = A f\,. \tag{4.6.8}$$

An interrelation between the inverse problem (4.6.1)–(4.6.5) and the nonlinear equation (4.6.8) from the viewpoint of their solvability is revealed in the following assertion.

Theorem 4.6.1 *Let* $\Omega \subset \mathbf{R}^2$, $\mathbf{g} \in \mathbf{C}([0, T], \mathbf{L}_2(\Omega))$, $\mathbf{a} \in \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$ *and* $\varphi \in \mathbf{W}_2^1(0, T)$,

$$\left| \int_\Omega \mathbf{g}(x, t) \cdot \boldsymbol{\omega}(x)\, dx \right| \geq g_0 > 0\,, \qquad 0 \leq t \leq T\,.$$

Then the following assertions are valid:

(a) *if the inverse problem* (4.6.1)–(4.6.5) *is solvable, then so is equation* (4.6.8);

(b) *if equation* (4.6.8) *has a solution and the compatibility condition*

$$\int_\Omega \mathbf{a}(x) \cdot \boldsymbol{\omega}(x)\, dx = \varphi(0) \tag{4.6.9}$$

holds, then there exists a solution of the inverse problem (4.6.1)–(4.6.5).

Proof We proceed to prove item (a). Let the inverse problem (4.6.1)–(4.6.5) have a solution, say $\{\mathbf{v}, f\}$. By Definition 4.6.1 the pair $\{\mathbf{v}, f\}$ satisfies (4.6.6) with any $\boldsymbol{\Phi}$ from $\mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$. By merely setting $\boldsymbol{\Phi}(x) = \boldsymbol{\omega}(x)$ we obtain

$$\frac{d}{dt} \int_\Omega \mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx + \nu \int_\Omega \mathbf{v}_x(x,t) \cdot \boldsymbol{\omega}_x(x)\, dx + \int_\Omega (\mathbf{v}(x,t), \nabla)\mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx = f(t)\, g_1(t)\,, \tag{4.6.10}$$

where

$$g_1(t) = \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}(x)\, dx .$$

From definition (4.6.7) of the operator A and the overdetermination condition (4.6.5) it follows that the left-hand side of relation (4.6.10) is equal to $g_1(t)\,(A\,f)(t)$, leading by (4.6.10) to

$$A\,f = f .$$

This means that the function f solves equation (4.6.8) and thereby item (a) is completely proved.

We proceed to item (b). Let equation (4.6.8) have a solution belonging to the space $L_2(0,\,T)$. We denote this solution by f and substitute it into (4.6.1)–(4.6.2). After minor manipulations the system (4.6.1)–(4.6.4) may be treated in the context of Theorems 4.1.6–4.1.7 and a function $\mathbf{v}$ is obtained as a unique weak solution of the direct problem for the nonlinear nonstationary Navier–Stokes equations in the case when the domain Ω of space variables is plane. The system (4.6.1) is satisfied by the function $\mathbf{v}$ in the sense of identity (4.6.6). Under such an approach we have found the functions $\mathbf{v}$ and f, which belong to the class of functions from Definition 4.6.1 and satisfy relations (4.6.6) and (4.6.3) both. Let us show that the function $\mathbf{v}$ meets the integral overdetermination (4.6.5) as well. By inserting $\boldsymbol{\Phi}(x) = \boldsymbol{\omega}(x)$ in (4.6.6) we arrive at

$$\frac{d}{dt} \int_\Omega \mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx + \nu \int_\Omega \mathbf{v}_x(x,t) \cdot \boldsymbol{\omega}_x(x)\, dx + \int_\Omega (\mathbf{v}(x,t), \nabla)\mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx = f(t)\, g_1(t) . \tag{4.6.11}$$

On the other hand, the function f being a solution to equation (4.6.8) implies that

$$\varphi'(x) + \nu \int_\Omega \mathbf{v}_x(x,t) \cdot \boldsymbol{\omega}_x(x)\, dx + \int_\Omega (\mathbf{v}(x,t), \nabla)\mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx = f(t)\, g_1(t) . \tag{4.6.12}$$

Subtracting (4.6.12) from (4.6.11) yields

$$\frac{d}{dt} \int_\Omega \mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx - \varphi'(x) = 0 .$$

If, with the compatibility condition (4.6.9) in view, one integrates the preceding relation from 0 to t, then from the resulting expression it seems clear that the function $\mathbf{v}$ does follow the integral overdetermination (4.6.5). Because of this, the collection $\{\mathbf{v},\ f\}$ is just a weak solution of the inverse problem (4.6.1)–(4.6.5) and this proves the assertion of the theorem. ∎

We should raise the question of the solvability of the inverse problem (4.6.1)–(4.6.5). For this, we have occasion to use a closed ball D in the space $L_2(0,\ T)$ with center φ'/g_1 such that

$$\mathcal{D} = \left\{ f \in L_2(0,\ T) \colon \| f - \varphi'/g_1 \|_{2,(0,T)} \leq r \right\}, \tag{4.6.13}$$

where

$$g_1(t) = \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}(x)\ dx\ .$$

In what follows the symbol A^k will stand for the kth degree of the operator A for $k \in \mathbf{N}$.

Theorem 4.6.2 *Let* $\Omega \subset \mathbf{R}^2$, $\mathbf{g} \in \mathbf{C}([0,T], \mathbf{L}_2(\Omega))$, $\mathbf{a} \in \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega}_x \in \mathbf{L}_\infty(\Omega)$, $\varphi \in \mathbf{W}_2^1(0,\ T)$ *and*

$$\left| \int_\Omega \mathbf{g}(x,\ t) \cdot \boldsymbol{\omega}(x)\ dx \right| \geq g_0 > 0 \quad (g_0 \equiv \text{const}), \quad 0 \leq t \leq T\ .$$

One assumes, in addition, that the nonlinear operator A carries the ball $\mathcal{D}$ into itself. Then there exists a positive integer k such that the operator A^k is a contraction mapping in the ball $\mathcal{D}$.

Proof Let either of the functions f_1 and f_2 belong to the ball $\mathcal{D}$. By the definition of the operator A,

$$\| A f_1 - A f_2 \|_{2,(0,t)} = \int_0^t | A f_1 - A f_2 |^2\ d\tau \tag{4.6.14}$$

$$\leq \frac{1}{g_0} \int_0^t \bigg| \int_\Omega \big[\nu(\mathbf{v}_1 - \mathbf{v}_2)_x \cdot \boldsymbol{\omega}_x(x)$$

$$+ (\mathbf{v}_1 - \mathbf{v}_2, \nabla)\, \mathbf{v}_1 \cdot \boldsymbol{\omega}(x)$$

$$+ (\mathbf{v}_2, \nabla)(\mathbf{v}_1 - \mathbf{v}_2) \cdot \boldsymbol{\omega}(x) \big]\ dx \bigg|^2\ d\tau\ ,$$

where the functions $\mathbf{v}_1$ and $\mathbf{v}_2$ are associated with the coefficients f_1 and f_2, respectively.

The absolute value on the right-hand side of (4.6.14) can be estimated as follows:

$$\tag{4.6.15} \begin{aligned} &\Big| \int_\Omega \Big[\nu\, (\mathbf{v}_1 - \mathbf{v}_2)_x \cdot \boldsymbol{\omega}_x(x) + (\mathbf{v}_1 - \mathbf{v}_2, \nabla)\, \mathbf{v}_1 \cdot \boldsymbol{\omega}(x) \\ &\qquad + (\mathbf{v}_2, \nabla)\, (\mathbf{v}_1 - \mathbf{v}_2) \cdot \boldsymbol{\omega}(x) \Big]\, dx \Big| \\ &\le \nu\, \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|_{2,\Omega} \cdot \| \Delta \boldsymbol{\omega} \|_{2,\Omega} \\ &\quad + \Big| \int_\Omega \sum_{k=1}^{2} (v_{1k} - v_{2k})\, (\mathbf{v}_1 \cdot \boldsymbol{\omega}_{x_k})\, dx \Big| \\ &\quad + \Big| \int_\Omega \sum_{k=1}^{2} v_{2k}\, [(\mathbf{v}_1 - \mathbf{v}_2) \cdot \boldsymbol{\omega}_{x_k}]\, dx \Big| \\ &\le \nu\, \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|_{2,\Omega} \cdot \| \Delta \boldsymbol{\omega} \|_{2,\Omega} \\ &\quad + \Big| \int_\Omega | \mathbf{v}_1 - \mathbf{v}_2 |\, | \mathbf{v}_1 |\, | \boldsymbol{\omega}_x |\, dx \Big| \\ &\quad + \Big| \int_\Omega | \mathbf{v}_1 - \mathbf{v}_2 |\, | \mathbf{v}_2 |\, | \boldsymbol{\omega}_x |\, dx \Big| , \end{aligned}$$

where v_{1k} and v_{2k} are the components of the vector functions $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. We substitute (4.6.15) into (4.6.14) and, by obvious rearrangements, led to

$$\tag{4.6.16} \begin{aligned} \| A f_1 - A f_2 \|_{2,(0,t)} \le \frac{1}{g_0} \int_0^t & \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, \tau) \|_{2,\Omega}^2 \\ & \times \Big[\nu\, \| \Delta \boldsymbol{\omega} \|_{2,\Omega} + \operatorname*{ess\,sup}_{x \in \Omega} | \boldsymbol{\omega}_x | \\ & \times \big(\| \mathbf{v}_1(\,\cdot\,, \tau) \|_{2,\Omega} + \| \mathbf{v}_2(\,\cdot\,, \tau) \|_{2,\Omega} \big) \Big]^2 \, d\tau \, . \end{aligned}$$

In subsequent calculations we shall need as yet the estimate

$$(4.6.17)\ \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|_{2,\Omega} \le \int_0^t \| f_1(\tau)\, \mathbf{g}(\,\cdot\,, \tau) - f_2(\tau)\, \mathbf{g}(\,\cdot\,, \tau) \|_{2,\Omega} \times \exp \left\{ \frac{1}{\nu} \int_0^t \| (v_1)_x(\,\cdot\,, \xi) \|^2_{2,\Omega} \, d\xi \right\} d\tau , \quad 0 \le t \le T ,$$

which can be derived in just the same way as we obtained estimate (4.4.12). The only difference here lies in the presence of (4.5.4) instead of (4.5.8).

From (4.6.16) and (4.6.17) it follows that

$$(4.6.18) \qquad \| A f_1 - A f_2 \|^2_{2,(0,t)} \le m_1 \int_0^t \| f_1 - f_2 \|^2_{2,(0,\tau)} \, d\tau ,$$

where

$$m_1 = \frac{T}{g_0} \sup_{t \in [0,T]} \| g(\,\cdot\,, t) \|^2_{2,\Omega} \exp \left(\frac{2}{\nu} \| (\mathbf{v}_1)_x \|^2_{2,Q_T} \right) \times \left[\nu \| \Delta \boldsymbol{\omega} \|_{2,\Omega} + \operatorname*{ess\,sup}_{x \in \Omega} | \boldsymbol{\omega}_x | \left(\sup_{t \in [0,T]} \| \mathbf{v}_1(\,\cdot\,, t) \|_{2,\Omega} + \sup_{t \in [0,T]} \| \mathbf{v}_2(\,\cdot\,, t) \|_{2,\Omega} \right) \right]^2 .$$

The quantity m_1 can be bounded by

$$(4.6.19) \qquad m_1 \le \frac{T}{g_0} \Big\{ \nu \| \Delta \boldsymbol{\omega} \|_{2,\Omega} + \operatorname*{ess\,sup}_{x \in \Omega} | \boldsymbol{\omega}_x | \, \Big[2 \| \mathbf{a} \|_{2,\Omega} + \sqrt{T} \sup_{t \in [0,T]} \| \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} \times \big(\| f_1 \|_{2,(0,T)} + \| f_2 \|_{2,(0,T)} \big) \Big] \Big\}^2 \exp \Big\{ \Big(2 \| \mathbf{a} \|^2_{2,\Omega} + 3T \times \sup_{t \in [0,T]} \| \mathbf{g}(\,\cdot\,, t) \|^2_{2,\Omega} \, \| f_1 \|^2_{2,(0,T)} \Big) \nu^{-2} \Big\}^2 .$$

Here we used also the following estimates with reference to (4.1.16)–(4.1.17):

$$(4.6.20)\qquad \sup_{t\in[0,T]} \| \mathbf{v}(\cdot, t) \|_{2,\Omega} \le \| \mathbf{a} \|_{2,\Omega} + \int_0^t \| f(t)\, \mathbf{g}(\cdot, t) \|_{2,\Omega}\, d\tau ,$$

$$(4.6.21)\qquad 2\nu \, \| \mathbf{v}_x \|^2_{2, Q_T} \le 2 \| \mathbf{a} \|^2_{2,\Omega} + 3 \left(\int_0^t \| f(t)\, \mathbf{g}(\cdot, t) \|_{2,\Omega}\, d\tau \right)^2 .$$

Since either of the functions f_1 and f_2 lies within the ball $\mathcal{D}$, the combination of inequalities (4.6.18) and (4.6.19) gives the estimate

$$(4.6.22)\quad \| A f_1 - A f_2 \|_{2,(0,t)} \le \left(m_2 \int_0^t \| f_1 - f_2 \|^2_{2,(0,\tau)}\, d\tau \right)^{1/2} , \qquad 0 \le t \le T ,$$

where

$$\begin{aligned} m_2 = \frac{T}{g_0} \Bigg[& \nu \, \| \Delta \boldsymbol{\omega} \|_{2,\Omega} + \operatorname*{ess\,sup}_{x\in\Omega} | \boldsymbol{\omega}_x | \\ & \times \left(2 \| \mathbf{a} \|_{2,\Omega} + 2\sqrt{T}\, \tilde{r} \sup_{t\in[0,T]} \| g(\cdot, t) \|_{2,\Omega} \right) \Bigg]^2 \\ & \times \exp \Bigg\{ \nu^{-2} \Bigg(2 \| \mathbf{a} \|^2_{2,\Omega} \\ & + 3 T \tilde{r}^2 \sup_{t\in[0,T]} \| g(\cdot, t) \|^2_{2,\Omega} \Bigg) \Bigg\} , \end{aligned}$$

$$\tilde{r} = r + \| \varphi' / g_1 \|_{2,(0,T)}$$

and r is the radius of the ball $\mathcal{D}$.

It is worth noting here that m_2 is expressed only in terms of input data and does not depend on t.

By assumption, the operator A carries the ball $\mathcal{D}$ into itself that makes it possible to define for any positive integer k the kth degree of the operator A. In what follows this operator will be denoted by the symbol A^k. Via the mathematical induction on k inequality (4.6.22) assures us of the validity of the estimate

$$(4.6.23) \qquad \| A^k f_1 - A^k f_2 \|_{2,(0,T)} \le \left(\frac{m_2^k T^k}{k!} \right)^{1/2} \| f_1 - f_2 \|_{2,(0,T)} .$$

It is clear that

$$\left(m_2^k T^k \right) / k! \longrightarrow 0$$

as $k \longrightarrow \infty$ and, therefore, there exists a positive integer k_0 such that

$$(m_2^{k_0} T^{k_0} / k_0 !)^{1/2} < 1 .$$

Due to estimate (4.6.23) the operator A^{k_0} is a contracting mapping in the ball $\mathcal{D}$. This proves the assertion of the theorem. ■

Sufficient conditions under which the operator A will carry the closed ball $\mathcal{D}$ into itself are established in the following theorem.

Theorem 4.6.3 *Let* $\Omega \subset \mathbf{R}^2$, $\mathbf{g} \in \mathbf{C}([0, T]; \mathbf{L}_2(\Omega))$, $\mathbf{a} \in \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega}_x \in \mathbf{L}_\infty(\Omega)$, $\varphi \in \mathbf{W}_2^1(0, T)$ *and*

$$\left| \int_\Omega \mathbf{g}(x, t) \cdot \boldsymbol{\omega}(x) \, dx \right| \ge g_0 > 0 \quad (g_0 \equiv \text{const}), \quad 0 \le t \le T .$$

If

$$(4.6.24) \qquad m_3 < r ,$$

where

$$m_3 = \frac{T}{g_0} \Bigg[\nu \, \| \Delta \boldsymbol{\omega} \|_{2,\Omega} \left(\| \mathbf{a} \|_{2,\Omega} + \sqrt{T} \, \tilde{r} \sup_{t \in [0,T]} \| g(\cdot, t) \|_{2,\Omega} \right)$$

$$+ \operatorname*{ess\,sup}_{x \in \Omega} | \boldsymbol{\omega}_x | \left(2 \| \mathbf{a} \|_{2,\Omega}^2 + 2 T \tilde{r}^2 \sup_{t \in [0,T]} \| g(\cdot, t) \|_{2,\Omega}^2 \right) \Bigg] ,$$

$$\tilde{r} = r + \| \varphi' / g_1 \|_{2,(0,T)}$$

and r is the radius of the ball $\mathcal{D}$, then the operator A carries the closed ball $\mathcal{D}$ into itself.

Proof We proceed as usual. This amounts to fixing an arbitrary function f from $\mathcal{D}$ and stating, by definition (4.6.13) of the ball $\mathcal{D}$, that

$$\| f \|_{2,(0,T)} \le \tilde{r} . \tag{4.6.25}$$

The norm of the function $A f - \varphi'/g_1$ can be estimated as follows:

$$\| A f - \varphi'/g_1 \|_{2,(0,T)} = \Big(\int_0^t | A f - \varphi'/g_1 |^2 \, dt \Big)^{1/2} \tag{4.6.26}$$

$$\le \frac{1}{g_0} \Big[\int_0^t \Big(\nu \, \| \mathbf{v}(\cdot, t) \|_{2,\Omega} \cdot \| \Delta \boldsymbol{\omega} \|_{2,\Omega}$$

$$+ \operatorname*{ess\,sup}_{x\in\Omega} | \boldsymbol{\omega}_x | \, \| \mathbf{v}(\cdot, t) \|_{2,\Omega}^2 \Big)^2 \, dt \Big]^{1/2} .$$

By appeal to (4.6.20) and (4.6.25) one can readily see that (4.6.26) yields

$$\| A f - \varphi'/g_1 \|_{2,(0,T)} \le m_3 , \tag{4.6.27}$$

where m_3 is the same as before.

By virtue of (4.6.24), we have $m_3 < r$. With this relation in view, estimate (4.6.27) immediately implies that the nonlinear operator A carries the ball $\mathcal{D}$ into itself, thereby completing the proof of the theorem. ∎

Regarding the unique solvability of the inverse problem concerned, we obtain the following result.

Theorem 4.6.4 *Let* $\Omega \subset \mathbf{R}^2$, $\mathbf{g} \in \mathbf{C}([0, T], \mathbf{L}_2(\Omega))$, $\mathbf{a} \in \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega}_x \in \mathbf{L}_\infty(\Omega)$, $\varphi \in \mathbf{W}_2^1(0, T)$ *and*

$$\Big| \int_\Omega \mathbf{g}(x, t) \cdot \boldsymbol{\omega}(x) \, dx \Big| \ge g_0 > 0 \quad (g_0 \equiv \text{const}) , \quad 0 \le t \le T .$$

If the compatibility condition (4.6.9) *and estimate* (4.6.24) *hold, then the following assertions are valid:*

(a) *the inverse problem* (4.6.1)–(4.6.5) *has a solution* $\{\mathbf{v}, f\}$ *with* $f \in \mathcal{D}$ *incorporated;*

(b) *there are no two distinct solutions* $\{\mathbf{v}_i, f_i\}$, $i = 1, 2$, *of the inverse problem* (4.6.1)–(4.6.5) *such that both satisfy the condition* $f_i \in \mathcal{D}$.

Proof In proving item (a) we begin by placing the operator equation (4.6.8) and note in passing that within the input assumptions the nonlinear operator A carries the closed ball $\mathcal{D}$ into itself on account of Theorem 4.6.3, whose use is justified. Consequently, the operator A does follow the conditions of Theorem 4.6.2 and there exists a positive integer k such that the operator A^k is a contraction on $\mathcal{D}$. By the well-known generalization of the principle of contracting mappings we conclude that A has a unique fixed point in $\mathcal{D}$. In the language of operator equations, the nonlinear equation (4.6.8) possesses a solution lying within the ball $\mathcal{D}$ and, moreover, this solution is unique in $\mathcal{D}$. In this framework Theorem 4.6.1 implies the existence of a solution of the inverse problem (4.6.1)–(4.6.5) and item (a) is completely proved.

We proceed to item (b). Assume to the contrary that there were two distinct solutions $\{\mathbf{v}_1,\, f_1\}$ and $\{\mathbf{v}_2,\, f_2\}$ of the inverse problem such that f_1 and f_2 both lie within the ball $\mathcal{D}$.

It is necessary to emphasize that under the present agreement f_1 cannot coincide with f_2 almost everywhere on $[0, T]$. Indeed, if f_1 is equal to f_2 almost everywhere on $[0, T]$, then $\mathbf{v}_1$ coincide with $\mathbf{v}_2$ in Q_T in accordance with the uniqueness theorem for the direct problem solution.

Initially, look at the first pair $\{\mathbf{v}_1,\, f_1\}$. Arguing as in the proof of item (a) from Theorem 4.6.1 we draw the conclusion that the function f_1 represents a solution to equation (4.6.8). Similar arguments serve to motivate that the function f_2 solves the same equation (4.6.8). But we have just established that equation (4.6.8) possesses in $\mathcal{D}$ only one solution. Thus, we have shown that the assumption about the existence of two distinct solutions $\{\mathbf{v}_i,\, f_i\}$, $i = 1, 2$, fails to be true, thereby completing the proof of the theorem. ■

Before concluding this section, we demonstrate that the class of functions satisfying the conditions of Theorem 4.6.4 is not empty.

Example 4.6.1 Let $\Omega \subset \mathbf{R}^2$, $\mathbf{g} = \mathbf{g}(x)$, $\mathbf{g} \in \mathbf{L}_2(\Omega)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega}_x \in \mathbf{L}_\infty(\Omega)$, $\mathbf{a} \in \overset{\circ}{\mathbf{J}}(\Omega)$ and

$$\left| \int\limits_\Omega \mathbf{g}(x) \cdot \boldsymbol{\omega}(x)\, dx \right| = g_0 > 0\,.$$

If we agree to consider

$$\varphi(t) \equiv \int\limits_\Omega \mathbf{a}(x)\, \boldsymbol{\omega}(x)\, dx\,,$$

that is, $\varphi \equiv$ const and set $r = 1$, it is easily seen that $\tilde{r} = 1$ and inequality (4.6.24) takes the form

$$\frac{\sqrt{T}}{g_0} \Big[\nu \, \| \Delta \boldsymbol{\omega} \|_{2,\Omega} \left(\| \mathbf{a} \|_{2,\Omega} + \sqrt{T} \, \| g \|_{2,\Omega} \right) + \operatorname*{ess\,sup}_{x \in \Omega} \, | \boldsymbol{\omega}_x | \left(2 \, \| \mathbf{a} \|^2_{2,\Omega} + 2T \, \| g \|^2_{2,\Omega} \right) \Big] < 1 . \tag{4.6.28}$$

Obviously, the left-hand side of (4.6.28) approaches zero as $T \to 0+$. Consequently, there exists a time T^* such that for any T from the half-open interval $(0, T^*]$ estimate (4.6.28) will be true. From such reasoning it seems worthwhile to consider the inverse problem (4.6.1)–(4.6.5), keeping $T \in (0, T^*]$ and taking $r = 1$, the radius of the ball $\mathcal{D}$. It is easily comprehended that the inverse problem with these input data satisfies the conditions of Theorem 4.6.4.

4.7 Nonstationary linearized system of Navier–Stokes equations: adopting a linearization via recovering a coefficient

The main object of investigation is the system consisting of the nonstationary linearized Navier–Stokes equations in the general form and the incompressibility equation

$$\begin{aligned} \mathbf{v}_t - \nu \Delta \mathbf{v} + \sum_{k=1}^{n} B_k(x,t) \, \mathbf{v}_{x_k} + A(x,t) \, \mathbf{v} &= -\nabla p + \mathbf{F}(x,t) , \\ \operatorname{div} \mathbf{v} &= 0 , \qquad (x,t) \in Q_T , \end{aligned} \tag{4.7.1}$$

where B_k and A are given $(n \times n)$-matrices with entries b_k^{ij} and a^{ij}, $n = 2, 3$, the function $\mathbf{F}$ is known in advance and the velocity $\mathbf{v}$ meets the initial condition

$$\mathbf{v}(x,0) = \mathbf{a}(x) , \qquad x \in \Omega ,$$

and the boundary condition

$$\mathbf{v}(x,t) = 0 , \qquad (x,t) \in S_T .$$

In the present problem statement with the available functions $\mathbf{a}$ and $\mathbf{F}$ it is interesting to ask whether the choice of a special type is possible within the general class of linearizations (4.7.1) that enables the fluid flow to satisfy certain conditions in addition to the initial and boundary data. One needs to exercise good judgment in deciding which to consider. When more information is available, there arises a problem of adopting a linearization and recovering the corresponding coefficients on the left-hand side of the first equation (4.7.1).

We offer one of the approaches to adopting a linearization through a common setting with further careful analysis of the corresponding coefficient inverse problem. Here the subsidiary information about the flow is of the integral overdetermination form

$$\int_\Omega \mathbf{v}(x,t)\,\omega(x)\,dx = \varphi(t)\,, \qquad 0 \le t \le T\,,$$

where the functions ω and φ are known in advance. In that case one might expect the unique recovery only of a single scalar function depending on t.

There are many ingredients necessary for the well-posedness of the initial problem. One should make a number of assumptions in achieving this property. Let A be a diagonal matrix, whose elements a^{ij} of the main diagonal are independent of x and coincide. Consequently, we might attempt the matrix A in the form

$$A(x,t) = \alpha(t)\ I\,,$$

where I is the unit matrix of the appropriate size and $\alpha(t)$ is a scalar coefficient. This provides support for an alternative form of writing

$$A(x,t)\,\mathbf{v} \equiv \alpha(t)\ \mathbf{v}\,,$$

where the unknown coefficient $\alpha(t)$ is sought. In order to simplify some of the subsequent manipulations we set $B_k(x,t) \equiv \beta_k(x,t)\,I$, where $\beta_k(x,t)$ is a scalar function. It follows from the foregoing that

$$\sum_{k=1}^{n} B_k(x,t)\ \mathbf{v}_{x_k} \equiv \sum_{k=1}^{n} \beta_k(x,t)\ \frac{\partial}{\partial x_k}\ \mathbf{v} \equiv \big(\boldsymbol{\beta}(x,t), \nabla\,\big)\,\mathbf{v}\,,$$

where $\boldsymbol{\beta}$ stands for a vector function with the components β_k.

Being concerned with the coefficients ν and β and the functions $\mathbf{a}$, $\mathbf{F}$, ω and φ we are now in a position to set up the nonlinear inverse problem of finding the velocity $\mathbf{v}(x,t)$, the pressure gradient $\nabla p(x,t)$ and the coefficient $\alpha(t)$ from the system of equations

$$\begin{aligned} &\mathbf{v}_t - \nu\,\Delta\,\mathbf{v} + \big(\boldsymbol{\beta}(x,t), \nabla\big)\,\mathbf{v} + \alpha(t)\,\mathbf{v} = -\nabla\,p + \mathbf{F}(x,t)\,, \\ &\operatorname{div}\,\mathbf{v} = 0\,, \qquad (x,t) \in Q_T\,, \end{aligned} \tag{4.7.2}$$

the initial condition

$$(4.7.3) \qquad \mathbf{v}(x,0) = \mathbf{a}(x), \qquad x \in \Omega,$$

the boundary condition

$$(4.7.4) \qquad \mathbf{v}(x,t) = 0, \qquad (x,t) \in S_T,$$

and the condition of **integral overdetermination**

$$(4.7.5) \qquad \int\limits_{\Omega} \mathbf{v}(x,t)\,\omega(x)\,dx = \varphi(t), \qquad 0 \le t \le T.$$

In the sequel a solution of the inverse problem (4.7.2)–(4.7.5) is sought in the class of functions

$$\mathbf{v} \in \mathbf{W}_{2,0}^{2,1}(Q_T), \qquad \nabla p \in \mathbf{L}_2(Q_T), \qquad \alpha \in C([0,T]),$$

where $C([0,T])$ is the space of all continuous on $[0,T]$ functions. Recall that the norm of the space $C([0,T])$ is defined by

$$(4.7.6) \qquad \|\alpha\|_C = \sup_{t\in[0,T]} |\alpha(t)\exp\{-\gamma t\}|,$$

where a proper choice of the constant γ will be justified below. While studying the inverse problem at hand we employ once again certain devices for deriving several a priori estimates for the norms of the functions sought in terms of input data and follow Prilepko and Vasin (1993).

Assuming $\mathbf{F} \in \mathbf{L}_2(Q_T)$, $\boldsymbol{\beta} \in \mathbf{C}([0,T]; \mathbf{L}_4(\Omega))$ and $\mathbf{a} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, we choose an arbitrary function α from $C(0,T)$ and substitute it into (4.7.2). Being concerned with the functions $\boldsymbol{\beta}$, $\mathbf{a}$ and $\mathbf{F}$ we involve the system for finding a set of the vector functions $\{\mathbf{v}, \nabla p\}$. The corresponding theorem on existence and uniqueness of the solution of the direct problem (4.7.2)–(4.7.4) asserts that the pair $\{\mathbf{v}, \nabla p\}$, which interests us (see Ladyzhenskaya (1967) and Solonnikov (1973)), can uniquely be recovered.

Upon receipt of the function $\mathbf{v}$ we can refer to the nonlinear operator

$$A\colon C([0,T]) \mapsto C([0,T])$$

with the values

$$(4.7.7) \quad (A\alpha)(t) = \left\{ \int\limits_{\Omega} \left[\nu\,\mathbf{v}\cdot\Delta\omega - (\boldsymbol{\beta},\nabla)\,\mathbf{v}\cdot\omega + \mathbf{F}\cdot\omega \right] dx \right.$$

$$\left. - \varphi'(x) \right\} \Big/ \varphi(t), \qquad 0 \le t \le T,$$

where the new functions $\boldsymbol{\omega}$ and φ obey the restrictions

$$\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega), \quad \varphi \in C^1(0, T), \quad |\varphi(t)| \geq \varphi_T > 0, \quad \varphi_T \equiv \text{const}.$$

Common practice involves the nonlinear operator equation of the second kind for the function $\alpha(t)$ over the space $C([0, T])$:

$$\alpha = A\alpha. \tag{4.7.8}$$

The result we present below establishes an interrelation between the solvability of the inverse problem (4.7.2)–(4.7.5) and solvability of the operator equation (4.7.8).

Theorem 4.7.1 *Let* $\mathbf{F} \in \mathbf{L}_2(Q_T)$, $\boldsymbol{\beta} \in \mathbf{C}([0, T]; \mathbf{L}_4(\Omega)) \cap \mathbf{J}(Q_T)$, $\mathbf{a} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\varphi \in C^1([0, T])$ *and* $|\varphi(t)| \geq \varphi_T > 0$ *(*$\varphi_T \equiv \text{const}$*)*, $0 \leq t \leq T$. *Then the following assertions are valid:*

(a) *if the inverse problem* (4.7.2)–(4.7.3) *has a solution* $\{\mathbf{v}, \nabla p, \alpha\}$, *then the function* α *involved satisfies equation* (4.7.8);

(b) *if equation* (4.7.8) *is solvable and the compatibility condition*

$$\int_\Omega \mathbf{a}(x)\, \boldsymbol{\omega}(x)\, dx = \varphi(0) \tag{4.7.9}$$

holds, then the inverse problem (4.7.2)–(4.7.5) *is solvable.*

Proof We proceed to prove item (a). Let the inverse problem (4.7.2)–(4.7.5) possess a solution, say $\{\mathbf{v}, \nabla p, \alpha\}$. Multiplying the first equation (4.7.2) by the function $\boldsymbol{\omega}$ scalarly in $\mathbf{L}_2(\Omega)$ and making a standard rearrangement, we arrive at the identity

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx &- \nu \int_\Omega \mathbf{v}(x,t) \cdot \Delta\boldsymbol{\omega}(x)\, dx \\
&+ \int_\Omega \big(\boldsymbol{\beta}(x,t), \nabla\big)\, \mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx \\
&+ \alpha(t) \int_\Omega \mathbf{v}(x,t) \cdot \boldsymbol{\omega}(x)\, dx \\
&= \int_\Omega \mathbf{F}(x,t) \cdot \boldsymbol{\omega}(x)\, dx, \quad 0 \leq t \leq T.
\end{aligned} \tag{4.7.10}$$

Substituting (4.7.5) into (4.7.10) and taking into account (4.7.7), we establish the relation

$$\alpha = A\,\alpha\,,$$

meaning that α gives a solution to equation (4.7.8). Thus, the assertion of item (a) is completely proved.

We proceed to item (b). Let (4.7.8) have a solution belonging to the space $C([0,T])$. We denote it by α and substitute into (4.7.2). Having resolved the system (4.7.2)–(4.7.4) one can find a pair of the functions $\{\mathbf{v},\,\nabla p\}$ as a unique solution of the direct problem. It is necessary to prove that the function $\mathbf{v}$ thus obtained satisfies the overdetermination condition (4.7.5) as well. For further motivations it is convenient to deal with

$$\int\limits_{\Omega} \mathbf{v}(x,t)\cdot\boldsymbol{\omega}(x)\,dx = \varphi_1(t)\,, \qquad 0\le t\le T\,.$$

From (4.7.3) it follows that

$$\int\limits_{\Omega} \mathbf{a}(x)\cdot\boldsymbol{\omega}(x)\,dx = \varphi_1(0)\,. \tag{4.7.11}$$

By the same reasoning as in the derivation of (4.7.10) we establish the relation

$$\begin{aligned}\varphi_1'(t) - \nu \int\limits_{\Omega} \big[\mathbf{v}\cdot\Delta\boldsymbol{\omega} + (\boldsymbol{\beta},\nabla)\,\mathbf{v}\cdot\boldsymbol{\omega}\big]\,dx&\\ + \alpha(t)\,\varphi_1(t) = \int\limits_{\Omega} \mathbf{F}\cdot\boldsymbol{\omega}\,dx\,, &\qquad 0\le t\le T\,,\end{aligned} \tag{4.7.12}$$

showing the notation $\varphi_1(t)$ to be a sensible one. The function $\alpha(t)$ being a solution of (4.7.8) implies that

$$\begin{aligned}\varphi'(t) - \nu \int\limits_{\Omega} \big[\mathbf{v}\cdot\Delta\boldsymbol{\omega} + (\boldsymbol{\beta},\nabla)\,\mathbf{v}\cdot\boldsymbol{\omega}\big]\,dx&\\ + \alpha(t)\,\varphi(t) = \int\limits_{\Omega} \mathbf{F}\cdot\boldsymbol{\omega}\,dx\,, &\qquad 0\le t\le T\,,\end{aligned} \tag{4.7.13}$$

Subtracting (4.7.12) from (4.7.13) yields the differential equation

$$(\varphi-\varphi_1)' + \alpha\,(t)\,(\varphi-\varphi_1) = 0\,,$$

whose general solution is of the form

$$(4.7.14)\qquad (\varphi-\varphi_1)(t) = c\,\exp\left\{-\int\limits_0^t \alpha(\tau)\,d\tau\right\},\qquad c=\text{const}\,.$$

From (4.7.9) and (4.7.11) we deduce that

$$(\varphi-\varphi_1)(0) = 0\,,$$

leading by formula (4.7.14) to

$$(\varphi-\varphi_1)(t) \equiv 0\,.$$

This means that the function $\mathbf{v}$ satisfies (4.7.5) and thereby the triple $\{\mathbf{v},\, \nabla p,\, \alpha\}$ with these members is just a solution of the inverse problem (4.7.2)–(4.7.5). Thus, the theorem is completely proved. ∎

Remark 4.7.1 One circumstance involved should be taken into account with regard to the inverse problem (4.7.2)–(4.7.5). If the functions **a** and **F** both are equal to zero almost everywhere in Ω and Q_T, respectively, then, in complete agreement with the corresponding theorem, the direct problem (4.7.2)–(4.7.4) has only a trivial solution for any $\boldsymbol{\beta}$ and α from the indicated classes. For this reason it would be impossible to recover the coefficient α. However, the presence of relation (4.7.5) in the input conditions and the inequality $|\,\varphi(t)\,| \ge \varphi_T > 0$ in the assumptions of Theorem 4.7.1 excludes that case from further consideration in the present statement of the inverse problem.

At the next stage we examine the properties of the nonlinear operator A, whose use permits us to justify a possibility of applying the **contraction mapping principle**. In preparation for this, we consider in the space $C([0,\,T])$ the closed ball

$$D_r = \left\{\, \alpha\in C([0,\,T])\colon\ \|\,\alpha\,\|_C \le r\,\right\}.$$

Lemma 4.7.1 *Let* $\mathbf{F}\in \mathbf{L}_2(Q_T)$, $\boldsymbol{\beta}\in \mathbf{C}([0,T];\mathbf{L}_4(\Omega))\cap \mathbf{J}(Q_T)$, $\mathbf{a}\in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)\cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega}\in \mathbf{W}_2^2(\Omega)\cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)\cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\varphi\in C^1([0,\,T])$ *and* $|\varphi(t)|\ge \varphi_T>0$ *(*$\varphi_T\equiv$ const*),* $0\le t\le T$. *If the radius of* D_r *is taken to be*

$$(4.7.15)\qquad r = (2T)^{-1}\exp\{-\gamma\, t\}\,,$$

then the nonlinear operator A *admits the estimate*

$$(4.7.16)\qquad \|\,A\,\alpha_1 - A\,\alpha_2\,\|_C \le m\,\gamma^{-1}\|\,\alpha_1-\alpha_2\,\|_C\,,\qquad \alpha_1,\,\alpha_2\in D_r\,,$$

where

$$m = 4\,\varphi_T^{-1}(\|\mathbf{a}\|_{2,\Omega} + \|\mathbf{F}\|_{2,1,Q_T}) \times \left(\nu\,\|\Delta\boldsymbol{\omega}\|_{2,\Omega} + \sup_{t\in[0,T]} \|\boldsymbol{\beta}(\,\cdot\,,t)\|_{4,\Omega}\,\|\boldsymbol{\omega}_x\|_{4,\Omega}\right).$$

Proof Let α_1 and α_2 be arbitrary distinct elements of D_r. By the definition of the operator A,

$$(4.7.17)\qquad \left|A\,\alpha_1 - A\,\alpha_2\right| = \frac{1}{|\varphi(t)|}\left|\int_\Omega \big[\nu\,(\mathbf{v}_1-\mathbf{v}_2)\cdot\Delta\boldsymbol{\omega} - (\boldsymbol{\beta},\nabla)(\mathbf{v}_1-\mathbf{v}_2)\cdot\boldsymbol{\omega}\big]\,dx\right|$$
$$\le \frac{1}{\varphi_T}\left(\nu\,\|\Delta\boldsymbol{\omega}\|_{2,\Omega} + \|\boldsymbol{\beta}(\,\cdot\,,t)\|_{4,\Omega}\cdot\|\boldsymbol{\omega}_x\|_{4,\Omega}\right) \times \|(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,t)\|_{2,\Omega}\,,\quad 0\le t\le T\,,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the solutions of the direct problems (4.7.2)–(4.7.4) with the coefficients α_1 and α_2, respectively.

Let us estimate the last factor on the right-hand side of (4.7.17).

Obviously, the function $\mathbf{v}_1-\mathbf{v}_2$ gives a solution of the direct problem

$$(4.7.18)\qquad (\mathbf{v}_1-\mathbf{v}_2)_t - \nu\Delta(\mathbf{v}_1-\mathbf{v}_2) + (\boldsymbol{\beta},\nabla)(\mathbf{v}_1-\mathbf{v}_2) + \alpha_2(\mathbf{v}_1-\mathbf{v}_2) + (\alpha_1-\alpha_2)\,\mathbf{v}_1 = -\nabla(p_1-p_2)\,,$$
$$\operatorname{div}(\mathbf{v}_1-\mathbf{v}_2) = 0\,,\qquad (x,t)\in Q_T\,,$$

$$(4.7.19)\qquad (\mathbf{v}_1-\mathbf{v}_2)(x,0) = 0\,,\qquad x\in\Omega\,,$$

$$(4.7.20)\qquad (\mathbf{v}_1-\mathbf{v}_2)(x,t) = 0\,,\qquad (x,t)\in S_T\,.$$

Let us write the corresponding energy identity

$$(4.7.21)\qquad \frac{1}{2}\,\frac{d}{dt}\|(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,t)\|_{2,\Omega}^2 + \nu\,\|(\mathbf{v}_1-\mathbf{v}_2)_x(\,\cdot\,,t)\|_{2,\Omega}^2 + \alpha_2(t)\,\|(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,t)\|_{2,\Omega}^2 = (\alpha_2-\alpha_1)\int_\Omega \mathbf{v}_1\cdot(\mathbf{v}_1-\mathbf{v}_2)\,dx$$

with reference to the relation

$$\int_{\Omega} (\boldsymbol{\beta}, \nabla)(\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2)\, dx = 0\,,$$

which is valid for any $\boldsymbol{\beta} \in \mathbf{J}(Q_T)$.

Combination of identity (4.7.21) and the Poincaré–Friedrichs inequality (1.2.21) gives the estimate

$$\begin{aligned}(4.7.22)\qquad \frac{d}{dt}\, \|(\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t)\|_{2,\Omega} &+ \frac{\nu}{c_1(\Omega)}\, \|(\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t)\|_{2,\Omega} \\ &\le |\alpha_2(t)|\, \|(\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t)\|_{2,\Omega} \\ &+ \|\mathbf{v}_1(\,\cdot\,, t)\|_{2,\Omega}\, |(\alpha_1 - \alpha_2)(t)|\,, \\ &0 \le t \le T\,,\end{aligned}$$

with constant $c_1(\Omega)$ arising from (4.2.21).

Estimate (4.7.21), in turn, leads to

$$\begin{aligned}(4.7.23)\qquad \|(\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, \tau)\|_{2,\Omega} &\le \int_0^{\tau} |\alpha_2(\xi)|\, \|(\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, \xi)\|_{2,\Omega}\, d\xi \\ &+ \int_0^{\tau} \|(\mathbf{v}_1)(\,\cdot\,, \xi)\|_{2,\Omega}\, |(\alpha_1 - \alpha_2)(\xi)|\, d\xi\,, \\ &0 \le \tau \le t \le T\,,\end{aligned}$$

yielding

$$\begin{aligned}(4.7.24)\qquad &\|(\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, \tau)\|_{2,\Omega} \\ &\le \sup_{\xi \in [0,\, t]} \|(\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, \xi)\|_{2,\Omega} \int_0^t |\alpha_2(\xi)|\, d\xi \\ &+ \sup_{\xi \in [0,\, t]} \|\mathbf{v}_1(\,\cdot\,, \xi)\|_{2,\Omega} \int_0^t |(\alpha_1 - \alpha_2)(\xi)|\, d\xi\,, \\ &0 \le \tau \le t \le T\,.\end{aligned}$$

From (4.7.24) it follows that

$$\begin{aligned}(4.7.25)\qquad \sup_{\tau\in[0,\,t]} \|\,(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,\,\tau)\,\|_{2,\,\Omega} &\le \sup_{\tau\in[0,\,t]} \|\,(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,\,\tau)\,\|_{2,\,\Omega} \\ &\quad\times \|\,\alpha_2\,\|_C\ T\ \exp\{\gamma T\} \\ &\quad+ \sup_{\tau\in[0,\,t]} \|\,\mathbf{v}_1(\,\cdot\,,\,\tau)\,\|_{2,\,\Omega} \\ &\quad\times \int\limits_0^t |(\alpha_1-\alpha_2)(\tau)|\ d\tau\,, \quad 0\le t\le T\,.\end{aligned}$$

Since $\alpha_2\in D_r$, having stipulated (4.7.25), the estimate becomes valid:

$$\begin{aligned}(4.7.26)\qquad \sup_{\tau\in[0,\,t]} \|\,(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,\,\tau)\,\|_{2,\,\Omega} &\le 2 \sup_{\tau\in[0,\,t]} \|\,\mathbf{v}_1(\,\cdot\,,\,\tau)\,\|_{2,\,\Omega} \\ &\quad\times \int\limits_0^t |(\alpha_1-\alpha_2)(\tau)|\ d\tau\,, \\ &\quad 0\le t\le T\,.\end{aligned}$$

Here we used also the relation

$$r\,T\ \exp\{\gamma T\} = \tfrac{1}{2}$$

as an immediate implication of (4.7.15).

If the system (4.7.2)–(4.7.4) is written for α_1 and $\mathbf{v}_1$, then by a similar reasoning as before we derive the estimate

$$\begin{aligned}(4.7.27)\qquad \sup_{\tau\in[0,\,t]} \|\,\mathbf{v}_1(\,\cdot\,,\,\tau)\,\|_{2,\,\Omega} &\le \sup_{\tau\in[0,\,t]} \|\,\mathbf{v}_1(\,\cdot\,,\,\tau)\,\|_{2,\,\Omega}\,\|\,\alpha_1\,\|_C\ T\ \exp\{\gamma T\} \\ &\quad+ \int\limits_0^t \|\,\mathbf{F}(\,\cdot\,,\,\xi)\,\|_{2,\,\Omega}\ d\xi + \|\,\mathbf{a}\,\|_{2,\,\Omega}\,, \\ &\quad 0\le t\le T\,.\end{aligned}$$

Since $\alpha_1 \in D_r$, we might have

$$\sup_{\tau\in[0,\,t]} \|\mathbf{v}_1(\,\cdot\,,\,\tau)\|_{2,\Omega} \le 2\left(\|\mathbf{a}\|_{2,\Omega} + \|\mathbf{F}\|_{2,1,Q_t}\right), \quad 0\le t\le T. \tag{4.7.28}$$

Substituting (4.7.28) into (4.7.26) yields the estimate

$$\sup_{\tau\in[0,\,t]} \|(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,\,\tau)\|_{2,\Omega} \le 4\left(\|\mathbf{a}\|_{2,\Omega} + \|\mathbf{F}\|_{2,1,Q_t}\right) \times \int_0^t |(\alpha_1-\alpha_2)(\tau)|\,d\tau, \quad 0\le t\le T, \tag{4.7.29}$$

which is valid for any α_1 and α_2 taken from the ball D_r.

From (4.7.17) and (4.7.29) it follows that

$$\left|(A\,\alpha_1 - A\,\alpha_2)(t)\right| \le m\int_0^t |(\alpha_1-\alpha_2)(\tau)|\,d\tau, \quad 0\le t\le T, \tag{4.7.30}$$

where m is the same as in (4.7.16). In the light of definition (4.7.6) we establish from (4.7.30) the desired estimate

$$\begin{aligned} \|A\,\alpha_1 - A\,\alpha_2\|_C &\le m \sup_{t\in[0,\,T]} \left|\exp\{-\gamma t\}\int_0^t |(\alpha_1-\alpha_2)(\tau)|\,d\tau\right| \\ &\le m\,\|\alpha_1-\alpha_2\|_C \sup_{t\in[0,\,T]}\left[\exp\{-\gamma t\}\int_0^t \exp\{\gamma\tau\}\,d\tau\right] \\ &\le \frac{m}{\gamma}\,\|\alpha_1-\alpha_2\|_C, \qquad \forall\alpha_1,\,\alpha_2\in D_r, \end{aligned} \tag{4.7.31}$$

thereby completing the proof of the lemma. ■

Corollary 4.7.1 *Let all the conditions of Lemma* 4.7.1 *hold. One assumes, in addition, that the constant γ involved in definition* (4.7.6) *of the norm is of the form*

$$\gamma = m+\varepsilon, \tag{4.7.32}$$

where ε is any positive number. Then the operator A is a contraction in the ball D_r of radius

$$r = (2T)^{-1} \exp\{-(m+\varepsilon)T\}$$

and the corresponding contraction coefficient is equal to $m/(m+\varepsilon)$.

Proof The proof amounts to substituting relation (4.7.32) into (4.7.15) and (4.7.16) both. ■

Let us find out under what assumptions the operator A carries the ball D_r into itself. In the following lemma we try to give a definite answer.

Lemma 4.7.2 *Let* $\mathbf{F} \in \mathbf{L}_2(Q_T)$, $\boldsymbol{\beta} \in \mathbf{C}([0,T]; \mathbf{L}_4(\Omega)) \cap \mathbf{J}(Q_T)$, $\mathbf{a} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega} \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\varphi \in \dot{C}^1([0,T])$ *and* $|\varphi(t)| \geq \varphi_T > 0$ ($\varphi_T \equiv$ const), $0 \leq t \leq T$. *If the radius of D_r is taken from* (4.7.15), *then*

$$\| A\alpha \|_C \leq \tfrac{1}{2} m + \varphi_T^{-1} \| f - \varphi' \|_C, \qquad \forall \alpha \in D_r, \tag{4.7.33}$$

where

$$f(t) = \int_\Omega \mathbf{F}(x,t) \cdot \boldsymbol{\omega}(x)\, dx$$

and m is the same as in (4.7.16).

Proof Because of (4.7.7),

$$\begin{aligned} | A\alpha | \leq \varphi_T^{-1} \Big[\big(\nu \| \Delta\boldsymbol{\omega} \|_{2,\Omega} \\ + \sup_{t\in[0,T]} \| \boldsymbol{\beta}(\,\cdot\,,t) \|_{4,\Omega} \, \| \boldsymbol{\omega}_x \|_{4,\Omega} \\ \times \| \mathbf{v}(\,\cdot\,,t) \|_{2,\Omega} + | f(t) - \varphi'(t) | \Big], \end{aligned} \tag{4.7.34}$$

where

$$f(t) = \int_\Omega \mathbf{F}(x,t) \cdot \boldsymbol{\omega}(x)\, dx\,.$$

On the other hand, any solution of (4.7.2)–(4.7.4) admits the estimate

$$\sup_{\tau\in[0,t]} \| \mathbf{v}(\,\cdot\,,\tau) \|_{2,\Omega} \leq 2 \Big(\| \mathbf{a} \|_{2,\Omega} + \| \mathbf{F} \|_{2,1,Q_T} \Big), \quad 0 \leq t \leq T, \tag{4.7.35}$$

which can be derived in the same manner as we did in the consideration of (4.7.28) by appeal to the coefficient α from the ball D_r.

Estimating the right-hand side of (4.7.34) by means of (4.7.35) and substituting the result into (4.7.6), we obtain (4.7.33) and this proves the lemma. ■

For the sake of definiteness, we accept $\gamma = m + 1$ in subsequent arguments. Joint use of Lemmas 4.7.1–4.7.2 and Theorem 4.7.1 makes it possible to establish sufficient conditions under which a solution of the inverse problem at hand exists and is unique.

Theorem 4.7.2 *Let* $\mathbf{F} \in \mathbf{L}_2(Q_T)$, $\beta \in \mathbf{C}([0,T]; \mathbf{L}_4(\Omega)) \cap \mathbf{J}(Q_T)$, $\mathbf{a} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\omega \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\varphi \in C^1([0,T])$ *and* $|\varphi(t)| \geq \varphi_T > 0$ ($\varphi_T \equiv \mathrm{const}$), $0 \leq t \leq T$. *One assumes, in addition, that the compatibility condition* (4.7.9) *holds and the radius of the ball* D_r *is equal to*

$$r = (2T)^{-1} \exp\{-(m+1)T\}. \tag{4.7.36}$$

If the input data of the inverse problem satisfy the inequality

$$T\left(m + 2\varphi_T^{-1} \| f - \varphi' \|_C\right) \leq \exp\{-(m+1)T\}, \tag{4.7.37}$$

where

$$m = 4\varphi_T^{-1} \left(\| \mathbf{a} \|_{2,\Omega} + \| \mathbf{F} \|_{2,1,Q_T} \right) \Bigg(\nu \| \Delta\omega \|_{2,\Omega}$$

$$+ \sup_{t\in[0,T]} \| \beta(\,\cdot\,,t) \|_{4,\Omega} \, \| \omega_x \|_{4,\Omega} \Bigg),$$

$$f(t) = \int_\Omega \mathbf{F}(x,t) \cdot \omega(x)\, dx,$$

then the following assertions are valid:

(a) *the inverse problem* (4.7.2)–(4.7.5) *has a solution* $\{\mathbf{v}, \nabla p, \alpha\}$ *with* $\alpha \in D_r$;

(b) *there are no two distinct solutions* $\{\mathbf{v}_i, \nabla p_i, \alpha_i\}$, $i = 1, 2$, *of the inverse problem* (4.7.1)–(4.7.5) *such that both satisfy the condition* $\alpha_i \in D_r$, $i = 1, 2$.

Proof First, we are going to show that the nonlinear operator A specified by (4.7.7) can be considered in the context of the contraction mapping principle. Indeed, by Corollary 4.7.1 the operator A appears to be a contraction in the closed ball D_r of radius (4.7.36) and the contraction coefficient therewith is equal to $m/(m+1)$ (recall that $\gamma = m+1$).

On the other hand, since $\gamma = m+1$, Lemma 4.7.2 and estimate (4.7.37) together imply that the operator A carries D_r into itself and, therefore,

A has a unique fixed point in the ball D_r. To put it differently, we have proved that (4.7.8) has in the same ball D_r a unique solution, say α. Under the compatibility condition (4.7.9) Theorem 4.7.1 implies that there exists a solution $\{\mathbf{v}, \nabla p, \alpha\}$ of the inverse problem (4.7.2)–(4.7.5), where α is exactly the same function which solves (4.7.8) and belongs to D_r. Item (a) is proved.

We proceed to item (b). Assume to the contrary that there were two distinct solutions

$$\{\mathbf{v}_1, \nabla p_1, \alpha_1\} \quad \text{and} \quad \{\mathbf{v}_2, \nabla p_2, \alpha_2\} \tag{4.7.38}$$

of the inverse problem (4.7.2)–(4.7.5) such that α_1 and α_2 both lie within the ball D_r. As noted above, if the collections in (4.7.38) are different, then so are the functions α_1 and α_2. Indeed, if $\alpha_1 = \alpha_2$ then, due to the uniqueness theorem for the direct problem (4.7.2)–(4.7.4), the functions $\mathbf{v}_1$ and ∇p_1 coincide almost everywhere in Q_T with $\mathbf{v}_2$ and ∇p_2, respectively.

The collection $\{\mathbf{v}_1, \nabla p_1, \alpha_1\}$, whose functions obey (4.7.2)–(4.7.5), comes first. By Theorem 4.7.1 the function α_1 is just a solution to equation (4.7.8). A similar remark shows that α_2 also satisfies (4.7.8). At the very beginning both functions α_1 and α_2 were taken from the ball D_r. Therefore, we have found in D_r two distinct functions satisfying one and the same equation (4.7.8). But this disagrees with the uniqueness property of the fixed point of the operator A in the ball D_r that has been proved earlier.

Consequently, the assertion of item (b) is true and this completes the proof of the theorem. ∎

Let us consider an example illustrating the result obtained.

Example 4.7.1 Let $\mathbf{F} \equiv 0$ and $\boldsymbol{\beta} \equiv 0$. The functions $\mathbf{a}$ and $\boldsymbol{\omega}$ are arbitrarily chosen in the corresponding classes so that

$$\left| \int_\Omega \mathbf{a} \cdot \boldsymbol{\omega} \, dx \right| > 0 .$$

If we agree to consider

$$\varphi(t) \equiv \left| \int_\Omega \mathbf{a} \cdot \boldsymbol{\omega} \, dx \right| > 0 ,$$

then $\varphi_T \equiv \varphi(t) \equiv \text{const} > 0$, the compatibility condition (4.7.9) holds and inequality (4.7.37) takes now the form

$$T m \le \exp\{-(m+1) T\} , \tag{4.7.39}$$

where

$$m = 4\,\nu \,\| \mathbf{a} \|_{2,\Omega} \cdot \| \Delta \omega \|_{2,\Omega} \left(\int_{\Omega} \mathbf{a} \cdot \omega \, dx > 0 \right)^{-1} .$$

Obviously, for any $m > 0$ the left-hand side of (4.7.39) tends to 0 as $T \to 0+$, while the right-hand side has 1 as its limit. Consequently, there exists a time T_1, for which (4.7.39) becomes true. For $T = T_1$ and the input data we have imposed above it makes sense to turn to the inverse problem (4.7.2)–(4.7.5), which, by Theorem 4.7.2, possesses a solution $\{\mathbf{v},\, \nabla p,\, \alpha\}$ with $\alpha \in D_r$ and

$$r = (2\,T_1)^{-1} \exp\{-(m+1)\,T_1\} .$$

Moreover, there are no two distinct solutions $\{\mathbf{v}_i, \nabla p_i, \alpha_i\}$, $i = 1, 2$, such that both satisfy the condition $\alpha_i \in D_r$.

The example cited permits us to recognize that Theorem 4.7.2 is, generally speaking, of local character. The uniqueness in item (b) was proved as a corollary to the principle of contracting mapping. However, by another reasoning we obtain the following global uniqueness result.

Theorem 4.7.3 *Let* $\beta \in \mathbf{C}([0,\, T];\, \mathbf{L}_4(\Omega))$, $\omega \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\varphi \in C([0,\, T])$ *and* $|\,\varphi(t)\,| \ge \varphi_T > 0$ $(\varphi_T \equiv \text{const}\,)$, $0 \le t \le T$. *Then the inverse problem* (4.7.2)–(4.7.5) *can have at most one solution.*

Proof Let us prove this assertion by reducing to a contradiction. Assume that there were two distinct solutions

$$\{\, \mathbf{v}_1,\, \nabla p_1,\, \alpha_1 \,\}$$

and

$$\{\, \mathbf{v}_2,\, \nabla p_2,\, \alpha_2 \,\}$$

of the inverse problem (4.7.2)–(4.7.5). As noted above, if these are different, then so are the functions α_1 and α_2 involved.

Let us subtract the system (4.7.2)–(4.7.5) written for $\{\mathbf{v}_1,\, \nabla p_1,\, \alpha_1\}$ from the same system but written for $\{\mathbf{v}_2,\, \nabla p_2,\, \alpha_2\}$. By introducing the new functions $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2$, $\nabla q = \nabla p_1 - \nabla p_2$ and $\mu(t) = \alpha_1(t) - \alpha_2(t)$ it is plain to show that the members of the collection $\{\mathbf{u},\, \nabla q,\, \mu\}$ solve the linear inverse problem

$$\mathbf{u}_t - \nu\Delta\,\mathbf{u} + (\beta, \nabla)\,\mathbf{u} + \alpha_1\,\mathbf{u} = -\nabla q + \mu(t)\,\mathbf{v}_2(x,t)\,, \tag{4.7.40}$$

$$\operatorname{div}\mathbf{u} = 0\,, \qquad (x,t) \in Q_T\,,$$

$$\mathbf{u}(x,0) = 0\,, \qquad x \in \Omega\,, \tag{4.7.41}$$

$$\mathbf{u}(x,t) = 0\,, \qquad (x,t) \in S_T\,, \tag{4.7.42}$$

$$\int_\Omega \mathbf{u}(x,t)\,\omega(x)\;dx = 0\,, \qquad t \in [0,\,T]\,. \tag{4.7.43}$$

Obviously, it has at least the trivial solution $\mathbf{u} \equiv 0$, $\nabla q \equiv 0$, $\mu \equiv 0$. To decide for yourself whether the solution thus obtained is the unique solution of (4.7.40)–(4.7.43), a first step is to appeal to Theorem 4.3.5. Indeed, the restrictions on the smoothness of input data are satisfied for the linear inverse problem at hand and so it remains to analyze the situation with condition (4.3.30) taking now the form

$$\left| \int_\Omega \mathbf{v}_2(x,t)\,\omega(x)\;dx \right| \geq g_0 > 0\,, \qquad t \in [0,\,T]\,. \tag{4.7.44}$$

Recall that the function $\mathbf{v}_2$ is involved in the solution $\{\mathbf{v}_2,\,\nabla p_2,\,\alpha_2\}$ of the inverse problem (4.7.2)–(4.7.5) and, consequently, satisfies the integral overdetermination (4.7.5), what means that

$$\mathbf{v}_2 \in \mathbf{C}\big([0,\,T];\,\overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)\big)$$

and

$$\int_\Omega \mathbf{v}_2(x,t)\,\omega(x)\;dx = \varphi(t)\,, \qquad t \in [0,\,T]\,.$$

By assumption,

$$|\varphi(t)| \geq \varphi_T > 0$$

for any $t \in [0,\,T]$ and inequality (4.7.44) holds true. Thus, the inverse problem (4.7.40)–(4.7.43) meets all the requirements of Theorem 4.3.5 from which another conclusion can be drawn saying that there are no solutions other than the trivial solution of (4.7.40)–(4.7.43). Consequently, the assumption about the existence of two distinct solutions of the inverse problem (4.7.2)–(4.7.5) fails to be true, thereby completing the proof of the theorem. ■

4.8 Nonstationary linearized system of Navier–Stokes equations: the combined recovery of two coefficients

This section is devoted once again to the linearized Navier–Stokes equations of the general form (4.7.1). For the moment, the set of unknowns contains not only the differential operator on the left-hand side of (4.7.1), for which at least one coefficient needs recovery, but also the vector function of the external force by means of which the motion of a viscous incompressible fluid is produced.

For more a detailed exploration we deal with the system of equations

$$
\begin{aligned}
&\mathbf{v}_t - \nu\,\Delta\,\mathbf{v} + \alpha(t)\,\mathbf{v} = -\nabla\,p + \mathbf{F}(x,t)\,, \\
&\operatorname{div}\mathbf{v} = 0\,, \qquad (x,t) \in Q_T = \Omega \times (0,T)\,,
\end{aligned}
\tag{4.8.1}
$$

where Ω is a bounded domain in the space $\mathbf{R}^n$, $n = 2, 3$, with boundary $\partial\Omega$ of class C^2.

Let the function $\mathbf{F}$ admit the representation

$$
\mathbf{F} = f(t)\,\mathbf{g}(x,t)\,,
$$

where $\mathbf{g}$ is a given vector function and f is an unknown coefficient. Assume that the coefficient α is also unknown and is sought along with the coefficient f, the velocity $\mathbf{v}$ and the pressure gradient ∇p.

A common setting of the inverse problem concerned necessitates imposing more information on the solution in addition to the initial and boundary conditions. On the other hand, in trying to treat the inverse problem at hand as a well-posed one it is worth bearing in mind that, since the total number of unknown coefficients is equal to 2, the number of available overdeterminations should be the same.

For example, one is to measure the velocity and the pressure gradient by the data units making a certain averaging with weights over the domain of space variables x. In such a case the conditions of integral overdetermination as a mathematical description of such measurements are good enough for the purposes of the present section.

So, the nonlinear inverse problem of the combined recovery of the evolution of the coefficients α and f amounts to recovering the vector functions $\mathbf{v}$ and ∇p and the scalar coefficients α and f from the system of equations

$$
\begin{aligned}
&\mathbf{v}_t - \nu\,\Delta\,\mathbf{v} + \alpha(t)\,\mathbf{v} = -\nabla p + f(t)\,\mathbf{g}(x,t)\,, \\
&\operatorname{div}\mathbf{v} = 0\,, \qquad (x,t) \in Q_T\,,
\end{aligned}
\tag{4.8.2}
$$

the initial condition

$$\mathbf{v}(x,0) = \mathbf{a}(x)\,, \qquad x \in \Omega\,, \tag{4.8.3}$$

the boundary condition

$$\mathbf{v}(x,t) = 0\,, \qquad (x,t) \in S_T = \partial\Omega \times [0,\,T]\,, \tag{4.8.4}$$

and the conditions of **integral overdeterminations**

$$\int_\Omega \mathbf{v}(x,t) \cdot \boldsymbol{\omega}_1(x)\, dx = \varphi(t)\,, \qquad t \in [0,\,T]\,, \tag{4.8.5}$$

$$\int_\Omega \nabla p(x,t) \cdot \boldsymbol{\omega}_2(x)\, dx = \psi(t)\,, \qquad t \in [0,\,T]\,, \tag{4.8.6}$$

where the functions $\mathbf{g}$, $\mathbf{a}$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, φ, ψ and the coefficient ν are known in advance.

Definition 4.8.1 *A collection of the functions* $\{\,\mathbf{v},\ \nabla p,\ \alpha,\ f\,\}$ *is said to be a generalized solution of the inverse problem* (4.8.2)–(4.8.6) *if*

$$\mathbf{v} \in \mathbf{W}^{2,\,1}_{2,\,0}(Q_T) \cap \overset{\circ}{\mathbf{J}}(Q_T)\,, \qquad \alpha \in C([0,\,T])\,,$$

$$f \in C([0,\,T])\,, \qquad \nabla p \in \mathbf{G}(Q_T)\,,$$

the integral

$$\int_\Omega \nabla p(x,t) \cdot \boldsymbol{\Phi}(x)\, dx$$

is continuous with respect to t on the segment $[0,\,T]$ *for any* $\boldsymbol{\Phi} \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega) \cap \mathbf{G}(\Omega)$ *and all of the relations* (4.8.2)–(4.8.6) *occur.*

It is worth recalling here that the spaces $\overset{\circ}{\mathbf{J}}(Q_T)$ and $\mathbf{G}(Q_T)$ comprise all the vectors of the space $\mathbf{L}_2(Q_T)$ belonging, respectively, to $\overset{\circ}{\mathbf{J}}(\Omega)$ and $\mathbf{G}(\Omega)$ for almost all $t \in [0,\,T]$ and that the space $\mathbf{C}([0,\,T]) = C([0,\,T]) \times C([0,\,T])$ is equipped with the norm

$$\uparrow \mathbf{u} \uparrow_{\mathbf{C}([0,\,T])} = \|\, u_1 \,\|_C + \|\, u_2 \,\|_C\,,$$

where u_1 and u_2 are the components of the vector $\mathbf{u}$ and

$$\|\, u_i \,\|_C = \sup_{t \in [0,\,T]} |\, \exp\{-\gamma\, t\}\, u_i(t)|\,, \qquad \gamma = \text{const}\,.$$

Taking the input data from the classes

$$
\begin{gathered}
\text{(4.8.7)} \qquad \mathbf{g} \in \mathbf{C}\big([0,T], \mathbf{L}_2(\Omega)\big)\,, \quad \mathbf{a} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \\
\omega_1 \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \overset{\circ}{\mathbf{J}}(\Omega)\,, \quad \omega_2 \in \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \bigcap \mathbf{G}(\Omega)\,, \\
\varphi \in C^1([0,T])\,, \quad \psi \in C([0,T])\,, \quad \left| \int_\Omega \mathbf{g}(x,t) \cdot \omega_2(x)\, dx \right| \ge g_T > 0\,, \\
|\,\varphi(t)| \ge \varphi_T > 0\,, \qquad t \in [0,T] \qquad \big(g_T, \varphi_T \equiv \text{const}\,\big)\,,
\end{gathered}
$$

and keeping the notations

$$
\begin{aligned}
\chi &= \{\alpha(t),\, f(t)\}\,, \\
g_1(t) &= \int_\Omega \mathbf{g}(x,t) \cdot \omega_1(x)\, dx\,, \\
g_2(t) &= \int_\Omega \mathbf{g}(x,t) \cdot \omega_2(x)\, dx\,,
\end{aligned}
$$

we begin the study of the inverse problem concerned by deriving a system of two operator equations of the second kind for the coefficients $\alpha(t)$ and $f(t)$. To that end, we choose an arbitrary vector $\{\alpha(t),\, f(t)\}$ from the space $\mathbf{C}([0,T])$ and substitute then α and f into (4.8.2). Since other input functions involved in (4.8.2)–(4.8.4) meet (4.8.7), the velocity $\mathbf{v} \in \mathbf{W}_{2,0}^{2,1}(Q_T) \cap \overset{\circ}{\mathbf{J}}(Q_T)$ and the pressure gradient $\nabla p \in \mathbf{G}(Q_T)$ are determined as a unique solution of the direct problem (4.8.2)–(4.8.4). As such, it also will be useful to refer the nonlinear operator

$$
\mathbf{A}\colon\ \mathbf{C}([0,T]) \mapsto \mathbf{C}([0,T])
$$

acting on every vector $\chi = \{\alpha(t),\, f(t)\}$ as follows:

$$
\text{(4.8.8)} \qquad [\mathbf{A}(\chi)](t) = \big\{ [A_1(\alpha,f)](t)\,, [A_2(\alpha,f)](t) \big\}\,, \qquad t \in [0,T]\,,
$$

where

$$[A_1(\alpha, f)](t) = [\widetilde{A}_1(\alpha, f)](t) + \frac{g_1(t)}{\varphi(t)}$$

$$\times \left\{ [\widetilde{A}_2(\alpha, f)](t) + \frac{\psi(t)}{g_2(t)} \right\} - \frac{\varphi'(t)}{\varphi(t)} ,$$

$$[A_2(\alpha, f)](t) = [\widetilde{A}_2(\alpha, f)](t) + \frac{\psi(t)}{g_2(t)} ,$$

$$[\widetilde{A}_1(\alpha, f)](t) = \frac{1}{\varphi(t)} \left[\nu \int_\Omega \mathbf{v}(x,t) \cdot \Delta \boldsymbol{\omega}_1 \, dx \right] ,$$

$$[\widetilde{A}_2(\alpha, f)](t) = - \frac{1}{g_2(t)} \left[\nu \int_\Omega \mathbf{v}(x,t) \cdot \Delta \boldsymbol{\omega}_2 \, dx \right] .$$

Let us consider a nonlinear operator equation of the second kind over the space $\mathbf{C}([0, T])$:

(4.8.9) $$\chi = A \chi ,$$

which, obviously, is equivalent to the system of two nonlinear equations related to the two unknowns functions

(4.8.10) $$\begin{cases} \alpha = \widetilde{A}_1(\alpha, f) + \dfrac{g_1}{\varphi} \, \widetilde{A}_2(\alpha, f) + \dfrac{g_1 \, \psi}{g_2 \, \varphi} - \dfrac{\varphi'}{\varphi} , \\ f = \widetilde{A}_2(\alpha, f) + \dfrac{\psi}{g_2} . \end{cases}$$

Theorem 4.8.1 *Let the input data of the inverse problem* (4.8.2)-(4.8.6) *comply with* (4.8.7). *Then the following assertions are valid:*

(a) *if the inverse problem* (4.8.2)-(4.8.6) *has a solution* $\{\mathbf{v}, \nabla p, \alpha, f\}$, *then the vector* $\chi = \{\alpha, f\}$ *gives a solution to equation* (4.8.9);

(b) *if equation* (4.8.9) *is solvable and the compatibility condition*

(4.8.11) $$\varphi(0) = \int_\Omega \mathbf{a}(x) \cdot \boldsymbol{\omega}_1(x) \, dx$$

holds, then the inverse problem (4.8.2)-(4.8.6) *is solvable.*

Proof We proceed to prove item (a). Let $\{\mathbf{v},\, \nabla p,\, \alpha,\, f\}$ constitute a solution of problem (4.8.2)–(4.8.6). Taking the scalar product of both sides of the first equation (4.8.2) and the function $\boldsymbol{\omega}_1 \in \overset{\circ}{\mathbf{J}}(\Omega) \cap \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ from the space $\mathbf{L}_2(\Omega)$, we arrive at

$$\frac{d}{dt} \int_\Omega \mathbf{v} \cdot \boldsymbol{\omega}_1 \, dx - \nu \int_\Omega \mathbf{v} \cdot \Delta \boldsymbol{\omega}_1 \, dx + \alpha(t) \int_\Omega \mathbf{v} \cdot \boldsymbol{\omega}_1 \, dx = f(t) \int_\Omega \mathbf{g} \cdot \boldsymbol{\omega}_1 \, dx . \tag{4.8.12}$$

Via a similar transform with

$$\boldsymbol{\omega}_2 \in \mathbf{G}(\Omega) \bigcap \mathbf{W}_2^2(\Omega) \bigcap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$$

we obtain the identity

$$-\nu \int_\Omega \mathbf{v} \cdot \Delta \boldsymbol{\omega}_2 \, dx = - \int_\Omega \nabla p \cdot \boldsymbol{\omega}_2 \, dx + f(t) \int_\Omega \mathbf{g} \cdot \boldsymbol{\omega}_2 \, dx . \tag{4.8.13}$$

Provided the overdetermination conditions (4.8.5)–(4.8.6) hold, we deduce from (4.8.12)–(4.8.13) that

$$\begin{cases} \alpha(t) = \widetilde{A}_1(\alpha, f) + \dfrac{1}{\varphi(t)} \left[f(t)\, g_1 - \varphi'(t) \right], \\ f(t) = \widetilde{A}_2(\alpha, f) + \dfrac{\psi(t)}{g_2(t)} , \end{cases} \tag{4.8.14}$$

where $\widetilde{A}_1(\alpha, f)$ and $\widetilde{A}_2(\alpha, f)$ take the same values as in (4.8.8).

Upon substituting the second equation (4.8.14) into the first it is easily seen that the functions $\alpha(t)$ and $f(t)$ solve the system (4.8.10) and, consequently, the vector $\chi = \{\alpha,\, f\}$ satisfies the operator equation (4.8.9).

We proceed to item (b). Let χ be a solution to equation (4.8.9). For later use, we denote by $\alpha(t)$ and $f(t)$ its first and second components, respectively. So, in what follows we operate with $\chi = \{\alpha,\, f\}$.

Let us substitute α and f into (4.8.2) and look for $\mathbf{v} \in \mathbf{W}_{2,\,0}^{2,\,1}(Q_T) \cap \overset{\circ}{\mathbf{J}}(Q_T)$ and $\nabla p \in \mathbf{G}(Q_T)$ as a unique solution of the direct problem (4.8.2)–(4.8.4). It is easy to verify that for the above function $\nabla p(x,t)$ the integral

$$\int_\Omega \nabla p(x,t) \cdot \boldsymbol{\Phi}(x) \, dx$$

is continuous with respect to t on the segment $[0, T]$ for any $\mathbf{\Phi} \in \mathbf{G}(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$. Indeed, the first equation (4.8.2) implies the identity

$$\int_\Omega \nabla p(x,t) \cdot \mathbf{\Phi}(x)\, dx = -\nu \int_\Omega \mathbf{v}_x \cdot \mathbf{\Phi}_x(x)\, dx + \int_\Omega \mathbf{g} \cdot \mathbf{\Phi}\, dx\,,$$

yielding

$$\begin{aligned} &\left| \int_\Omega [\nabla p(x,t+\Delta t) - \nabla p(x,t)] \cdot \mathbf{\Phi}(x)\, dx \right| \\ &\qquad \le \nu \, \| \mathbf{\Phi}_x \|_{2,\Omega} \cdot \| \mathbf{v}_x(\,\cdot\,, t+\Delta t) - \mathbf{v}_x(\,\cdot\,,t) \|_{2,\Omega} \\ &\qquad\quad + | f(t+\Delta t) | \, \| \mathbf{\Phi} \|_{2,\Omega} \cdot \| \mathbf{g}(\,\cdot\,, t+\Delta t) - \mathbf{g}(\,\cdot\,,t) \|_{2,\Omega} \\ &\qquad\quad + | f(t+\Delta t) - f(t) | \, \| \mathbf{g}(\,\cdot\,,t) \|_{2,\Omega} \cdot \| \mathbf{\Phi} \|_{2,\Omega}\,, \end{aligned}$$

whence the desired property follows immediately.

Let us find out whether the above functions $\mathbf{v}$ and ∇p satisfy the overdetermination conditions (4.8.5)–(4.8.6). For later use, put

$$\int_\Omega \mathbf{v}(x,t) \cdot \boldsymbol{\omega}_1(x)\, dx = \varphi_1(t)\,, \quad t \in [0, T]\,, \tag{4.8.15}$$

$$\int_\Omega \nabla p(x,t) \cdot \boldsymbol{\omega}_2(x)\, dx = \psi_1(t)\,, \quad t \in [0, T]\,. \tag{4.8.16}$$

Since $\mathbf{v}$ satisfies the initial condition (4.8.3), we thus have

$$\varphi_1(0) = \int_\Omega \mathbf{a}(x) \cdot \boldsymbol{\omega}_1(x)\, dx\,. \tag{4.8.17}$$

By exactly the same reasoning as in the derivation of (4.8.12)–(4.8.13) we arrive at

$$\alpha(t)\, \varphi_1(t) = \nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_1(x)\, dx + f(t)\, g_1(t) - \varphi_1'(t) \tag{4.8.18}$$

and

$$f(t) = \frac{1}{g_2(t)} \left[-\nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_2(x)\, dx + \psi_1(t) \right], \tag{4.8.19}$$

where

$$g_1(t) = \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}_1(x)\, dx\,, \qquad g_2(t) = \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}_2(x)\, dx\,.$$

Substitution of (4.8.19) into (4.8.18) yields

$$\begin{aligned} (4.8.20)\quad \alpha(t)\,\varphi_1(t) = \nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_1(x)\, dx + \frac{g_1(t)}{g_2(t)} \\ \times \left[-\nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_2(x)\, dx + \psi_1(t) \right] - \varphi_1'(t)\,, \end{aligned}$$

$$(4.8.21)\qquad f(t) = \frac{1}{g_2(t)} \left[-\nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_2(x)\, dx + \psi_1(t) \right],$$

showing the notations $\varphi_1(t)$ and $\psi_1(t)$ to be sensible ones.

On the other hand, since the vector $\chi = \{\alpha,\, f\}$ gives a solution to equation (4.8.9), both functions α and f satisfy the system

$$\begin{aligned} (4.8.22)\quad \alpha(t)\,\varphi(t) = \nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_1(x)\, dx + \frac{g_1(t)}{g_2(t)} \\ \times \left[-\nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_2(x)\, dx + \psi(t) \right] - \varphi'(t)\,, \end{aligned}$$

$$(4.8.23)\qquad f(t) = \frac{1}{g_2(t)} \left[-\nu \int_\Omega \mathbf{v} \cdot \Delta\boldsymbol{\omega}_2(x)\, dx + \psi(t) \right].$$

Subtracting (4.8.22) and (4.8.23) from (4.8.20) and (4.8.23), respectively, leads to the system

$$\begin{aligned} (4.8.24)\qquad \alpha(t)\left[\varphi_1(t) - \varphi(t)\right] &= \frac{g_1(t)}{g_2(t)} \left[\psi_1(t) - \psi(t)\right] \\ &\quad - \left[\varphi_1'(t) - \varphi'(t)\right], \end{aligned}$$

$$(4.8.25)\qquad \psi_1(t) = \psi(t)\,, \qquad t \in [0,\, T]\,,$$

followed by the differential equation

$$\frac{d}{dt}\left[\varphi(t) - \varphi_1(t)\right] + \alpha(t)\left[\varphi(t) - \varphi_1(t)\right] = 0\,, \qquad t \in [0,\, T]\,,$$

having the general solution in the form

$$(4.8.26) \qquad \varphi(t) - \varphi_1(t) = c \exp\left\{ -\int\limits_0^t \alpha(\tau)\, d\tau \right\}, \qquad c \equiv \text{const}\,.$$

From (4.8.11), (4.8.17) and (4.8.26) we find that

$$(4.8.27) \qquad \varphi_1(t) = \varphi(t)\,, \qquad \forall t \in [0,\, T]\,.$$

With relations (4.8.25) and (4.8.27) in view, it is straightforward to verify that the functions $\mathbf{v}$ and ∇p satisfy the overdetermination conditions (4.8.5)–(4.8.6), respectively. This provides support for decision-making that the collection $\{\mathbf{v},\, \nabla p,\, \alpha,\, f\}$ gives a solution of the inverse problem (4.8.2)–(4.8.6), thereby completing the proof of the theorem. ■

Before proceeding to deeper study, it is reasonable to touch upon the properties of the nonlinear operator $\mathbf{A}$, which complements special investigations. Let D_r be a closed ball in the space $\mathbf{C}([0,\, T])$ such that

$$D_r = \left\{ \chi \in \mathbf{C}([0,\, T]) \colon \uparrow \chi \uparrow_{\mathbf{C}([0,\,T])} \le r \right\}.$$

Lemma 4.8.1 *Let the input data of the inverse problem* (4.8.2)–(4.8.6) *comply with* (4.8.7). *If the radius of the ball D_r is taken to be*

$$(4.8.28) \qquad r = (2\,T)^{-1} \exp\{-\gamma\, T\}\,,$$

then the nonlinear operator $\mathbf{A}$ admits in the ball D_r the estimate

$$(4.8.29) \qquad \uparrow \mathbf{A}\chi_1 - \mathbf{A}\chi_2 \uparrow_{\mathbf{C}([0,\,T])} \le \gamma^{-1}\, m \uparrow \chi_1 - \chi_2 \uparrow_{\mathbf{C}([0,\,T])}\,,$$

where the constant γ arose from the definition of the norm of the space $\mathbf{C}([0,\, T])$ *and*

$$m = 2\,\nu \Big[\sup_{t \in [0,\,T]} \|\, \mathbf{g}(\,\cdot\,,\, t)\,\|_{2,\,\Omega} + 2\,\|\,\mathbf{a}\,\|_{2,\,\Omega} \Big]$$

$$\times \Big[\varphi_T^{-1}\, \|\, \Delta\omega_1\,\|_{2,\,\Omega} + g_T^{-1}\, \big(1 + \varphi_T^{-1}$$

$$\times \sup_{t \in [0,\,T]} |\, g_1(t) |\big)\; \|\, \Delta\omega_2\,\|_{2,\,\Omega} \Big]\,,$$

$$g_1(t) = \int\limits_\Omega \mathbf{g}(x,t) \cdot \omega_1(x)\; dx\,.$$

Proof For more clear understanding of relation (4.8.29) we first derive some auxiliary estimates for the system (4.8.2) solutions. In particular, we are going to show that

$$(4.8.30)\qquad \sup_{\tau\in[0,t]} \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega} \le \sup_{\tau\in[0,t]} \|\mathbf{g}(\cdot,\tau)\|_{2,\Omega} + 2\,\|\mathbf{a}\|_{2,\Omega}, \qquad t\in[0,T],$$

if $\chi = \{\alpha(t),\, f(t)\}$ lies within the ball D_r of radius r specified by (4.8.28). Taking the scalar product of both sides of the first equation (4.8.2) and the function $\mathbf{v}$ from the space $\mathbf{L}_2(\Omega)$, we get

$$(4.8.31)\qquad \frac{d}{dt}\,\|\mathbf{v}(\cdot,t)\|^2_{2,\Omega} + \nu\,\|\mathbf{v}_x(\cdot,t)\|^2_{2,\Omega} + \alpha(t)\,\|\mathbf{v}(\cdot,t)\|^2_{2,\Omega} = f(t)\int_\Omega \mathbf{g}(x,t)\cdot\mathbf{v}(x,t)\,dx, \quad t\in[0,T].$$

Identity (4.8.31) implies that

$$(4.8.32)\qquad \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega} \le \int_0^\tau |\alpha(\xi)|\,\|\mathbf{v}(\cdot,\xi)\|_{2,\Omega}\,d\xi + \int_0^\tau |f(\xi)|\,\|\mathbf{g}(\cdot,\xi)\|_{2,\Omega}\,d\xi + \|\mathbf{a}\|_{2,\Omega}.$$

By the same token,

$$(4.8.33)\qquad \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega} \le \sup_{\tau\in[0,t]} \|\mathbf{v}(\cdot,\xi)\|_{2,\Omega} \int_0^t |\alpha(\xi)|\,d\xi + \sup_{\tau\in[0,t]} \|\mathbf{g}(\cdot,\xi)\|_{2,\Omega} \int_0^t |f(\xi)|\,d\xi + \|\mathbf{a}\|_{2,\Omega}, \qquad 0\le\tau\le t\le T.$$

Estimate (4.8.33) allows us to derive the inequality

$$(4.8.34)\qquad \sup_{\tau\in[0,t]} \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega} \le \sup_{\tau\in[0,t]} \|\mathbf{v}(\cdot,\tau)\|_{2,\Omega}\,\|\alpha\|_C\,T\exp\{\gamma T\} + \sup_{\tau\in[0,t]} \|\mathbf{g}(\cdot,\tau)\|_{2,\Omega}\,\|f\|_C\,T\exp\{\gamma T\} + \|\mathbf{a}\|_{2,\Omega}, \qquad t\in[0,T].$$

If the vector $\chi = \{\alpha(t),\, f(t)\}$ lies within the ball D_r of radius r specified by (4.8.28), then (4.8.34) immediately implies the first auxiliary estimate (4.8.30).

The second auxiliary estimate for the solutions of (4.8.2) is as follows:

$$(4.8.35)\qquad \sup_{\tau\in[0,\,t]} \|(\mathbf{v}_1-\mathbf{v}_2)(\,\cdot\,,\tau)\|_{2,\Omega} \le 2\,\gamma^{-1}\left[\exp\{\gamma t\}-1\right] \times \Big[\sup_{\tau\in[0,\,t]} \|\mathbf{g}(\,\cdot\,,\tau)\|_{2,\Omega} + \|\mathbf{a}\|_{2,\Omega}\Big] \times \big(\|\alpha_1-\alpha_2\|_C + \|f_1-f_2\|_C\big),\quad t\in[0,T],$$

where

$$\chi_1 = \{\alpha_1,\, f_1\}$$

and

$$\chi_2 = \{\alpha_2,\, f_2\}$$

both lie within the ball D_r of radius r specified by (4.8.28). Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are the solutions of the direct problems

$$(4.8.36)\qquad (\mathbf{v}_1)_t - \nu\,\Delta\mathbf{v}_1 + \alpha_1(t)\,\mathbf{v}_1 = -\nabla p_1 + f_1(t)\,\mathbf{g}(x,t),\qquad \operatorname{div}\mathbf{v}_1 = 0,\quad (x,t)\in Q_T,$$

$$(4.8.37)\qquad \mathbf{v}_1(x,0)=\mathbf{a}(x),\quad x\in\Omega;\qquad \mathbf{v}_1(x,t)=0,\quad (x,t)\in S_T;$$

and

$$(4.8.38)\qquad (\mathbf{v}_2)_t - \nu\,\Delta\mathbf{v}_2 + \alpha_2(t)\,\mathbf{v}_2 = -\nabla p_2 + f_2(t)\,\mathbf{g}(x,t),\qquad \operatorname{div}\mathbf{v}_2 = 0,\quad (x,t)\in Q_T;$$

$$(4.8.39)\qquad \mathbf{v}_2(x,0)=\mathbf{a}(x),\quad x\in\Omega;\qquad \mathbf{v}_2(x,t)=0,\quad (x,t)\in S_T;$$

where the coefficients

$$\{\alpha_1,\, f_1\}$$

and

$$\{\alpha_2,\, f_2\}$$

are suitably chosen in the space $\mathbf{C}([0,\, T])$ subject to condition (4.8.35).

A simple observation may be of help in establishing estimate (4.8.35) saying that (4.8.36)-(4.8.39) lead to the system

$$
\begin{aligned}
(4.8.40)\qquad & (\mathbf{v}_1 - \mathbf{v}_2)_t - \nu\, \Delta(\mathbf{v}_1 - \mathbf{v}_2) \\
&\qquad + [\alpha_1(t) - \alpha_2(t)]\, \mathbf{v}_1 + \alpha_2(t)\, (\mathbf{v}_1 - \mathbf{v}_2) \\
&\qquad = -\nabla(p_1 - p_2) + [f_1(t) - f_2(t)]\, \mathbf{g}(x,t)\,, \\
&\operatorname{div}(\mathbf{v}_1 - \mathbf{v}_2) = 0\,, \qquad (x,t) \in Q_T\,,
\end{aligned}
$$

supplied by the initial and boundary conditions

$$
(4.8.41)\qquad (\mathbf{v}_1 - \mathbf{v}_2)(x,0) = 0\,, \qquad x \in \Omega\,,
$$

$$
(4.8.42)\qquad (\mathbf{v}_1 - \mathbf{v}_2)(x,t) = 0\,, \qquad (x,t) \in S_T\,.
$$

The first equation of the system (4.8.40) implies the energy identity

$$
\begin{aligned}
\frac{1}{2}\, \frac{d}{dt}\, \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|^2_{2,\Omega} &+ \nu\, \| (\mathbf{v}_1 - \mathbf{v}_2)_x(\,\cdot\,, t) \|^2_{2,\Omega} \\
&+ \alpha_2(t)\, \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|^2_{2,\Omega} \\
&+ [\alpha_1(t) - \alpha_2(t)] \int_\Omega \mathbf{v}_1 \cdot (\mathbf{v}_1 - \mathbf{v}_2)\, dx \\
&= [f_1(t) - f_2(t)] \int_\Omega \mathbf{g} \cdot (\mathbf{v}_1 - \mathbf{v}_2)\, dx\,, \\
& t \in [0,\, T]\,,
\end{aligned}
$$

which assures us the validity of the estimate for $t \in [0,\, T]$

$$
\begin{aligned}
(4.8.43)\qquad \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|_{2,\Omega} \le & \int_0^t |\, \alpha_2(\tau)|\; \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, \tau) \|_{2,\Omega}\, d\tau \\
&+ \int_0^t |\, \alpha_1(\tau) - \alpha_2(\tau)|\; \| \mathbf{v}_1(\,\cdot\,, \tau) \|_{2,\Omega}\, d\tau \\
&+ \int_0^t |\, f_1(\tau) - f_2(\tau)|\; \| \mathbf{g}(\,\cdot\,, \tau) \|_{2,\Omega}\, d\tau\,.
\end{aligned}
$$

Here the initial condition (4.8.41) was taken into account as well.

It follows from (4.8.43) that

$$
\begin{aligned}
(4.8.44)\quad & \sup_{\tau\in[0,\,t]} \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,,\tau) \|_{2,\Omega} \\
& \le \sup_{\tau\in[0,\,t]} \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,,\tau) \|_{2,\Omega} \, \| \alpha_2 \|_C \, T \exp\{\gamma T\} \\
& \quad + \int_0^T \exp\{\gamma T\}\, d\tau \Bigg[\sup_{\tau\in[0,\,t]} \| \mathbf{v}_1(\,\cdot\,,\tau) \|_{2,\Omega} \, \| \alpha_1 - \alpha_2 \|_C \\
& \quad + \sup_{\tau\in[0,\,t]} \| \mathbf{g}(\,\cdot\,,\tau) \|_{2,\Omega} \, \| f_1 - f_2 \|_C \Bigg] .
\end{aligned}
$$

Recalling that the pairs

$$\chi_1 = \{ \alpha_1, f_1 \}$$

and

$$\chi_2 = \{ \alpha_2, f_2 \}$$

belong to the ball D_r, whose radius r is specified by (4.8.28), we place estimate (4.8.30) in (4.8.40) as we did for the function $\mathbf{v}_1$, thereby justifying the inequality

$$
\begin{aligned}
\sup_{\tau\in[0,\,t]} \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,,\tau) \|_{2,\Omega} \le 2 \int_0^t & \exp\{\gamma T\}\, d\tau \\
& \times \Big[\sup_{\tau\in[0,\,t]} \| \mathbf{g}(\,\cdot\,,\tau) \|_{2,\Omega} + \| \mathbf{a} \|_{2,\Omega} \Big] \\
& \times \Big(\| \alpha_1 - \alpha_2 \|_C + \| f_1 - f_2 \|_C \Big) , \\
& t \in [0, T] ,
\end{aligned}
$$

which immediately implies the second auxiliary estimate (4.8.35).

Returning to the proof of estimate (4.8.29) we find that

(4.8.45)
$$
\begin{aligned}
\uparrow \mathbf{A}\chi_1 - \mathbf{A}\chi_2 \uparrow_{\mathbf{C}([0,T])} &= \| A_1(\alpha_1, f_1) - A_1(\alpha_2, f_2) \|_C \\
&\quad + \| A_2(\alpha_1, f_1) - A_2(\alpha_2, f_2) \|_C \\
&\le \| \widetilde{A}_1(\alpha_1, f_1) - \widetilde{A}_1(\alpha_2, f_2) \|_C \\
&\quad + \Big[1 + \sup_{t\in[0,T]} | g_1(t) | \, \varphi_T^{-1} \Big] \\
&\quad \times \| \widetilde{A}_2(\alpha_1, f_1) - \widetilde{A}_2(\alpha_2, f_2) \|_C
\end{aligned}
$$

and, in turn,

(4.8.46)
$$
\begin{aligned}
&\| \widetilde{A}_1(\alpha_1, f_1) - \widetilde{A}_1(\alpha_2, f_2) \|_C \\
&\le \nu\, \varphi_T^{-1} \, \| \Delta \boldsymbol{\omega}_1 \|_{2,\Omega} \sup_{t\in[0,T]} \Big[\exp\{-\gamma t\} \, \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|_{2,\Omega} \Big]
\end{aligned}
$$

and

(4.8.47)
$$
\begin{aligned}
&\| \widetilde{A}_2(\alpha_1, f_1) - \widetilde{A}_2(\alpha_2, f_2) \|_C \\
&\le \nu\, g_T^{-1} \, \| \Delta \boldsymbol{\omega}_2 \|_{2,\Omega} \sup_{t\in[0,T]} \Big[\exp\{-\gamma t\} \, \| (\mathbf{v}_1 - \mathbf{v}_2)(\,\cdot\,, t) \|_{2,\Omega} \Big] .
\end{aligned}
$$

Recall that the functions A_1, A_2, $\widetilde{A}_1$ and $\widetilde{A}_2$ were introduced earlier in (4.8.8) and help motivate what is done.

Having involved (4.8.35) in estimation of the right-hand sides of (4.8.46)–(4.8.47) and substituted the final result into (4.8.45) we can be pretty sure that the operator $\mathbf{A}$ acts on D_r by the governing rule (4.8.29), thereby completing the proof of the lemma. ■

Remark 4.8.1 Let us show that $m \neq 0$ because of (4.8.7). At first glance, the value

$$
\sup_{t\in[0,T]} \| \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} + 2 \| \mathbf{a} \|_{2,\Omega}
$$

is positive, since

$$
\Big| \int_{\Omega} \mathbf{g}(x,t) \cdot \boldsymbol{\omega}_2(x) \, dx \Big| \ge g_T > 0
$$

for any $t \in [0,\, T]$. A similar remark is still valid for the second multiplier in the formula for m (see (4.8.29)). This is due to the fact that $\| \Delta \omega_1 \|_{2,\,\Omega} \neq 0$ in the context of (4.8.7). Indeed, having preassumed the contrary we obtain $\omega_1(x) = 0$ almost everywhere in Ω as a corollary to the basic restriction on the input data $\omega_1 \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ and the maximum principle with regard to the components of the function ω_1. In this line,

$$\varphi(t) = \int_{\Omega} \mathbf{v}(x,t) \cdot \omega_1(x) \; dx \equiv 0 \,,$$

which disagrees with the constraint $| \varphi(t) | \geq \varphi_T > 0$ imposed at the very beginning for any $t \in [0,\, T]$.

Corollary 4.8.1 *Let the conditions of Lemma* 4.8.1 *hold. If γ is representable by*

$$\dot{\gamma} = m + \varepsilon \,, \tag{4.8.48}$$

where ε is an arbitrary positive number, then the operator $\mathbf{A}$ is a contraction in the ball D_r of radius

$$r = (2\,T)^{-1} \exp \{ -(m + \varepsilon)\, T \}$$

and the relevant contraction coefficient is equal to $m/(m + \varepsilon)$.

Proof The proof reduces to inserting (4.8.48) in (4.8.28)–(4.8.29). After that, the statement we must prove is simple to follow. ■

The next step is to find out under what sufficient conditions the operator $\mathbf{A}$ carries the ball D_r into itself. We are in receipt of the answer from the following lemma.

Lemma 4.8.2 *Let the input data of the inverse problem* (4.8.2)–(4.8.6) *satisfy* (4.8.7). *If the radius of D_r is given by formula* (4.8.28), *then the operator $\mathbf{A}$ admits the estimate*

$$\begin{aligned} \uparrow \mathbf{A} \chi \uparrow_{C([0,\,T])} \leq \; & 2^{-1}\, m + \varphi_T^{-1} \, \| \varphi' \|_C \\ & + g_T^{-1} \, \| \psi \|_C + \varphi_T^{-1} \, g_T^{-1} \, \| g_1 \, \psi \|_C \,, \end{aligned} \tag{4.8.49}$$

where $\chi \in D_r$ and m is of the same form as we approved in Lemma 4.8.1.

Proof In conformity with (4.8.8) we write down

$$
\begin{aligned}
(4.8.50)\qquad \uparrow \mathbf{A}\chi \uparrow_{\mathbf{C}([0,T])} &= \| A_1(\alpha, f) \|_C + \| A_2(\alpha, f) \|_C \\
&\leq \| \widetilde{A}_1(\alpha, f) \|_C + \Bigg(1 + \varphi_T^{-1} \\
&\quad \times \sup_{t\in[0,T]} | g_1(t) | \Bigg) \| \widetilde{A}_2(\alpha, f) \|_C \\
&\quad + \varphi_T^{-1} \| \varphi' \|_C + g_T^{-1} \| \psi \|_C \\
&\quad + (\varphi_T\, g_T)^{-1} \| g_1 \psi \|_C
\end{aligned}
$$

and, in turn,

$$
(4.8.51)\qquad | \widetilde{A}_1(\alpha, f) | \leq \nu\, \varphi_T^{-1} \| \Delta \boldsymbol{\omega}_1 \|_C \cdot \| \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega}, \quad t \in [0, T],
$$

$$
(4.8.52)\qquad | \widetilde{A}_2(\alpha, f) | \leq \nu\, g_T^{-1} \| \Delta \boldsymbol{\omega}_2 \|_C \cdot \| \mathbf{v}(\,\cdot\,, t) \|_{2,\Omega}, \quad t \in [0, T].
$$

Estimating the right-hand sides of inequalities (4.8.51)–(4.8.52) on the basis of (4.8.30) and substituting the resulting expressions into (4.8.50), we obtain (4.8.49) and the lemma is completely proved. ■

The principal result of this section is the following.

Theorem 4.8.2 *Let* $\mathbf{g} \in \mathbf{C}([0,T], \mathbf{L}_2(\Omega))$, $\mathbf{a} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega}_1 \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \overset{\circ}{\mathbf{J}}(\Omega)$, $\boldsymbol{\omega}_2 \in \mathbf{W}_2^2(\Omega) \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \cap \mathbf{G}(\Omega)$, $\varphi \in C^1([0,T])$, $\psi \in C([0,T])$, $\left| \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}_2(x)\, dx \right| \geq g_T > 0$, $|\varphi(t)| \geq \varphi_T > 0$ $(g_T, \varphi_T \equiv$ const $)$ *for* $t \in [0,T]$ *and let the compatibility condition* (4.8.11) *hold. With respect to the ball* D_r *of radius*

$$
(4.8.53)\qquad r = (2T)^{-1} \exp\{ -(m+1)\, T \},
$$

where

$$
\begin{aligned}
m = 2\nu &\left[\sup_{t\in[0,T]} \| \mathbf{g}(\,\cdot\,, t) \|_{2,\Omega} + 2 \| \mathbf{a} \|_{2,\Omega} \right] \\
&\times \Bigg[\varphi_T^{-1} \| \Delta \boldsymbol{\omega}_1 \|_{2,\Omega} + g_T^{-1} \Big(1 + \varphi_T^{-1} \\
&\times \sup_{t\in[0,T]} | g_1(t) | \Big) \| \Delta \boldsymbol{\omega}_2 \|_{2,\Omega} \Bigg],
\end{aligned}
$$

$$
g_1(t) = \int_\Omega \mathbf{g}(x,t) \cdot \boldsymbol{\omega}_1(x)\, dx,
$$

the estimate

$$(4.8.54)\quad 2T\left(2^{-1}m+\varphi_T^{-1}\,\|\,\varphi'\,\|_C+g_T^{-1}\,\|\,\psi\,\|_C+(\varphi_T\,g_T)-1\,\|\,g_1\,\psi\,\|_C\right)$$
$$\leq \exp\{-(m+1)\,T\}$$

is supposed to be true. Then:

(a) *the inverse problem* (4.8.2)–(4.8.6) *has a solution*

$$\{\mathbf{v},\,\nabla p,\,\alpha,\,f\}$$

and the vector $\chi=\{\alpha,\,f\}$ *lies within* D_r*;*

(b) *there are no two distinct solutions* $\{\mathbf{v}_i,\nabla p_i,\alpha_i,f_i\}$*,* $i=1,2$*, of the inverse problem* (4.8.2)–(4.8.6) *such that both satisfy the condition* $\chi_i=\{\alpha_i,f_i\}\in D_r$*,* $i=1,2$*.*

Proof First, we are going to show that the principle of contracting mapping is acceptable for the nonlinear operator $\mathbf{A}$ specified by (4.8.8) on account of Corollary 4.8.1 with $\varepsilon=1$ and, consequently, $\gamma=m+1$ (see (4.8.48)). Having this remark done, it is straightforward to verify that the operator $\mathbf{A}$ is a contraction on the closed ball D_r of radius r specified by (4.8.53). If so, the contraction coefficient equals $m/(m+1)$.

On the other hand, since $\gamma=m+1$, Lemma 4.8.2 and estimate (4.8.54) together imply that the operator $\mathbf{A}$ carries the ball D_r into itself. Therefore, the operator $\mathbf{A}$ has a unique fixed point in D_r. To put it differently, we have established earlier that the nonlinear equation (4.8.9) has in D_r a unique solution, say χ, whose first and second components will be denoted by $\alpha(t)$ and $f(t)$, respectively, that is,

$$\chi\equiv\{\alpha,\,f\}\,.$$

Since the compatibility condition (4.8.11) holds, Theorem 4.8.1 yields that there exists a solution $\{\mathbf{v},\nabla p,\alpha,f\}$ of the inverse problem (4.8.2)–(4.8.6). The location of the vector $\chi=\{\alpha,\,f\}$ in the ball D_r was established before. Thus, item (a) is completely proved.

We proceed to item (b). Assume to the contrary that there were two distinct solutions

$$\{\mathbf{v}_1,\,\nabla p_1,\,\alpha_1,\,f_1\}$$

and

$$\{\mathbf{v}_2,\,\nabla p_2,\,\alpha_2,\,f_2\}$$

of the inverse problem (4.8.2)–(4.8.6) such that both vectors

$$\chi_1 = \{\, \alpha_1 ,\, f_1 \}$$

and

$$\chi_2 = \{\, \alpha_2 ,\, f_2 \}$$

lie within the ball D_r.

We claim that if the collections

$$\{\, \mathbf{v}_1 ,\, \nabla p_1 ,\, \alpha_1 ,\, f_1 \}$$

and

$$\{\, \mathbf{v}_2 ,\, \nabla p_2 ,\, \alpha_2 ,\, f_2 \}$$

are different, then so are the vectors χ_2 and χ_2. Indeed, let $\chi_1 \equiv \chi_2$. Then $\alpha_1 \equiv \alpha_2$ and $f_1 \equiv f_2$. Consequently, by the uniqueness theorem for the direct problem (4.8.2)–(4.8.4) the functions $\mathbf{v}_1$ and ∇p_1 should coincide almost everywhere in Q_T with $\mathbf{v}_2$ and ∇p_2, respectively.

Consider the first collection $\{\mathbf{v}_1, \nabla p_1, \alpha_1, f_1\}$. By assumption, these satisfy the system (4.8.2)–(4.8.6). Then the first assertion of Theorem 4.8.1 ensures that the vector

$$\chi_1 = \{\alpha_1 ,\, f_1\}$$

gives a solution to equation (4.8.9). A similar reasoning shows that the vector

$$\chi_2 = \{\, \alpha_2 ,\, f_2 \}$$

satisfies the same equation (4.8.9).

On the other hand, we preassumed that both vectors χ_1 and χ_2 lie within the ball D_r, thus causing the appearance of two distinct solutions to equation (4.8.9) that belong to D_r. But this disagrees with the uniqueness of the fixed point of the operator $\mathbf{A}$ in D_r that has been established before. Consequently, an assumption violating the assertion of item (b) fails be true and thereby the theorem is completely proved. ■

In conclusion we give an example illustrating the result obtained.

Example 4.8.1 We are concerned with functions $\mathbf{g}$, $\mathbf{a}$, ω_1 and ω_2 suitably chosen in the appropriate classes in such a way that

$$\left| \int\limits_{\Omega} \mathbf{g}(x,t) \cdot \omega_2(x) \, dx \right| \geq g_T > 0 \,, \qquad \left| \int\limits_{\Omega} \mathbf{a}(x) \cdot \omega_1(x) \, dx \right| > 0 \,.$$

Accepting $\varphi(t) \equiv \int_\Omega \mathbf{a}(x) \cdot \boldsymbol{\omega}_1(x)\, dx$ and $\psi(t) \equiv 0$ we obtain $\varphi(t) \equiv \text{const}$ and check that the compatibility condition (4.8.11) is satisfied. Estimate (4.8.54) is rewritten as

$$T\, m \le \exp\{-(m+1)\, T\}\,, \tag{4.8.55}$$

where m is of the form (4.8.53). Obviously, as $T \to 0+$, the left-hand side of (4.8.53) tends to 0, while the right-hand side has 1 as its limit. Consequently, for any $m > 0$ there exists a time moment T_1, at which estimate (4.8.55) becomes true. We consider (4.8.2)–(4.8.6) with $T = T_1$ and the input data being still subject to the conditions imposed above. In such a setting Theorem 4.8.2 asserts that the inverse problem (4.8.2)–(4.8.6) possesses a solution

$$\{\, \mathbf{v},\ \nabla p,\ \alpha,\ f\, \}$$

with $\{\alpha,\ f\} \in D_r$, where

$$r = (2\, T_1)^{-1} \exp\{-(m+1)\, T_1\}\,.$$

What is more, there are no two distinct solutions $\{\mathbf{v}_i, \nabla p_i, \alpha_i, f_i\}$, $i = 1, 2$, such that both satisfy the condition $\{\alpha_i, f_i\} \in D_r$, $i = 1, 2$.

Chapter 5

Some Topics from Functional Analysis and Operator Theory

5.1 The basic notions of functional analysis and operator theory

Contemporary methods for solving inverse problems are gaining the increasing popularity. They are being used more and more in solving applied problems not only by professional mathematicians but also by investigators working in other branches of science. In order to make this book accessible not only to specialists but also to graduate and post-graduate students, we give a complete account of notions and definitions which will be used in the sequel. The concepts and theorems presented below are of an auxiliary nature and are included for references rather than for primary study. For this reason the majority of statements are quoted without proofs. We will also cite bibliographical sources for further, more detailed, information.

A set V is called a **vector space** over the field of real numbers $\mathbf{R}$ or complex numbers $\mathbf{C}$ if

(A) *the sum* $x + y \in V$ *is defined for any pair of elements* x, $y \in V$;

(B) *the operation of multiplication* $\alpha x \in V$ *is defined for any element*

$x \in V$ and any number $\alpha \in \mathbf{R}$;

(C) *the operations of addition and multiplication just introduced are subject to the following conditions:*

(1) $x + y = y + x$ *for any* $x, y \in V$;
(2) $(x + y) + z = x + (y + z)$ *for any* $x, y, z \in V$;
(3) *in* V *there exists a zero element* θ *such that* $x + \theta = x$ *for any* $x \in X$;
(4) *every element* $x \in V$ *has a negative element* $(-x) \in V$ *such that* $x + (-x) = \theta$;
(5) $1 \cdot x = x$ *for any* $x \in V$;
(6) $\alpha(\beta x) = (\alpha\beta)x$ *for any* $x \in V$ *and any* $\alpha, \beta \in \mathbf{R}$;
(7) $(\alpha + \beta)x = \alpha x + \beta x$ *for any* $x \in V$ *and any* $\alpha, \beta \in \mathbf{R}$;
(8) $\alpha(x + y) = \alpha x + \beta x$ *for any* $x, y \in V$ *and any* $\alpha \in \mathbf{R}$.

In this context, let us stress that in any vector space the zero and negative elements are unique. Also, by the difference $x - y$ we mean the sum of elements x and $(-y)$. An element y is called a **linear combination** of elements $x_1, x_2, \dots, x_n$ with coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ if

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n .$$

Any subset V_1 of the vector space V is termed a **subspace** of this space if it can be treated as a vector space once equipped with the usual operations of addition and multiplication by a number. For any subset V_1 of the vector space V a **linear span** $L(V_1)$ is defined as the set of all linear combinations of elements from V_1. It is clear that the span $L(V_1)$ is always a linear subspace of the space V.

Elements $x_1, x_2, \dots, x_n$ from the space V are said to be linearly dependent if some linear combination with at least one non-zero coefficient gives a zero element of this space. A system of elements $x_1, x_2, \dots, x_n$ is said to be linearly independent if there is no linear combination of this type. Likewise, a subset V_1 of the vector space V is said to be linearly independent if each finite system of its elements is linearly independent. A subset V_1 of the space V is called complete if $L(V_1) = V$. Any linearly independent and complete subset is termed a **Hamel** or an **algebraic basis**. Cardinalities of various basises of one and the same space coincide. By the choice axiom the existence of an algebraic basis is established in any vector space. If such a basis contains only a finite number of elements, the basic space is said to be finite-dimensional. Otherwise, it turns out to be infinite-dimensional.

Banach spaces, being the most general ones, are given first. A space V is called **normed** if with each element $x \in V$ one can associate a real-valued function $\| x \|$ with the following properties:

(1) $\| x \| \geq 0$ *for any* $x \in V$ *and* $\| x \| = 0$ *if and only if* $x = \theta$;

(2) *for any* $x,\, y \in V$ *the triangle inequality* $\| x + y \| \leq \| x \| + \| y \|$ *holds;*

(3) $\| \alpha x \| = |\, \alpha \,| \cdot \| x \|$ *for any* $x \in V$ *and any number* $\alpha \in \mathbf{R}$.

Any such function is called a **norm** on the space V. The number $\| x \|$ with the indicated properties refers to the norm of an element x. The proximity between elements of a normed space V is well-characterized by means of the function

$$\rho(x, y) = \| x - y \| ,$$

possessing the standard three properties of the **distance** ρ:

(1) $\rho(x, y) \geq 0$ *for all* $x,\, y \in V$ *and* $\rho(x, y) = 0$ *if and only if* $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ *for any* $x,\, y \in V$;

(3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ *for any* $x,\, y,\, z \in V$.

In any normed space the convergence of a sequence $\{ x_n \}$ to an element x amounts to the convergence in norm: $x = \lim_{n \to \infty} x_n$ if

$$\lim_{n \to \infty} \| x_n - x \| = 0 .$$

This limit is always unique (if it exists). What is more, any convergent sequence $\{ x_n \}$ is bounded, that is, there exists a constant $M > 0$ such that $\| x_n \| \leq M$ for all $n = 1, 2, \ldots$. All linear operations and the associated norm are continuous in the sense that

$$x_n \to x\,,\ y_n \to y \implies x_n + y_n \to x + y\,;$$

$$x_n \to x\,,\ \alpha_n \to \alpha \implies \alpha_n\, x_n \to \alpha\, x\,;$$

$$x_n \to x \implies \| x_n \| \to \| x \| .$$

A sequence $\{ x_n \}$ is termed a **Cauchy sequence** if the convergence $\rho(x_n, x_m) \to 0$ occurs as $n,\, m \to \infty$. Any normed space in which every Cauchy sequence has a limit is referred to as a **Banach space**.

In a normed space V the next object of investigation is a **series**

$$x = \sum_{n=1}^{\infty} x_n . \tag{5.1.1}$$

We say that the series in (5.1.1) converges to an element x if x is a limit of the sequence of partial sums $s_n = x_1 + x_2 + \cdots + x_n$. In the case of Banach spaces the convergence of the series

$$\sum_{n=1}^{\infty} \| x_n \|$$

implies the convergence of the series in (5.1.1). Using the notion of series it will be sensible to introduce the concept of **topological basis** known in the modern literature as **Shauder's basis**. A sequence of elements $\{ x_n \}$ of the space V constitutes what is called a topological or Shauder basis if any element $x \in V$ can uniquely be representable by

$$x = \sum_{n=1}^{\infty} \alpha_n x_n .$$

The availability of Shauder's basis is one of the principal peculiarities of Banach spaces. Common practice involves a Shauder basis when working in Banach spaces. A necessary condition for the existence of Shauder's basis is the separability of the Banach space V, that is, the existence of a countable and everywhere dense set in the space V. Due to Shauder it is interesting to learn whether any separable Banach space possesses a topological (Shauder) basis. This question has been open for a long time. The negative answer is now known. It was shown that there exists a separable Banach space without Shauder basis.

Subsequent studies need as yet the notion of **Euclidean space**. A real vector space V becomes an Euclidean space upon receipt of a function (x, y) of two variables. This function known as an **inner product** possesses the following properties:

(1) $(x, y) = (y, x)$ *for any* $x, y \in V$;

(2) $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$ *for any* $x, y, z \in V$ *and all numbers* α, β;

(3) $(x, x) \geq 0$ *for any* $x \in V$ *and* $(x, x) = 0$ *if and only if* $x = 0$.

When operating in a complex vector space V, the equality in condition (1) should be replaced by $(x, y) = \overline{(y, x)}$, where over-bar denotes conjunction. Being elements of an Euclidean space, $x, y \in V$ are subject to the **Cauchy–Schwartz inequality**:

$$|(x, y)| \leq \sqrt{(x, x)} \sqrt{(y, y)} .$$

The function $\| x \| = \sqrt{(x, x)}$ complies with the properties of the norm we have mentioned above. In view of this, it is always presupposed that any

Euclidean space is normed with respect to that norm. When an Euclidean space becomes a Banach space, we call it a **Hilbert space**. Any inner product is continuous in the sense of the following relation: $(x_n, y_n) \to (x, y)$ as $x_n \to x$ and $y_n \to y$.

It is worth noting here that for any normed space V there exists a Banach space V_1, containing V as a subspace and relating to a **completion** of the space V. In turn, the space V is everywhere dense in V_1 and the norms of each element in the spaces V and V_1 coincide. Since any normed space has a completion, such a trick permits the reader to confine yourself to Banach spaces only. In what follows by a normed space we shall mean a Banach space and by an Euclidean space – a Hilbert space. The notion of **orthogonal decomposition** in a Hilbert space is aimed at constructing **orthogonal projections**. Two elements x and y of a Hilbert space V are said to be orthogonal if $(x, y) = 0$. Let V_1 be a closed subspace of V. We initiate the construction of the set of elements from the space V that are orthogonal to all of the elements of the space V_1. Such a set is also a closed subspace of V and is called the **orthogonal complement** to V_1 in the space V. With the orthogonal complement V_2 introduced, we may attempt the Hilbert space V in the form

$$V = V_1 \oplus V_2 .$$

The meaning of **direct sum** is that each element $x \in V$ can uniquely be decomposed as

$$x = y + z ,$$

where $y \in V_1$, $z \in V_2$. These members are orthogonal and are called the orthogonal projections of the element x on the subspaces V_1 and V_2, respectively.

Separable Hilbert spaces will appear in later discussions. Unlike Banach spaces, any separable Hilbert space possesses a Shauder basis. Moreover, in any separable Hilbert space there exists an **orthonormal basis**, that is, a basis in which all of the elements have the unit norm and are orthogonal as couples. This profound result serves as a background for some analogy between separable Hilbert spaces and finite-dimensional Euclidean spaces as further developments occur. With this aim, let us fix an orthogonal basis $\{ x_n \}$, that is, a basis in which all of the elements, being arranged in pairs, are orthogonal. Then each element can be represented in the **Fourier series** as follows:

$$x = \sum_{n=1}^{\infty} c_n x_n$$

with **Fourier coefficients**

$$c_n = \frac{(x, x_n)}{\| x_n \|^2} ,$$

giving

$$\| x \|^2 = \sum_{n=1}^{\infty} | c_n |^2 \| x_n \|^2 .$$

We now turn to the concept of **linear operator**. Recall that an operator A from X into Y is said to be linear if

$$A(\alpha x + \beta y) = \alpha A x + \beta A y$$

for any elements $x, y \in X$ and all numbers α, β. In the case of normed spaces X and Y we say that a linear operator $A\colon X \mapsto Y$ is **bounded** if there is a constant $C \geq 0$ such that the inequality

$$\| A x \| \leq C \| x \|$$

holds for each $x \in X$ and adopt the value

$$\| A \| = \sup_{x \neq 0} \frac{\| A x \|}{\| x \|} \tag{5.1.2}$$

as the **norm of the linear operator** A. Once equipped with norm (5.1.2) the space of all linear bounded operators acting from X into Y becomes a **normed vector space** and will be denoted by the symbol $\mathcal{L}(X, Y)$. If Y is a Banach space, then so is the space $\mathcal{L}(X, Y)$. We write, as usual, $\mathcal{L}(X)$ for the space $\mathcal{L}(X, X)$. If $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$, then $B A \in \mathcal{L}(X, Z)$, so that the estimate

$$\| B A \| \leq \| B \| \cdot \| A \|$$

is valid.

The symbol Ker A designates the **kernel** of a linear operator A, that is, the set of all elements on which the operator A equals the zero element. In the case of a bounded operator the kernel is always closed. By Im A we denote the **image** of an operator A, that is, the set of its possible values.

An operator A is said to be **continuous** at a point x if $A x_n \to A x$ as $x_n \to x$. Recall that any linear operator defined on an entire normed space X with values in a normed space Y is continuous at a certain fixed point if and only if this operator is continuous at each point. Furthermore, the continuity of such a linear operator is equivalent to being bounded.

The following results are related to linear operators and find a wide range of applications in functional analysis.

Theorem 5.1.1 (the **Banach–Steinhaus theorem**) *Let a sequence of linear operators $A_n \in \mathcal{L}(X, Y)$ be bounded in $\mathcal{L}(X, Y)$ and a subset $X_1 \subset X$ be dense. One assumes, in addition, that the limit $A_n x$ exists for each $x \in X_1$. Then the limit $A x = \lim_{n\to\infty} A_n x$ exists for each $x \in X$ and the inclusion $A \in \mathcal{L}(X, Y)$ occurs.*

Theorem 5.1.2 (the **principle of uniform boundedness**) *Let $\mathcal{A} \subset \mathcal{L}(X, Y)$ be such that for any $x \in X$ the set $\{ A x \colon A \in \mathcal{A} \}$ is bounded in the space Y. Then $\mathcal{A}$ is bounded in the space $\mathcal{L}(X, Y)$.*

The property that an operator A is **invertible** means not only the existence of the **inverse** A^{-1} in a sense of mappings, but also the boundedness of the operator A^{-1} on the entire space Y. If the operator A^{-1} is invertible and the inequality

$$\| B - A \| < \| A^{-1} \|^{-1}$$

holds, then the operator B is invertible, too. Moreover, the representation

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} \Big((A - B) A^{-1} \Big)^n \tag{5.1.3}$$

takes place. The series on the right-hand side of (5.1.3) converges in the space $\mathcal{L}(Y, X)$.

Being concerned with a linear operator $A \in \mathcal{L}(X)$, we call $\rho(A)$ the **resolvent set** of the operator A if $\rho(A)$ contains all of the complex numbers λ for which the operator $\lambda I - A$ is invertible. In that case the operator $R(\lambda, A) = \big(\lambda I - A \big)^{-1}$ is called the **resolvent** of the operator A. Here I stands, as usual, for the **identity operator** in the space X. The complement of the resolvent set $\rho(A)$ is referred to as the **spectrum** of the operator A and is denoted in the sequel by $\sigma(A)$. The spectrum of the operator A lies within a closed circle with center at zero and radius $\| A \|$. We agree to consider

$$R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n$$

as long as $| \lambda | > \| A \|$. As can readily be observed, the resolvent set is always open. More specifically, if the inclusion $\lambda \in \rho(A)$ occurs, then a circle with center at the point λ and radius $\| R(\lambda, A) \|^{-1}$ is completely covered by the resolvent set and for any μ from that circle the representation

$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1} \tag{5.1.4}$$

can be established without difficulty. This provides enough reason to conclude that the spectrum of each bounded operator A is closed and its resolvent set is not empty. It is easy to show that its spectrum is not empty, too. The value

$$r(A) = \sup_{\lambda \in \sigma(A)} | \lambda |$$

is called the **spectral radius** of the operator A. The spectral radius of any linear bounded operator A is given by the formula

$$r(A) = \lim_{n\to\infty} \| A^n \|^{1/n}. \tag{5.1.5}$$

It is worth noting here that it is possible to split up the spectrum of the operator A into three nonintersecting parts: the point spectrum, the continuous spectrum and the residual spectrum. The **point spectrum** is formed by all the eigenvalues of the operator A. In the case of a finite-dimensional space the spectrum of the operator A coincides with its point spectrum. But this property may be violated in an arbitrary Banach space and so the remaining part of the spectrum can be separated once again into two nonintersecting parts: the **continuous spectrum** $\sigma_c(A)$ and the **residual spectrum** $\sigma_r(A)$. In so doing the continuous spectrum $\sigma_c(A)$ is formed by all of the points λ for which the range of the operator $\lambda I - A$ is dense in the space X.

It is fairly common to call linear operators from a normed space X into real numbers **R** or complex numbers **C** **linear functionals**. The collection of all linear continuous functionals over the space X with norm (5.1.2) is called the **dual space** and is denoted by the symbol X^*. It is worth mentioning here that the dual space X^* is always a Banach space even if the basic normed space X does not fall within the category of Banach spaces.

Let now $A \in \mathcal{L}(X)$ and $f \in X^*$. A linear operator A^* is defined by the relation

$$\left(A^* f\right)(x) = f(A\,x).$$

It is straightforward to verify that the functional $A^* f \in X^*$. The operator so defined is called the **adjoint** of A. It is plain to show that for any linear continuous operator A its own adjoint A^* is continuous with $\| A^* \| = \| A \|$. Moreover, the spectrums of the operators A^* and A coincide.

In the case of a Hilbert space for any element $f \in X$ the functional

$$\hat{f}(x) = (x,\, f) \tag{5.1.6}$$

belongs to the space X^* and $\| \hat{f} \| = \| f \|$. The converse is established in the following theorem.

Theorem 5.1.3 (the **Riesz theorem**) *Let X be a Hilbert space and a functional $\hat{f} \in X^*$. Then there exists an element $f \in X$ such that equality* (5.1.6) *holds for each $x \in X$. Moreover, this element is unique and $\| \hat{f} \| = \| f \|$.*

In agreement with the Riesz theorem formula (5.1.6) describes a general form of any linear continuous functional in a Hilbert space. Just for this reason the functional $\hat{f}$ in the Hilbert space X is associated with the element f given by formula (5.1.6). Because of this fact, the space X^* is identified with X and the adjoint A^* acts in the space X as well. With this in mind, the operator A^* can be defined by the relation

$$(Ax, f) = (x, A^* f),$$

which is valid for all $x, f \in X$. In this case the equality $\| A \| = \| A^* \|$ continues to hold, but the spectrum of the operator A^* becomes complex conjugate to the spectrum of the operator A.

Under such an approach it is possible to introduce a notion of self-adjoint operator. An operator A is said to be **self-adjoint** if $A^* = A$. The spectrum of any self-adjoint operator lies on the real line, while its residual spectrum is empty.

In the space $\mathcal{L}(X)$ of all linear bounded operators in a Banach space X it will be sensible to distinguish a subspace of compact operators. A linear operator $A \in \mathcal{L}(X)$ is said to be **compact** or **completely continuous** if any bounded subset of the space X contains a sequence which is carried by the operator A into a converging one. The subspace of compact operators is closed in the space $\mathcal{L}(X)$. By multiplying a compact operator from the left as well as from the right by a bounded operator belonging to the space $\mathcal{L}(X)$ we obtain once again a compact operator. The last properties of compact operators are often formulated as follows: the set of all compact operators forms in the space $\mathcal{L}(X)$ a **closed two-sided ideal**. If the operator A is compact, then so is the operator A^*. The structure of the spectrum of a compact operator is known from the following assertion.

Theorem 5.1.4 *When an operator A from the space $\mathcal{L}(X)$ is compact, its spectrum is at most countable and does not have any non-zero limiting points. Furthermore, each number $\lambda \in \sigma(A)$, $\lambda \neq 0$, falls within eigenvalues of finite multiplicity for the operator A and is one of the eigenvalues of the same multiplicity for the operator A^*. If the basic space X is infinite-dimensional, then the inclusion $0 \in \sigma(A)$ occurs.*

One thing is worth noting in this context. As a matter of fact, when X becomes a Hilbert space, the dual X^* will be identified with X. The outcome of this is that the operator A^* acts in the space X and its spectrum passes through a procedure of complex conjugation. In this connection the second item of Theorem 5.1.4 should be restated as follows: *each number $\lambda \in \sigma(A)$, $\lambda \neq 0$, is one of the eigenvalues of finite multiplicity for the operator A and the number $\bar{\lambda}$ is one of the eigenvalues of the same multiplicity for the operator A^*.*

The theories of compact self-adjoint operators in a Hilbert space X and of symmetric operators in a finite-dimensional space reveal some analogy. If the space X is separable, then the collection of eigenvectors of any self-adjoint compact operator constitutes an orthonormal basis in the space X. But sometimes the space X is not separable, thus causing difficulties. This issue can be resolved following established practice. Let A be a linear self-adjoint operator. Then the orthogonal complement to its kernel coincides with the closure of its image. If one assumes, in addition, that the operator A is compact, then the closure of its image will be separable. That is why the subspace generated by all the eigenvectors associated with non-zero eigenvalues of any compact self-adjoint operator A will be also separable and may be endowed with an orthonormal basis consisting of its eigenvectors. At the same time the orthogonal complement to the subspace includes all of its eigenvectors relating to the zero eigenvalue. The facts we have outlined above can be formulated in many ways. We cite below one possible statement known as the Hilbert–Schmidt theorem.

Theorem 5.1.5 (the **Hilbert–Schmidt theorem**) *Let A be a compact self-adjoint operator in a Hilbert space X. Then for each $x \in X$ the element Ax is representable by a convergent Fourier series with respect to an orthonormal system formed by eigenvectors of the operator A.*

The Hilbert–Schmidt theorem can serve as a basis for solving the following equation by the **Fourier method**:

$$Ax = y. \tag{5.1.7}$$

Equation (5.1.7) as well as the integral equation

$$\int_a^b A(t,s)\, x(s)\, ds = y(t)$$

fall within the category of **equations the first kind**. Observe that a solution to equation (5.1.7) can be found up to elements from the kernel $\operatorname{Ker} A$ of the operator A. So, the requirement $\operatorname{Ker} A = 0$ is necessary for this solution to be unique. If the operator A is compact and self-adjoint and its kernel consists of a single zero point only, then the space X can be equipped with the orthonormal basis $\{ e_n \}$ formed by eigenvectors of the operator A, that is,

$$A e_n = \lambda_n e_n, \qquad \lambda_n \neq 0.$$

Involving such a basis we turn to (5.1.4) and write down the **Fourier expansions** for the elements x and y:

$$x = \sum_{n=1}^{\infty} x_n e_n , \qquad x_n = (x, e_n) , \tag{5.1.8}$$

$$y = \sum_{n=1}^{\infty} y_n e_n , \qquad y_n = (x, e_n) . \tag{5.1.9}$$

All this enables us to accept the decomposition

$$A x = \sum_{n=1}^{\infty} x_n e_n .$$

From (5.1.7) we deduce by the uniqueness of the Fourier series that $\lambda_n x_n = y_n$ for any element y and, therefore,

$$x_n = \frac{y_n}{\lambda_n} . \tag{5.1.10}$$

For the series in (5.1.8) to be convergent in the space X it is necessary and sufficient that the series $\sum_{n=1}^{\infty} | x_n |^2$ is convergent. Summarizing, we obtain the following result.

Theorem 5.1.6 *Let an operator A in a Hilbert space X be compact and self-adjoint with zero kernel and expansion* (5.1.9) *hold. Then equation* (5.1.7) *has a solution if and only if*

$$\sum_{n=1}^{\infty} \left| \frac{y_n}{\lambda_n} \right|^2 < \infty .$$

Moreover, this solution is unique and is given by formulae (5.1.8) *and* (5.1.10).

Any equation of the form

$$x - A x = y \tag{5.1.11}$$

falls within the category of **equations of the second kind**. If $A \in \mathcal{L}(X)$ and $\| A \| < 1$, then the operator $B = I - A$ is invertible, so that

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n, \tag{5.1.12}$$

thereby justifying that a solution to equation (5.1.11) exists and is unique for any $y \in X$. Furthermore, this solution is given by the formula

$$x = \sum_{n=0}^{\infty} A^n y . \tag{5.1.13}$$

Both series from expansions (5.1.12) and (5.1.13) are called the **Neumann series.**

The method of **successive approximations** may be of help in studying equation (5.1.11) with the aid of a sequence

$$x_{n+1} = A x_n + y , \qquad n = 0, 1, 2, \ldots . \tag{5.1.14}$$

The essence of the matter would be clear from the following assertion.

Theorem 5.1.7 *Let $A \in \mathcal{L}(X)$ and $\| A \| < 1$. Then a solution to equation* (5.1.11) *exists and is unique for any $y \in X$. Moreover, for any initial data $y \in X$ the successive approximations specified by* (5.1.14) *converge to an exact solution x to equation* (5.1.11) *and the estimate is valid:*

$$\| x_n - x \| \leq \frac{\| A \|^n}{1 - \| A \|} \, \| A x_0 - x_0 \| . \tag{5.1.15}$$

Observe that the partial sums of the Neumann series in (5.1.13) will coincide with the successive approximations specified by (5.1.14) for the case $x_0 = y$. In some cases it is possible to weaken the condition $\| A \| < 1$. This is due to the fact from the following proposition.

Theorem 5.1.8 *Let an operator $A \in \mathcal{L}(X)$ and there exist a positive integer k such that $\| A^k \| < 1$. Then a solution to equation* (5.1.11) *exists and is unique for any $y \in X$. Furthermore, for any initial data $x_0 \in X$ the successive approximations*

$$x_{n+1} = A^k x_n + \sum_{s=0}^{k-1} A^s y , \qquad n = 0, 1, 2, \ldots ,$$

converge to an exact solution x of equation (5.1.11) *and the estimate is valid:*

$$\| x_n - x \| \leq \frac{\| A^k \|^n}{1 - \| A^k \|} \, \| x_1 - x_0 \| .$$

The method of successive approximations being used in solving equation (5.1.11) applies equally well to equations with a **nonlinear operator** A. Let an operator A (generally speaking, nonlinear) map a Banach space X into itself. We say that the operator A is a **contraction** on a set Y if there exists a number $q \in (0, 1)$ such that the inequality

$$\| A x - A y \| \leq q \, \| x - y \| \tag{5.1.16}$$

holds true for any $x, y \in Y$.

Theorem 5.1.9 (the **contraction mapping principle**) *Let Y be a closed subset of the Banach space X, the inclusion $Ax + y \in Y$ occur for any $x \in Y$ and an operator A be a contraction on the space Y. Then a solution x to equation* (5.1.11) *that lies within Y exists and is unique. Moreover, the successive approximations specified by* (5.1.14) *converge to x and the estimate*

$$\| x_n - x \| \leq \frac{q^n}{1-q} \| x_1 - x_0 \|$$

is valid with constant g arising from (5.1.16).

Equation (5.1.11) is an **abstract counterpart** of the Fredholm integral equation of the second kind

$$x(t) - \int_a^b A(t,s)\, x(s)\, ds = y(t) . \tag{5.1.17}$$

Fredholm's theory of integral equations of the type (5.1.17) is based on the fact that the integral operator in (5.1.17) is compact in an appropriate functional space. F. Riesz and J. Schauder have extended the results of Fredholm's theory to cover the abstract equation (5.1.11) with a compact operator $A \in \mathcal{L}(X)$ involved. Along with (5.1.11) we shall need as yet the following equations:

$$x - Ax = 0 , \tag{5.1.18}$$

$$x - A^* x = y , \tag{5.1.19}$$

$$x - A^* x = 0 . \tag{5.1.20}$$

Some basic results concerning Fredholm's theory of operator equations of the second kind are quoted in the following proposition.

Theorem 5.1.10 *Let an operator $A \in \mathcal{L}(X)$ be compact. Then the following assertions are true:*

(1) *equation* (5.1.11) *is solvable for any $y \in X$ if and only if equation* (5.1.18) *has a trivial solution only;*

(2) *the spaces of solutions to equations* (5.1.18) *and* (5.1.20) *have the same finite dimension;*

(3) *for a given element $y \in X$ equation* (5.1.11) *is solvable if and only if each solution to equation* (5.1.20) *equals zero at the element y on the right-hand side of* (5.1.11).

Theorem 5.1.10 implies that the solvability of equation (5.1.11) for any right-hand side element is equivalent to being unique for its solution. Because of this fact, the problem of existence of a solution amounts to the problem of its uniqueness and could be useful in applications. As we have mentioned above, if the operator A is compact, then so is the operator A^*. In view of this, the solvability of equation (5.1.11) for any right-hand side element is equivalent not only to being unique for a solution to equation (5.1.11), but also to being solvable for each right-hand side $y \in X^*$ of equation (5.1.19).

In addition to Theorem 5.1.10 we might indicate when equation (5.1.19) has a solution. For a given element $y \in X^*$ equation (5.1.19) is solvable if and only if this element equals zero at each solution of equation (5.1.18).

In the case of a Hilbert space X the dual X^* will be identified with X under the inner product structure (x, f), where (x, f) is the value of f at an element x if f is treated as a functional. With this in mind, one can restate assertion (3) of Theorem 5.1.10 by imposing the requirement for the element y to be orthogonal to each solution of equation (5.1.20). When the operator A happens to be self-adjoint, some procedures with equations (5.1.11) and (5.1.18) become much more simpler, making it possible to reformulate Theorem 5.1.10 in simplified form.

Theorem 5.1.11 *Let X be a Hilbert space and a linear operator A be compact and self-adjoint. Then the following assertions are true:*

(1) *equation* (5.1.11) *is solvable for any $y \in X$ if and only if equation* (5.1.18) *has a trivial solution only;*

(2) *the space of solutions to equation* (5.1.18) *is of finite dimension;*

(3) *for a given element $y \in X$ equation* (5.1.11) *is solvable if and only if this element is orthogonal to each solution of equation* (5.1.18).

From the viewpoint of applications, differential operators are of great importance. When these operators are considered within the framework of the theory of Banach spaces, they may be unbounded and their domains do not necessarily coincide with the entire space. In mastering difficulties connected with unbounded operators, some notions we have introduced above need generalization. With this aim, consider a linear operator A in a Banach space X. The symbol $\mathcal{D}(A)$ stands, as usual, for the domain of A. It is supposed that $\mathcal{D}(A)$ is a linear subspace of X and for all numbers

α, β and all elements $x, y \in \mathcal{D}(A)$ the equality

$$A(\alpha\, x + \beta\, y) = \alpha\, Ax + \beta\, Ay$$

holds. Also, we would be agreeable with the assumption that the subspace $\mathcal{D}(A)$ is always dense in the space X. In this case the operator A is said to be densely defined. The sum $A + B$ of operators A and B is defined by means of the relation

$$(A + B)(x) = A\, x + B\, x$$

and the domain here is

$$\mathcal{D}(A + B) = \mathcal{D}(A) \bigcap \mathcal{D}(B)\,.$$

The product $B\, A$ of operators A and B is defined by the equality

$$(B\, A)(x) = B\,(Ax)$$

and the domain of this operator consists of all elements $x \in \mathcal{D}(A)$ such that $A\, x \in \mathcal{D}(B)$. It should be taken into account that the domains of the operators $A + B$ and $B\, A$ may coincide with the zero subspace even if the domains of the operators A and B are dense in the space X.

Let an operator A, whose domain is dense, be bounded. In this case there is a constant C such that $\|\, A\, x\,\| \le C\,\|\, x\,\|$ for all $x \in \mathcal{D}(A)$. Therefore, the operator A can uniquely be extended up to the operator $\bar{A}$, which is defined on the entire space X and belongs to the space $\mathcal{L}(X)$. The operator $\bar{A}$ so constructed is called a continuous extension of the operator A.

Furthermore, the concept of closed operator takes the central place in the theory of linear unbounded operators. An operator A is said to be closed if $x \in \mathcal{D}(A)$ and $y = A\, x$ for a sequence $x_n \in \mathcal{D}(A)$ such that $x_n \to x$ and $A\, x_n \to y$. As can readily be observed, the concept of closeness permits one to give the extended concept of continuity in a certain sense. It is worth noting here that each operator A from the space $\mathcal{L}(X)$ is closed. As such, it also will be useful to give an equivalent definition of the operator closeness which will be used in the sequel. With this aim, let us consider the Cartesian product $X \times X$. That space equipped with the norm

$$\|(x,\, y)\| = \|\, x\,\| + \|\, y\,\|$$

becomes a Banach space. A manifold $\Gamma(A)$ regards to the graph of an operator A if, we set, by definition,

$$\Gamma(A) \;=\; \Big\{\, (x,\, y) \in X \times X\colon\; x \in \mathcal{D}(A),\; y = A\, x \,\Big\}\,.$$

Then the operator A is closed if and only if the graph $\Gamma(A)$ is closed in the space $X \times X$.

The theory of linear unbounded operators is based on several well-known results given below as the closed graph theorem, the open mapping theorem and the Banach theorem on the inverse operator.

Theorem 5.1.12 (the **closed graph theorem**) *Let A be a linear operator in a Banach space X. If the operator A is closed and the domain $\mathcal{D}(A) = X$, then A is bounded.*

Theorem 5.1.13 (the **open mapping theorem**) *If $A \in \mathcal{L}(X, Y)$ and the range of the operator A is the entire Banach space X, then for any open subset of the Banach space X its image obtained by the operator A is open in the space Y.*

Theorem 5.1.14 (the **Banach theorem on inverse**) *If a one-to-one correspondence between Banach spaces X and Y was established by means of an operator A so that the range of the operator A coincides with the space Y, then the inverse $A^{-1} \in \mathcal{L}(Y, X)$.*

Some spectral properties of closed operators need investigation. Let A be a closed operator in a Banach space X. By definition, the resolvent set $\rho(A)$ of the operator A consists of all complex numbers λ such that $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. Recall that the operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is known as the **resolvent** of the operator A and establishes a one-to-one correspondence between the space X and the domain of the operator A. The complement of the resolvent set is called the **spectrum** of the operator A and denoted by $\sigma(A)$. It may happen that the resolvent set or spectrum of an unbounded operator is empty. Nevertheless, its resolvent set is open, while the spectrum is closed. If $\lambda \in \rho(A)$, then the resolvent set contains the entire circle with radius $\| R(\lambda, A)\|^{-1}$ and center λ. Moreover, for any point μ from that circle the series in (5.1.4) converges in the space $\mathcal{L}(X)$ to the resolvent $R(\mu, A)$. Note that for all λ, $\mu \in \rho(A)$ the resolvent identity

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)\, R(\lambda, A)\, R(\mu, A) \tag{5.1.21}$$

holds true.

In the case of an unbounded operator it is possible to split up its spectrum again into three parts. By definition, the point spectrum of an operator A is formed by all of its eigenvalues and is denoted by $\sigma_p(A)$. The remaining part of the spectrum can be divided into two nonintersecting parts: the continuous spectrum $\sigma_c(A)$ and the residual spectrum $\sigma_r(A)$, where the continuous spectrum $\sigma_c(A)$ consists of all points λ, for which the range of the operator $\lambda I - A$ is dense in the space X.

As we will see later, it will be sensible to introduce the notion of adjoint operator in the case of an unbounded operator. In preparation for this, let A be a linear operator, whose domain $\mathcal{D}(A)$ is dense in a Banach space X. The domain of the adjoint is composed by such elements $f \in X^*$

that there exists an element $f^* \in X^*$ satisfying for each $x \in \mathcal{D}(A)$ the relation

$$f(Ax) = f^*(x).$$

Since the domain $\mathcal{D}(A)$ is dense in the space X, the unknown element f^* can uniquely be found. Set, by definition,

$$A^* f = f^*.$$

The adjoint A^* will always be linear. Moreover, A^* is closed even if the operator A is not closed. It is plain to explain when the operator A has a closure by means of its own adjoint. An operator $\bar{A}$ is called the **closure** of the operator A if $\bar{A}$ is a minimal (in a sense of closing domains) closed operator being an extension of the operator A. The operator A has the closure if and only if its own adjoint A^* is densely defined in the above sense.

In the case of a Hilbert space X the spaces X and X^* will be identified as usual and the adjoint acts in the space X. If the domain of A^* is dense, then a new operator $A^{**} = (A^*)^*$ coincides with the closure $\bar{A}$ of the operator A. We say that the operator A is self-adjoint if $A^* = A$. Some things are worth noting here:

each self-adjoint operator is densely defined and closed;

the spectrum of any self-adjoint operator lies on the real line;

the residual spectrum of any self-adjoint operator is empty;

each self-adjoint operator is associated with a certain operator function $E(\lambda)$ defined for all $\lambda \in \mathbf{R}$ with values in the space $\mathcal{L}(X)$. This function known as the **spectral family** or the **spectral resolution** of unity is subject to the following conditions:

(1) $E^*(\lambda) = E(\lambda)$ *for each* $\lambda \in \mathbf{R}$*;*

(2) $E(\lambda)\, E(\mu) = E\big(\min(\lambda, \mu)\big)$ *for all* $\lambda, \mu \in \mathbf{R}$*;*

(3) $\lim\limits_{\mu \to \lambda+0} E(\mu)\, x = E(\lambda)\, x$ *for any* $\lambda \in \mathbf{R}$ *and each* $x \in X$*;*

(4) $\lim\limits_{\lambda \to -\infty} E(\lambda)\, x = 0$ *and* $\lim\limits_{\lambda \to +\infty} E(\lambda)\, x = x$ *and each* $x \in X$.

For any $x, y \in X$ the function $\sigma(\lambda) = \big(E(\lambda)\, x, y\big)$ being of bounded variation satisfies the relation

$$\int_{-\infty}^{\infty} d\big(E(\lambda)\, x, y\big) = (x, y),$$

where the integral on the right-hand side is of Stieltjes' type. The domain of the operator A consists of all elements $x \in X$, for which

$$\int_{-\infty}^{\infty} \lambda^2 \, d(E(\lambda)x, x) < \infty .$$

If $x \in \mathcal{D}(A)$, then for each $y \in X$

$$(5.1.22) \qquad (Ax, y) = \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)x, y) .$$

By formula (5.1.22) an alternative symbolic form of writing the operator A looks like this:

$$(5.1.23) \qquad A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) .$$

For each resolution of unity there exists a unique self-adjoint operator A satisfying (5.1.22). In turn, any self-adjoint operator A specifies uniquely the corresponding resolution of unity. Formula (5.1.23) is called the **spectral decomposition** of the operator A. It is plain to show that items (1) and (2) of the above definition of the function $E(\lambda)$ imply the existence of the strong one-sided limits of $E(\lambda)$ from the right and from the left for any λ. Therefore, item (3) has a sense of scaling and is needed for further support of a one-to-one correspondence between self-adjoint operators and resolutions of the identity. Let us stress that item (3) can be replaced by the condition for the function to be continuous from the left.

The function $E(\lambda)$ may be of help in deeper study of the spectrum of the operator A. The inclusion $\lambda_0 \in \rho(A)$ occurs if and only if the function $E(\lambda)$ is constant in some neighborhood of the point λ_0. A number λ_0 is one of the eigenvalues of the operator A if and only if λ_0 is a discontinuity point of the function $E(\lambda)$.

The next step is to touch upon the functions of one real variable with values in a Banach space. As known, this notion is much applicable in functional analysis. Being concerned with a function $f\colon [a, b] \mapsto X$, we call an element A of a Banach space X a limit of the function f as $t \to t_0$ and write this fact as

$$A = \lim_{t \to t_0} f(t)$$

if for any sequence $t_n \in [a, b]$, $t_n \neq t_0$, $t_n \to t_0$, the sequence $f(t_n)$ converges to A in the space X. The function f is continuous at a point $t_0 \in [a, b]$ if

$$\lim_{t \to t_0} f(t) = f(t) .$$

The function f is said to be continuous on the segment $[a, b]$ if it is continuous at each point of this segment. Any continuous function f on the segment $[a, b]$ is bounded thereon, that is, there exists a constant $M > 0$ such that $\| f(t)\| \leq M$ for all $t \in [a, b]$. The collection of all functions with values in a Banach space X that are continuous on the segment $[a, b]$ is denoted by $C([a, b]; X)$. The set $C([a, b]; X)$ with the usual operations of addition and multiplication by a number forms a vector space. Under the norm structure

$$\| f \|_{C([a,b]; X)} = \sup_{t \in [a, b]} \| f(t)\| \tag{5.1.24}$$

the space $C([a, b]; X)$ becomes a Banach space. In what follows it is always preassumed that the norm on the space $C([a, b]; X)$ is defined by means of relation (5.1.24).

Let A be a closed linear operator in a Banach space X. When the subspace $\mathcal{D}(A)$ is equipped with the graph norm

$$\| x \|_\Gamma = \| x \| + \| A x \| ,$$

we operate in a Banach space. Further treatment of $\mathcal{D}(A)$ as a Banach space necessitates imposing the graph norm on that subspace. In particular, just this norm is presupposed in the notation $C([a, b]; \mathcal{D}(A))$. The presence of a function f in the space $C([a, b]; \mathcal{D}(A))$ means that for each $t \in [a, b]$ the value $f(t)$ belongs to $\mathcal{D}(A)$ and f, $A f \in C([a, b]; X)$.

Riemann integral is defined as a limit in the space X of the corresponding Riemann sums

$$\int_a^b f(t)\, dt = \lim_{n \to \infty} \sum_{k=1}^{n} (t_k - t_{k-1})\, f(\xi_k) .$$

If the function f is continuous on the segment $[a, b]$, then it is integrable thereon and

$$\left\| \int_a^b f(t)\, dt \right\| \leq \int_a^b \| f(t)\|\, dt . \tag{5.1.25}$$

Having at our disposal $f \in C([a, b]; X)$ and $A \in \mathcal{L}(X)$ we arrive at the relation

$$A \int_a^b f(t)\, dt = \int_a^b A f(t)\, dt . \tag{5.1.26}$$

This property can be generalized to cover closed linear operators as well. Let an operator A be closed and linear. If $f \in C([a,b];\, \mathcal{D}(A))$, then

$$\int_a^b f(t)\, dt \in \mathcal{D}(A)$$

and

$$A \int_a^b f(t)\, dt = \int_a^b A\, f(t)\, dt\,.$$

We say that the function f has a derivative at a point t_0 if there exists the limit

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}\,,$$

which is to be understood as a limit in the space X. If the function f has a derivative at a point t_0, then it is continuous at this point. Each function, whose derivative is continuous on the segment $[a,\, b]$, is said to be continuously differentiable on the segment $[a,\, b]$. The set of all continuously differentiable on $[a,\, b]$ functions with values in the space X forms a vector space which is denoted by the symbol $C^1([a,b];\, X)$. This space equipped with the norm

$$\| f \|_{C^1([a,b];\, X)} = \| f \|_{C([a,b];\, X)} + \| f' \|_{C([a,b];\, X)} \tag{5.1.27}$$

becomes a Banach space.

By analogy, the symbol $C^k([a,b];\, X)$ is used for the space of all functions with values in the space X, whose derivatives of the first k order exist and are continuous on the segment $[a,\, b]$. The space $C^k([a,b];\, X)$ becomes a Banach space upon receipt of the norm

$$\| f \|_{C^k([a,b];\, X)} = \sum_{n=0}^{k} \| f^{(n)} \|_{C([a,b];\, X)}\,.$$

It is known that the Newton–Leibnitz formula

$$\int_a^b f'(t)\, dt = f(b) - f(a) \tag{5.1.28}$$

is valid for any function $f \in C^1([a,b];\, X)$. If the function f is differentiable, that is, has a derivative at any point of the segment $[a,\, b]$, then the mean value formula reduces to

$$\| f(b) - f(a) \| \le \sup_{t \in (a,\, b)} \| f'(t) \|\, (b - a)\,. \tag{5.1.29}$$

Let $\alpha \in (0, 1)$. We say that a function f with values in the space X satisfies on the segment $[a, b]$ Hölder's condition with exponent α if there exists a constant C such that

$$\| f(t) - f(s) \| \le C\,|t - s|^{\alpha} \tag{5.1.30}$$

for all $t, s \in [a, b]$. If the function f satisfies on the segment $[a, b]$ Hölder's condition with exponent α, then it is continuous thereon. The set of all functions satisfying on $[a, b]$ Hölder's condition with exponent α forms a vector space $C^{\alpha}([a, b]; X)$. The norm on that space is defined by

$$\| f \|_{C^{\alpha}([a,b];\, X)} = \sup_{t,\, s \in [a,\, b],\, t \neq s} \frac{\| f(t) - f(s) \|}{|t - s|^{\alpha}}\,,$$

thus causing a Banach space. Any function f with values in the space X falls within the category of Lipschitz functions if it satisfies relation (5.1.30) with exponent $\alpha = 1$. The set of all Lipschitz functions on the segment $[a, b]$ forms a vector space $\mathrm{Lip}([a, b]; X)$. The space $\mathrm{Lip}([a, b]; X)$ becomes a Banach space once equipped with the norm

$$\| f \|_{\mathrm{Lip}([a,b];\, X)} = \sup_{t,\, s \in [a,\, b],\, t \neq s} \frac{\| f(t) - f(s) \|}{|t - s|}\,.$$

Of special interest is the concept of strongly continuous operator function. Let us consider on the segment $[a, b]$ an operator function $A(t)$ with values in the space $\mathcal{L}(X, Y)$. The function $A(t)$ is said to be strongly continuous at a point $t_0 \in [a, b]$ if for each $x \in X$ the function $A(t)\,x$ is continuous in the norm of the space Y at this point t_0. We say that the function $A(t)$ is strongly continuous on the segment $[a, b]$ if it is strongly continuous at any point of this segment, that is, for each $x \in X$ the function $A(t)\,x$ is continuous on the segment $[a, b]$ in the norm of the space Y. Theorem 5.1.2 implies that if the function $A(t)$ is strongly continuous on the segment $[a, b]$, then it is bounded in the space $\mathcal{L}(X, Y)$. Therefore, there exists a constant $M > 0$ such that $\| A(t) \| \le M$ for all $t \in [a, b]$. For any strongly continuous function A the integral

$$\mathcal{A} = \int_a^b A(t)\, dt$$

is well-defined. In this line we set, by definition,

$$\mathcal{A}\, x = \int_a^b A(t)\, x\, dt$$

for each $x \in X$. The operator $\mathcal{A}$ so constructed is called the strong integral or the integral in a strong sense. If the function $A(t)$ with values in the space $\mathcal{L}(X,Y)$ is strongly continuous on the segment $[a, b]$ and the inclusion $f \in C([a,b]; X)$ occurs, then the function $A(t)\,f(t)$ is continuous on the segment $[a, b]$ in the norm of the space Y.

If the derivative of the function $A(t)\,x$ exists for each $x \in X$, then an operator function $A'(t)$ acting in accordance with the rule

$$A'(t)\,x = \big(A(t)\,x\big)'$$

is called the strong derivative of $A(t)$. Suppose now that the operator function $A(t)$ with values in the space X has a strong derivative $A'(t_0)$ at some point $t_0 \in [a, b]$. Due to Theorems 5.1.1–5.1.2 the derivative $A'(t_0)$ belongs to the space $\mathcal{L}(X,Y)$. The function $A(t)$ is said to be strongly continuously differentiable on the segment $[a, b]$ if the strong derivative of the function $A(t)$ exists for all $t \in [a, b]$ and is strongly continuous theron. If the function $A(t)$ is strongly continuously differentiable on the segment $[a, b]$, then it is strongly continuous thereon.

In subsequent chapters we shall need, among other things, the following assertions.

Theorem 5.1.15 *Let a function $A(t)$ with values in the space $\mathcal{L}(X,Y)$ be strongly continuously differentiable on the segment $[a, b]$ and*

$$f \in C^1([a,b]; X)\,.$$

Then the function $A(t)\,f(t) \in C^1([a, b]; Y)$ and

$$\big(A(t)\,f(t)\big)' = A'(t)\,f(t) + A(t)\,f'(t)\,. \tag{5.1.31}$$

Theorem 5.1.16 *If a function $A(t)$ with values in the space $\mathcal{L}(X,Y)$ be strongly continuously differentiable on the segment $[a, b]$ and for each $t \in [a, b]$ the inclusion $A^{-1}(t) \in \mathcal{L}(Y,X)$ occurs and the function $A^{-1}(t)$ is strongly continuous on the segment $[a, b]$, then the function $A^{-1}(t)$ is strongly continuously differentiable on the segment $[a, b]$ and*

$$\big(A^{-1}(t)\big)' = -A^{-1}(t)\,A'(t)\,A^{-1}(t)\,. \tag{5.1.32}$$

In this context, it is worth noting that Theorem 5.1.15 implies the formula of integrating by parts

$$\int_a^b A(t)\, f'(t)\, dt = A(b)\, f(b) - A(a)\, f(a) - \int_a^b A'(t)\, f(t)\, dt\,, \tag{5.1.33}$$

which is valid for any strong continuously differentiable on $[a, b]$ operator function $A(t)$ with values in the space $\mathcal{L}(X, Y)$ and any function $f \in C^1([a, b]; X)$.

We are now interested in learning more about Volterra integral equations. Let X be a Banach space. Consider a function $A(t, s)$ defined in the triangle

$$\Delta = \{ (t, s) \in \mathbf{R}^2 \colon\ a \le t \le b\,,\ a \le s \le t \} \tag{5.1.34}$$

with values in the space $\mathcal{L}(X)$. The equations

$$\int_a^t A(t, s)\, f(s)\, ds = g(t)\,, \qquad a \le t \le b\,, \tag{5.1.35}$$

and

$$f(t) - \int_a^t A(t, s)\, f(s)\, ds = g(t)\,, \qquad a \le t \le b\,, \tag{5.1.36}$$

with the operator kernel $A(t, s)$ are called the **Volterra integral equations** of the first and second kind, respectively. In what follows we restrict ourselves only to continuous solutions of these equations. That is to say, the function f is always sought in the class of functions $C([a, b]; X)$.

We first consider equation (5.1.36) of the second kind under the agreement that the kernel $A(t, s)$ is strongly continuous in the triangle Δ. This means that for each $x \in X$ the function $A(t, s)\, x$ is continuous on the triangle Δ in the norm of the space X. In other words, for any sequence $(t_n, s_n) \in \Delta$ converging to $(t, s) \in \Delta$ as $n \to \infty$ the sequence $A(t_n, s_n)\, x$ would converge to $A(t, s)\, x$ in the norm of the space X. It should be taken into account that the properties of strongly continuous functions of two variables are similar to those of one variable. In particular, if the function $A(t, s)$ is strongly continuous on the triangle Δ, then it is bounded thereon in the norm of the space $\mathcal{L}(X)$.

If the operator kernel $A(t, s)$ is strongly continuous on the triangle Δ, then for any function $f \in C([a, b]; X)$ the left part of equation (5.1.36) is continuous. Therefore, the requirement $g \in C([a,b];\, X)$ is necessary for a solution of equation (5.1.36) to exist in the class of continuous functions. It is plain to show that the same condition is sufficient, too. Approach to solving equation (5.1.36) in the case of operator kernels is similar to that in the scalar case. In the Banach space

$$\mathcal{X} = C([a,b];\, X) \tag{5.1.37}$$

we consider the integral operator

$$(\mathcal{A} f)(t) = \int_a^t A(t,s)\, f(s)\, ds\,. \tag{5.1.38}$$

The operator $\mathcal{A}$ is bounded in the space $\mathcal{X}$. However, unlike the scalar case, this operator may be noncompact. Because of (5.1.38), the integral equation (5.1.36) acquires the form of a second kind operator equation over the Banach space $\mathcal{X}$:

$$f - \mathcal{A} f = g\,. \tag{5.1.39}$$

Each power of the operator $\mathcal{A}$ can be written as

$$(\mathcal{A}^k f)(t) = \int_a^t A_k(t,s)\, f(s)\, ds$$

and regards again to an integral operator whose kernel $A_k(t,s)$ is strongly continuous. Each such kernel $A_k(t,s)$ can be defined by the recurrence relations

$$A_k(t,s) = \int_s^t A(t,\tau)\, A_{k-1}(\tau,s)\, d\tau\,, \qquad A_1(t,s) = A(t,s)\,.$$

These functions $A_k(t,s)$ are called **iterated kernels**.

Since the kernel $A(t,s)$ is strongly continuous, the value

$$M = \sup_{(t,s)\in\Delta} \| A(t,s)\|$$

is finite and the iterated kernels $A_k(t,s)$ satisfy the estimate

$$\| A_k(t,s)\| \le M^k\, (t-s)^{k-1} / (k-1)!\,,$$

yielding

$$\| \mathcal{A}^k \| \le M^k (b-a)^k / k! . \tag{5.1.40}$$

From inequality (5.1.40) we deduce that there exists some k such that $\| \mathcal{A}^k \| < 1$. So, according to Theorem 5.1.8 equation (5.1.36) has a solution for any function $g \in C([a,b]; X)$ and this solution is unique in the class of functions $C([a,b]; X)$. Moreover, one can show that the successive approximations

$$f_{n+1}(t) = \int_a^t A(t,s) f_n(s)\, ds + g(t) \tag{5.1.41}$$

converge in the space $\mathcal{X}$ to a solution of equation (5.1.36) for any initial data $f_0 \in \mathcal{X}$.

The series

$$B(t,s) = \sum_{k=1}^{\infty} A_k(t,s) \tag{5.1.42}$$

composed by the iterated kernels converges in the space $\mathcal{L}(X)$ uniformly with respect to $(t,s) \in \Delta$. The sum of the series in (5.1.42) will be a strongly continuous function with values in the space X and a solution to equation (5.1.36) can be written as

$$f(t) = g(t) + \int_a^t B(t,s)\, g(s)\, ds . \tag{5.1.43}$$

Estimate (5.1.40) and formula (5.1.5) indicate that the spectrum of the operator $\mathcal{A}$ consists of the single point $\lambda = 0$ only. Consider in the space $\mathcal{X}$ one more integral operator

$$(\mathcal{B} f)(t) = \int_a^t B(t,s) f(s)\, ds . \tag{5.1.44}$$

As can readily be observed, for each $\lambda \ne 0$ the resolvent of the operator $\mathcal{A}$ is defined with the aid of the relation

$$R(\lambda, \mathcal{A}) = \frac{1}{\lambda} I + \mathcal{B} . \tag{5.1.45}$$

The next object of investigation is equation (5.1.35) of the first kind under the natural premise that the kernel $A(t,s)$ of this equation is strongly continuous. Because of this, we are led to an operator equation of the first kind

$$\mathcal{A} f = g \tag{5.1.46}$$

in the Banach space $\mathcal{X}$ defined by (5.1.37). The operator $\mathcal{A}$ involved in (5.1.46) acts in accordance with rule (5.1.38) and is bounded. The number $\lambda = 0$ belongs to the spectrum of the operator $\mathcal{A}$, thus causing some difficulties in solving equation (5.1.46). It should be noted that the solution of equation (5.1.35) with an arbitrary strongly continuous kernel is one of the most difficult problems in functional analysis.

The situation becomes much more simpler if the operator kernel $A(t,s)$ is strongly continuously differentiable in t. Due to this property the left part of (5.1.35) is continuously differentiable for each continuous function f and

$$\Big(\int_a^t A(t,s)\, f(s)\, ds \Big)' = A(t,t)\, f(t) + \int_a^t A_t(t,s)\, f(s)\, ds\,, \tag{5.1.47}$$

where $A_t(t,s)$ denotes the strong derivative of the operator kernel $A(t,s)$ with respect to t. This fact implies that the conditions

$$g \in C^1([a,b];\, X)\,, \qquad g(a) = 0 \tag{5.1.48}$$

are necessary for equation (5.1.35) to be solvable in the class of continuous functions.

Assume that the function $A(t,t)^{-1}$ with values in the space $\mathcal{L}(X)$ is strongly continuous on the segment $[a,\, b]$. Therefore, with the aid of relations (5.1.47)–(5.1.48) equation (5.1.35) reduces to the operator equation of the second kind

$$f(t) + \int_a^t A(t,t)^{-1} A_t(t,s)\, f(s)\, ds = A(t,t)^{-1} g'(t)\,,$$

which is equivalent to (5.1.35). Thus, we have occasion to use the preceding results. Summarizing, we formulate the following assertions.

Theorem 5.1.17 *Let a function $A(t,s)$ with values in the space $\mathcal{L}(X)$ be strongly continuous on the specified triangle Δ and $g \in C\big([a,b];\, X\big)$. Then a solution to equation* (5.1.36) *exists and is unique in the class of functions $f \in C\big([a,b];\, X\big)$. Moreover, the successive approximations* (5.1.41) *converge in the norm of the space $C\big([a,b];\, X\big)$ to a solution of equation* (5.1.36) *for any initial data $f_0 \in C\big([a,b];\, X\big)$.*

Theorem 5.1.18 *Let a function $A(t, s)$ with values in the space $\mathcal{L}(X)$ be strongly continuous and strongly continuously differentiable with respect to t on the specified triangle Δ. If for each $t \in [a, b]$*

$$A(t,t)^{-1} \in \mathcal{L}(X),$$

the operator function $A(t,t)^{-1}$ is strongly continuous on the segment $[a, b]$ and the function g is in line with (5.1.48), then a solution to equation (5.1.35) exists, is unique in the class of functions $f \in C([a, b]; X)$ and the successive approximations

$$f_{n+1}(t) = -\int_a^t A(t,t)^{-1} A_t(t,s) f_n(s)\, ds + A(t,t)^{-1} g'(t)$$

converge in the norm of the space $C([a, b]; X)$ to a solution of equation (5.1.35) for any initial data $f_0 \in C([a, b]; X)$.

The results concerning the solvability of the linear equation (5.1.36) can be generalized to cover the nonlinear Volterra equation

$$f(t) - \int_a^t A(t,s) F(s, f(s))\, ds = g(t), \qquad a \le t \le b. \tag{5.1.49}$$

However, we have to realize that equation (5.1.49) has some peculiarities. In particular, for each solution of equation (5.1.49) the point $(t, f(t))$ should belong to the domain of the function F for all $t \in [a, b]$. Moreover, even if the function F is defined on the entire manifold $[a, b] \times X$ equation (5.1.49) may not possess a continuous solution defined on the whole segment $[a, b]$. Therefore, one of the principal issues related to equation (5.1.49) is concerned with its local solvability. It is necessary to establish some conditions under which the existence of a continuous solution to equation (5.1.49) is ensured on the segment $[a, a+h]$, $h > 0$, if a number h is small enough. Denote by $\bar{S}(y, R) = \{ x \in X\colon \| x - y \| \le R \}$ a closed ball of radius R with center y.

The following assertions are true.

Theorem 5.1.19 *Let a function $A(t, s)$ with values in the space $\mathcal{L}(X)$ by strongly continuous on the specified triangle Δ and $g \in C([a, b]; X)$. If there is a number $R > 0$ such that the function F is continuous on the manifold $U = [a, b] \times \bar{S}(g(a), R)$ and satisfies thereon the Lipschitz* *condition with respect to the second variable, that is, there is a constant* $L > 0$ *such that for all* (t, u), $(t, v) \in U$

$$\| F(t, u) - F(t, v)\| \leq L \, \| u - v \| ,$$

then there exist a number $h > 0$ *and a function* $f \in C\big([a, a+h];\, X\big)$ *such that* f *solves equation* (5.1.49) *on the segment* $[a,\, a+h]$. *Moreover, if two functions* f_1 *and* f_2 *give solutions to equation* (5.1.49) *on a segment* $[a,\, c]$, *then they coincide on this segment. Furthermore, if a continuous function satisfies equation* (5.1.49) *on a segment* $[a,\, d]$ *and cannot be extended up to a continuous solution to equation* (5.1.49) *on any segment* $[a,\, e]$ *such that* $[a,\, d] \subset [a,\, e]$, *then the point* $\big(d,\, f(d)\big)$ *lies on the boundary of the specified set* U.

A study of properties of solutions of abstract differential equations is based on the notion of distribution with values in a Banach space. Let Ω be an open set on the real line $\mathbf{R}$ and $D(\Omega)$ denote the collection of all infinite differentiable functions whose supports are contained in Ω. Recall that the support of any continuous function is defined as the closure of the set of all points at which this function does not equal zero. With regard to convergence on $D(\Omega)$ we say that a sequence $\{ \varphi_n \}$ converges to a function φ if the following conditions hold:

(1) *there exists a compact set* $K \subset \Omega$ *such that it contains the supports of all functions* φ_n;

(2) *for each* k *the sequence of the* k*th derivatives* $\varphi_n^{(k)}$ *converges as* $n \to \infty$ *to the* k*th derivative of the function* φ *uniformly in* k.

A linear operator f acting from $D(\Omega)$ into a Banach space X is called a **distribution** with values in the space X if this operator is continuous in the sense that $f(\varphi_n) \to f(\varphi)$ as $n \to \infty$ in the space X for any sequence $\varphi_n \to \varphi$ in $D(\Omega)$. The set of all distributions with values in the space X will be denoted by $D'(\Omega;\, X)$. Any function $f \in C\big([a, b];\, X\big)$ can be put in correspondence with a linear operator from $D((a, b))$ into X

$$\hat{f}(\varphi) = \int_a^b \varphi(t)\, f(t)\; dt\,, \tag{5.1.50}$$

which falls within the distributions with values in the space X. Formula (5.1.50) could be useful in specifying a distribution even if the function f is not continuous. A distribution is said to be regular if we can attempt it in the form (5.1.50). Note that for any regular distribution $\hat{f}$ the function f involved in formula (5.1.50) can uniquely be recovered up to its values on a set of zero measure. Just for this reason the distribution $\hat{f}$ and the function f related to each other by (5.1.50) will be identified in the sequel. Under the approved identification the symbol $f(t)$ is in common usage to denote formally any distribution f (even if it is not regular), where t refers to the argument of test functions φ, which constitute the domain $\mathcal{D}(\Omega)$ of the operator f.

We give below one possible example of singular distributions in which the well-known Dirac delta function $\delta(t-t_0)\,x$ is specified by a point $t_0 \in \Omega$ and an element $x \in X$. By definition, the aforementioned function does follow the governing rule $\delta(t-t_0)\,x(\varphi) = \varphi(t_0)\,x$.

Every distribution $f \in \mathcal{D}'(\Omega; X)$ and its generalized derivative $f' \in \mathcal{D}'(\Omega; X)$ are related by $f'(\varphi) = -f(\varphi')$. If a distribution f is regular and continuously differentiable, then the above definition is consistent with the definition of standard derivative.

We now consider two Banach spaces X and X_1 equipped with norms $\|\cdot\|$ and $\|\cdot\|_1$, respectively. The space X_1 is supposed to be continuously embedded into the space X; meaning $X_1 \subset X$ and that there is a constant $c > 0$ such that the inequality

$$\|x\| \leq c\,\|x\|_1$$

holds for any $x \in X_1$. In each such case any distribution with values in the space X_1 falls within the category of distributions with values in the space X. In dealing with an operator $A \in \mathcal{L}(X, Y)$ related to the Banach spaces X and Y, the operator Af is a distribution with values in the space Y for every distribution f with values in the space X.

Let X be a Banach space and A be a linear closed operator in X. When X is equipped with the norm

$$\|x\|_\Gamma = \|x\| + \|Ax\|,$$

the domain $\mathcal{D}(A)$ becomes a Banach space embedded continuously into the space X. In so doing,

$$A \in \mathcal{L}(\mathcal{D}(A); X),$$

which justifies that for every distribution f with values in $\mathcal{D}(A)$ the operator Af is a distribution with values in X. Under such an approach the generalized derivative f' is treated as a distribution with values in $\mathcal{D}(A)$ and a distribution with values in X simultaneously.

Being concerned with a sequence of elements $c_n \in X$, we begin by investigating the power series

$$f(t) = \sum_{n=0}^{\infty} c_n\,(t-t_0)^n, \tag{5.1.51}$$

where t and t_0 are real or complex numbers. By definition, the domain of convergence of the series in (5.1.51) contains all of those points t at which the series in (5.1.51) is convergent, and will be denoted by the symbol D. The value

$$R = \sup_{t \in D} \| t - t_0 \|$$

is referred to as the **radius of convergence** of the series in (5.1.51). When the space X is considered over the field of complex numbers, the power series in (5.1.51) is meaningful for real and complex values of t with a common radius of convergence. Also, formula (5.1.51) gives one natural way of extending the function f from the real axis onto the complex plane. By the Cauchy–Hadamard formula,

$$R = \frac{1}{\overline{\lim_{n \to \infty}} \sqrt{\| c_n \|}} . \tag{5.1.52}$$

If the radius of convergence of the series in (5.1.51) differs from zero, then the sum f is infinite differentiable in the open circle (or interval) $| t - t_0 | < R$, so that

$$f^{(k)}(t) = \sum_{n=k}^{\infty} n\,(n-1) \cdots (n-k+1)\, c_n\, (t - t_0)^{n-k} .$$

In particular, this yields

$$c_n = \frac{1}{n\,!}\, f^{(n)}(t_0) .$$

We say that the function f is **analytic** at point t_0 if there is a neighborhood of this point within which f coincides with the sum of the power series from (5.1.51) with a nonzero radius of convergence. By definition, the function f is analytic on a set Ω if it is analytic at every point of this set.

We have occasion to use an operator $A \in \mathcal{L}(X)$, by means of which the exponential function is defined to be

$$e^{A\,t} = \sum_{n=0}^{\infty} \frac{A^n}{n!}\, t^n . \tag{5.1.53}$$

Observe that the series in (5.1.53) is convergent for every t regardless of the choice of a complex or a real value. For the exponential function we thus have

$$e^{A\,t}\, e^{A\,s} = e^{A\,t+s} ,$$

which serves to motivate the following expansion in power series:

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n e^{At_0}}{n!} (t-t_0)^n \tag{5.1.54}$$

with any t_0 incorporated. Since the series in (5.1.54) converges for any t, the exponent function is analytic everywhere in the complex plane.

In concluding this section we refer to a closed linear operator A acting in a Banach space. If the resolvent set of the operator A is nonempty, the resolvent of the operator A

$$R(t, A) = \big(t\,I - A\big)^{-1}$$

arranges itself into a power series

$$R(t, A) = \sum_{n=0}^{\infty} (-1)^n R(t_0, A)^{n+1} (t-t_0)^n , \tag{5.1.55}$$

whose radius of convergence is no less than $\| R(t_0, A)\|^{-1}$. Because of this fact, the resolvent of any closed linear operator is analytic on the corresponding resolvent set.

5.2 Linear differential equations of the first order in Banach spaces

This section is devoted to abstract differential equations of the first order. There is plenty of fine books in this theory such as Arendt et al. (1986), Babin and Vishik (1989), Balakrishnan (1976), Belleni-Morante (1979), Clément et al. (1987), Davies (1980), Fattorini (1969a,b, 1983), Gajewski et al. (1974), Goldstein (1969, 1985), Henry (1981), Hille and Phillips (1957), Ivanov et al. (1995), Kato (1953, 1956, 1961, 1966), Krein (1967), Krein and Khazan (1983), Ladas and Laksmihanthan (1972), Lions (1957, 1961), Mizohata (1977), Pazy (1983), Solomyak (1960), Sova (1977), Tanabe (1960, 1979), Trenogin (1980), Vishik and Ladyzhenskaya (1956), Yosida (1956, 1963, 1965).

We now consider in a Banach space X a closed linear operator A, whose domain $\mathcal{D}(A)$ is dense. With regard to an element $u_0 \in X$ and a function f: $[0, T] \mapsto X$ the object of investigation is the **abstract Cauchy problem**

$$u'(t) = A\,u(t) + f(t)\,, \qquad 0 < t < T\,, \tag{5.2.1}$$

$$u(0) = u_0\,. \tag{5.2.2}$$

There seem to be at least two principal approaches to the concept of the Cauchy problem (5.2.1)–(5.2.2) solution. This is concerned with notions of strong and weak solutions. One assumes that f is a **distribution** with values in the space X. A distribution u with values in the space X is said to be a **weak solution** of the Cauchy problem (5.2.1)–(5.2.2) if it satisfies equation (5.2.1) in the sense of the equality between elements of the space $\mathcal{D}'((0,T);\, X)$. It is required, in addition, that u is a **regular function** belonging to the space $\mathcal{C}([0,\, T];\, X)$ and satisfies the initial condition (5.2.2) as a continuous function with values in the space X. It is worth noting here that a weak solution of the Cauchy problem (5.2.1)–(5.2.2) as an element only of the space $\mathcal{C}([0,\, T];\, X)$ does not necessarily possess a standard derivative or take on the values from $\mathcal{D}(A)$.

A function $u \in \mathcal{C}^1([0,\, T];\, X) \cap \mathcal{C}([0,\, T];\, \mathcal{D}(A))$ is said to be a **strong solution** of the Cauchy problem (5.2.1)–(5.2.2) if it satisfies in a pointwise manner equation (5.2.1) supplied by the initial condition $u(0) = u_0$. When u is treated as a distribution, each strong solution of the Cauchy problem at hand turns out to be a weak solution of the same problem. A necessary condition for a strong solution of the Cauchy problem (5.2.1)–(5.2.2) to exist amounts to the two inclusions

$$u_0 \in \mathcal{D}(A)\,, \qquad f \in \mathcal{C}([0,\, T];\, X)\,.$$

However, in the general case these conditions are insufficient. In what follows we deal mainly with strong solutions. Therefore, the term "solution" will be meaningful for strong solutions unless otherwise is explicitly stated.

In the sequel especial attention is being paid to necessary and sufficient conditions under which one can determine a unique solution of the Cauchy problem at hand and find out when this solution depends continuously on the input data. In preparation for this, the abstract Cauchy problem for the **homogeneous equation** comes first:

$$u'(t) = A\,u(t)\,, \qquad 0 < t < T\,, \tag{5.2.3}$$

$$u(0) = u_0\,. \tag{5.2.4}$$

We say that the Cauchy problem (5.2.3)–(5.2.4) is **uniformly well-posed** if the following conditions hold:

(1) *there is a dense subspace D of the space X such that for any $u_0 \in D$ a strong solution of the Cauchy problem* (5.2.3)–(5.2.4) *exists and is unique;*

(2) *if a sequence of strong solutions $u_n(t)$ to equation* (5.2.3) *is such that $u_n(0) \to 0$ as $n \to \infty$ in the norm of the space X, then $u_n(t) \to 0$ as $n \to \infty$ in the norm of the space $\mathcal{C}([0,\, T];\, X)$.*

Necessary and sufficient conditions for the Cauchy problem concerned to be uniformly well-posed admit several alternative forms. The **Hille–Phillips–Yosida–Miyadera theorem** is in common usage for these conditions. One of the possible statements is as follows.

Theorem 5.2.1 *The Cauchy problem* (5.2.3)-(5.2.4) *is uniformly well-posed if and only if the resolvent set of the operator A contains a ray $\lambda > \omega$ of the real axis and the resolvent*

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

of the operator A satisfies for all $\lambda > \omega$ the system of inequalities

$$\| R(\lambda, A)^k \| \le \frac{M}{(\lambda - \omega)^k}, \qquad k = 1, 2, 3, \ldots . \tag{5.2.5}$$

Quite often one can encounter the situations in which the resolvent $R(\lambda, A)$ obeys for $\lambda > \omega$ the estimate

$$\| R(\lambda, A) \| \le \frac{1}{\lambda - \omega}, \tag{5.2.6}$$

which is sufficient for the validity of estimate (5.2.5) with $M = 1$ for all numbers k.

With regard to solving the abstract Cauchy problem (5.2.3)–(5.2.4) a key role is played by the notion of strongly continuous semigroup of the linear operator. A family of linear operators $V(t)$ belonging to $\mathcal{L}(X)$ and defined for all $t \ge 0$ is called a **strongly continuous semigroup** if the following conditions are satisfied:

(1) $V(0) = I$;

(2) $V(t+s) = V(t)\, V(s)$ *for all* $t,\ s \ge 0$;

(3) *for each $x \in X$ the function $V(t)\,x$ is continuous with respect to t in the norm of the space X for all $t \ge 0$.*

As we will see later, any solution of the abstract Cauchy problem (5.2.3)–(5.2.4) can be expressed in terms of strongly continuous semigroups. For any uniformly well-posed Cauchy problem of the type (5.2.3)–(5.2.4) there exists a strongly continuous semigroup $V(t)$ such that we are led by the formula

$$u(t) = V(t)\, u_0 \tag{5.2.7}$$

to either a weak solution (if $u_0 \in X$) or a strong solution (if $u_0 \in \mathcal{D}(A)$) of problem (5.2.3)–(5.2.4). Observe that for any $t \geq 0$ each operator of the semigroup $V(t)$ thus obtained and the operator A are commuting. Furthermore, the semigroup $V(t)$ itself is uniquely defined by the operator A. The converse is certainly true. That is to say, for any strongly continuous semigroup $V(t)$ there exists a closed linear operator known as the **semigroup generator** with a dense domain such that the related Cauchy problem (5.2.3)–(5.2.4) is uniformly well-posed and its solution is given by formula (5.2.7). What is more, the generator A can uniquely be recovered.

For the purposes of the present chapter we shall need as yet several useful formulae expressing the semigroup $V(t)$ via its generator A and vice versa. The operator and its semigroup $V(t)$ are involved in the following relationships:

$$\mathcal{D}(A) = \left\{ x \in X : \exists \lim_{t \to 0} \frac{V(t)\,x - x}{t} \right\},$$

$$A\,x = \lim_{t \to 0} \frac{V(t)\,x - x}{t}\,.$$

The semigroup $V(t)$ and its generator A are related by

$$V(t)\,x = \lim_{\lambda \to +\infty} \exp\left\{ \left(\lambda^2 R(\lambda,\, A) - \lambda\, I \right) t \right\} x\,.$$

For any strongly continuous semigroup $V(t)$ there are two constants M and ω such that for all $t \geq 0$ the estimate

$$\| V(t) \| \leq M \exp(\omega\, t) \tag{5.2.8}$$

is valid. Since $V(0) = I$, the bound $M \geq 1$ is attained. However, it may happen that the constant ω is negative. This provides support for the view that the semigroup $V(t)$ is exponentially decreasing. There is a direct link between the semigroup $V(t)$ and the **resolvent** $R(\lambda, A)$ of its generator A. True, it is to be shown that for each $\lambda > \omega$ the relation

$$R(\lambda, A) = \int_0^\infty V(t)\, e^{-\lambda\, t}\, dt$$

holds. Here the integral is meant in a strong sense.

Formula (5.2.7) serves as a basis for resolving the uniformly well-posed Cauchy problem at hand. A weak solution exists for each element $u_0 \in X$. The condition $u_0 \in \mathcal{D}(A)$ is necessary and sufficient for a strong solution to exist. It is not difficult to show that for the uniformly well-posed Cauchy problem a weak solution is unique. Then so is a strong solution.

One thing is worth noting here. When imposing the definition of the Cauchy problem (5.2.3)–(5.2.4) to be uniformly well-posed, it is presupposed that its solution is defined on the segment $[0, T]$ only. Nevertheless, formula (5.2.7) may be of help in giving a unique solution on the entire semi-axis. This fact is stipulated by the **autonomy** of equation (5.2.3), meaning the independence of the operator A on the variable t. If the Cauchy problem (5.2.3)–(5.2.4) is uniformly well-posed, then the same property will be true on any segment $[a, b]$ for another problem

$$u'(t) = A\,u(t)\,, \qquad a < t < b\,,$$

$$u(a) = u_0\,.$$

Furthermore, the function

$$u(t) = V(t-a)\,u_0$$

solves the Cauchy problem posed above. The solvability of the nonhomogeneous Cauchy problem (5.2.1)–(5.2.2) will be of special investigations in the sequel. This type of situation is covered by the following result.

Theorem 5.2.2 *Let the Cauchy problem* (5.2.3)–(5.2.4) *be uniformly well-posed. One assumes, in addition, that*

$$u_0 \in X \qquad \textit{and} \qquad f \in C([0, T]; X)\,.$$

Then a weak solution u of the Cauchy problem (5.2.1)–(5.2.2) *exists and is unique. Also, this solution is representable by*

$$u(t) = V(t)\,u_0 + \int_0^t V(t-s)\,f(s)\,ds\,, \tag{5.2.9}$$

where $V(t)$ refers to the semigroup generated by the operator A.

One might expect that formula (5.2.9) should specify a strong solution too, because any strong solution is, at the same time, a weak one. Indeed, as we have established above, the inclusions $u_0 \in \mathcal{D}(A)$ and $f \in C([0, T]; X)$ are necessary for a strong solution to exist. It is interesting to derive sufficient conditions acceptable for this case. Recall that for $u_0 \in \mathcal{D}(A)$ formula (5.2.7) gives a strong solution to the homogeneous equation (5.2.3). To decide for yourself whether or not a strong solution to equation (5.2.1) exists, a first step is to check the continuous differentiability of the integral term on the right-hand side of formula (5.2.9). On the other hand, there are examples showing that the unique restriction on the function f to be continuous would not be sufficient in that case. The following result confirms this statement and allows to formulate some conditions under which one can find a unique strong solution to a **nonhomogeneous equation**.

Theorem 5.2.3 *Let the Cauchy problem* (5.2.3)–(5.2.4) *be uniformly well-posed. One assumes, in addition, that* $u_0 \in \mathcal{D}(A)$ *and*

$$f \in C^1([0, T]; X) + C([0, T]; \mathcal{D}(A)).$$

Then a strong solution of the Cauchy problem (5.2.1)–(5.2.2) *exists and is unique. Moreover, this solution* u *is given by formula* (5.2.9).

We note in passing that for a strong solution $u(t)$ of the Cauchy problem (5.2.1)–(5.2.2) the functions $u'(t)$ and $A\,u(t)$ are continuous. This property may be of help in deriving their simple expressions in terms of the input data. Via the **decomposition** $f = f_1 + f_2$, where $f_1 \in C^1([0, T]; X)$ and $f_2 \in C([0, T]; \mathcal{D}(A))$, we finally get

$$u'(t) = V(t)\,\big[A\,u_0 + f_1(0)\big] + \int_0^t V(t-s)\,\big[f_1'(s) + A\,f_2(s)\big]\,ds + f_2(t) \tag{5.2.10}$$

and

$$A\,u(t) = V(t)\,\big[A\,u_0 + f_1(0)\big] + \int_0^t V(t-s)\,\big[f_1'(s) + A\,f_2(s)\big]\,ds - f_1(t). \tag{5.2.11}$$

Let us stress that representations (5.2.10)–(5.2.11) permit us to improve the **solution smoothness**. Let $f \in C^1([0, T]; X)$. By merely inserting $v = u'$ we rewrite (5.2.10) as

$$v(t) = V(t)\,v_0 + \int_0^t V(t-s)\,g(s)\,ds,$$

where $v_0 = A\,u_0 + f(0)$ and $g(t) = f'(t)$. The resulting expression shows that the function v gives a weak solution of the Cauchy problem

$$\begin{cases} v'(t) = A\,v(t) + g(t), & 0 \le t \le T, \\ v(0) = v_0 . \end{cases} \tag{5.2.12}$$

Applying the condition of the existence of a strong solution to the Cauchy problem (5.2.12) yields a test for a solution of the original Cauchy problem (5.2.1)–(5.2.2) to be twice continuously differentiable.

Theorem 5.2.4 *Let the Cauchy problem* (5.2.3)–(5.2.4) *be uniformly well-posed. One assumes, in addition, that the inclusions*

$$u_0 \in \mathcal{D}(A)\,, \qquad\qquad f \in C^2([0,\,T];\,X)$$

and $A\,u_0 + f(0) \in \mathcal{D}(A)$ *occur. Then a solution u of the Cauchy problem* (5.2.1)–(5.2.2) *satisfies the condition*

$$u \in C^2([0,\,T];\,X)\,.$$

It is worth noting here that following this procedure in just the same way as we did before, it is possible to achieve the desired smoothness of the solution. Such an approach may be of help in establishing some conditions known as the **"spatial" smoothness**. This terminology reflects a link between abstract differential equations and partial differential equations and is used to indicate when the solution obtained belongs to the domain for some power of the operator A. Let $f \in C\big([0,\,T];\,\mathcal{D}(A)\big)$. The meaning of this is that both functions f and $A\,f$ are continuous in the norm of the space X. With regard to the function $w(t) = A\,u(t)$ we recast (5.2.11) with regard to $w_0 = A\,u_0$ and $h(t) = A\,f(t)$ as

$$w(t) \;=\; V(t)\,w_0 + \int\limits_0^t V(t-s)\,h(s)\;ds\,,$$

which means that the function w is a weak solution of the Cauchy problem

$$\begin{cases} w'(t) = A\,w(t) + h(t)\,, & 0 \le t \le T\,, \\ w(0) = w_0\,. \end{cases} \tag{5.2.13}$$

As far as the functions $w_0 \in \mathcal{D}(A)$ and $h \in C\big([0,\,T];\,\mathcal{D}(A)\big)$ are concerned, the function w becomes a strong solution of the Cauchy problem (5.2.13), thereby justifying the following assertion.

Theorem 5.2.5 *Let the Cauchy problem* (5.2.3)–(5.2.4) *be uniformly well-posed and let*

$$u_0 \in \mathcal{D}(A^2)\,, \qquad\qquad f \in C\big([0,\,T];\,\mathcal{D}(A^2)\big)\,.$$

Then a solution u of the Cauchy problem (5.2.1)–(5.2.2) *satisfies the condition*

$$u \in C\big([0,\,T];\,\mathcal{D}(A^2)\big)\,.$$

The same procedure leads to the gain of the "spatial" smoothness of a solution as high as necessary. Combination of both directions provides proper guidelines for the derivation of conditions of the so-called "mixed" smoothness.

The **nonlinear Cauchy problem** helps motivate what is done and is completely posed as follows:

$$u'(t) = A\,u(t) + f\big(t,\, u(t)\big)\,, \qquad 0 \le t \le T\,, \tag{5.2.14}$$

$$u(0) = u_0\,. \tag{5.2.15}$$

A peculiarity of this problem is connected with the obstacle that, as a rule, its solution will not be defined on the whole segment $[0,\, T]$. The interval within which a solution exists depends on the operator A, the function f and the initial element u_0. In this regard, some modification of the concept of solution is needed to require the existence of a real number $T^* > 0$ such that a solution exists on the segment $[0,\, T^*]$ only.

When a generalized solution is considered in this context, one more problem arises naturally.

Since equation (5.2.14) contains the **substitution operator**, the properties of the function f need investigation. Let $\bar{S}(y, R)$ be a closed ball in the space X of radius $R > 0$ with center y and

$$U = [0,\, T] \times \bar{S}(u_0,\, R)\,.$$

We confine ourselves to the classical case only. For the reader's convenience this restriction is labeled by (5.2.16). Summarizing,

the function f is continuous on the set U with respect to the totality of variables and satisfies thereon the Lipschitz condition with respect to the second variable, thereby providing that there is a constant $L > 0$ such that for all (t, u), $(t, v) \in U$

$$\|\, f(t, u) - f(t, v)\| \le L\, \|\, u - v\,\|\,. \tag{5.2.16}$$

There are several ingredients necessary for further successful development. Denote by u an arbitrary function from the class of continuous on $[0,\, T^*]$ functions. We assume also that the graph of u belongs to the set U. As far as the function f is continuous on the set U, the superposition $f(t,\, u(t))$ is continuous on the segment $[0,\, T^*]$. All this enables us to carry over without any difficulty the concepts of weak and strong solutions to the Cauchy problem (5.2.14)–(5.2.15) with continuous nonlinearity.

The first stage is devoted to weak solutions of the nonlinear Cauchy problem under the agreement that the homogeneous Cauchy problem (5.2.3)–(5.2.4) is uniformly well-posed. By appeal to formula (5.2.9) it is not difficult to show that the Cauchy problem (5.2.3)–(5.2.4) is equivalent to the integral equation

$$u(t) = V(t)\, u_0 + \int_0^t V(t-s)\, f(s,\, u(s))\; ds\,, \tag{5.2.17}$$

where $V(t)$ refers to the semigroup generated by the operator A. By applying Theorem 5.1.19 to equation (5.2.17) we obtain the following result.

Theorem 5.2.6 *Let the Cauchy problem* (5.2.3)–(5.2.4) *be uniformly well-posed, $u_0 \in X$ and the function f comply with condition* (5.2.16). *Then there exists a number $T^* > 0$ such that the nonlinear Cauchy problem* (5.2.14)–(5.2.15) *has a weak solution u on the segment* $[0, T^*]$. *Also, if there is no way of extending this solution on a more wide segment, then the point $\big(T^*,\, u(T^*)\big)$ belongs to the boundary of the set U. Moreover, any two weak solutions of problem* (5.2.14)–(5.2.15) *will coincide on a common part of the segments of their existence.*

The question of existence of a strong solution of the Cauchy problem (5.2.14)–(5.2.15) amounts to the problem of differentiability of a solution to the integral equation (5.2.17). With this in mind, we may assume that the function f is Frechet differentiable. By definition, the function $f(t, u)$ is said to be **Frechet differentiable at point** $(t_0,\, u_0)$ if the increment of the function f can be split up as follows:

$$f(t,u) - f(t_0,u_0) \;=\; f_t\,(t-t_0) + f_u\,(u-u_0) + \alpha(t,u)\,\big(\,|\,t-t_0| + \|\,u-u_0\|\big)\,,$$

where $f_t \in X$, $f_u \in \mathcal{L}(X)$ and $\alpha(t,u) \to 0$ as $(t,u) \to (t_0,\, u_0)$. In this case the element f_t and the operator f_u are known as the **partial derivatives** of the function f at the point $(t_0,\, u_0)$. We say that the function f is **Frechet differentiable on the set** U if it is differentiable at each point of this set. The restrictions on the partial derivatives are imposed in just the same way as we did in the consideration of the functions themselves and are labeled by (5.2.18) for the reader's convenience. When done with this sense of purpose,

(5.2.18) *the function f is Frechet differentiable on the set U; its partial derivatives f_t and f_u are continuous on this set and satisfy thereon U the Lipschitz condition with respect to the second argument.*

If the function f is Frechet differentiable and the function $u(t)$ is differentiable in t, then the superposition $f(t,\, u(t))$ is differentiable in t, so that

$$\frac{d}{dt}\, f(t,\, u(t)) \;=\; f_t + f_u\, u' .$$

Exploiting this fact and allowing $u_0 \in \mathcal{D}(A)$, one can formally differentiate equation (5.2.17) with the aid of formula (5.2.10). As a final result we get the equation related to the function $v = u'$:

$$(5.2.19) \qquad v(t) \;=\; V(t)\, v_0 + \int\limits_0^t \dot{V}(t-s)\, f_t(s,\, u(s))\, ds$$

$$+ \int\limits_0^t V(t-s)\, f_u(s,\, u(s))\, v(s)\; ds\,,$$

where $v_0 \;=\; A\, u_0 + f(0,\, u_0)$. When the function u is kept fixed, equation (5.2.19) may be viewed as the linear Volterra integral equation of the second kind for the function v. Moreover, the segment on which equation (5.2.19) has a continuous solution coincides with the same segment for equation (5.2.17). In this case one succeeds in showing without any difficulty that the solution to equation (5.2.19) is just the derivative of the solution to equation (5.2.17), thereby revealing the local solvability of the Cauchy problem (5.2.14)–(5.2.15) in the class of strong solutions. Thus, we arrive at the following result.

Theorem 5.2.7 *Let the Cauchy problem* (5.2.3)–(5.2.4) *be uniformly well-posed, $u_0 \in \mathcal{D}(A)$ and the function f comply with conditions* (5.2.16) *and* (5.2.18). *Then there exists a number $T^* > 0$ such that the nonlinear Cauchy problem* (5.2.14)–(5.2.15) *has a strong solution u on the segment $[0,\, T^*]$. Also, if there is no way of extending this solution on a more wide segment, then the point $\big(T^*,\, u(T^*)\big)$ belongs to the boundary of the set U. Moreover, any two strong solutions of problem* (5.2.14)–(5.2.15) *will coincide on a common part of the segments of their existence.*

Our next step is to consider on the semi-axis $t \geq 0$ the Cauchy problem for the nonhomogeneous equation

$$(5.2.20) \qquad u'(t) \;=\; A\, u(t) + f(t)\,, \qquad t > 0\,,$$

$$(5.2.21) \qquad u(0) \;=\; u_0\,,$$

and the related Cauchy problem for the homogeneous equation

$$u'(t) = A\,u(t)\,, \qquad t > 0\,, \tag{5.2.22}$$

$$u(0) = u_0\,, \tag{5.2.23}$$

A continuous function u defined for all $t \geq 0$ is said to be a **weak solution** of problem (5.2.20)–(5.2.21) if for any $T > 0$ this function gives a weak solution of problem (5.2.1)–(5.2.2). We say that a function is a **strong solution** of problem (5.2.20)–(5.2.21) if this function is continuously differentiable on the semi-axis $t \geq 0$ and for any $T > 0$ satisfies problem (5.2.1)–(5.2.2) as one possible strong solution. The homogeneous Cauchy problem (5.2.22)–(5.2.23) is said to be **uniformly well-posed** if the Cauchy problem (5.2.3)–(5.2.4) is uniformly well-posed for any $T > 0$. Recall that the well-posedness of problem (5.2.3)–(5.2.4) will be ensured for all $T > 0$ if we succeed in showing this property at least for one value $T > 0$. Thus, in agreement with Theorem 5.2.1 the Cauchy problem (5.2.22)–(5.2.23) is uniformly well-posed if and only if the operator A generates a strongly continuous semigroup $V(t)$.

Suppose that for every $u_0 \in X$ the function f is continuous on the semi-axis $t \geq 0$. If problem (5.2.22)–(5.2.23) is uniformly well-posed, then a weak solution u of the Cauchy problem (5.2.20)–(5.2.21) exists, is unique and is given by formula (5.2.9). Under the additional assumptions that $u_0 \in \mathcal{D}(A)$ and f is a sum of two functions: the first is continuously differentiable in the norm of the space X for $t \geq 0$ and the second is continuous in the norm of $\mathcal{D}(A)$ for $t \geq 0$, a strong solution of problem (5.2.20)–(5.2.21) is specified by formula (5.2.9). We will not pursue analysis of this: the ideas needed to do so have been covered.

Among all uniformly well-posed problems it is possible to extract a class of problems possessing solutions with extra smoothness. It is fairly common to call the problems of this kind **parabolic**. Before proceeding to a rigorous definition of the parabolic equation, it will be sensible to introduce a notion of classical solution. We say that a function u is a **classical solution** of the Cauchy problem (5.2.1)–(5.2.2) if this function is continuous on the segment $[0, T]$ in the norm of the space X and is continuously differentiable on $(0, T]$ in the norm of the space X, the values of u on $(0, T]$ belong to $\mathcal{D}(A)$ and equation (5.2.1) is satisfied on $(0, T]$ with the supplementary condition $u(0) = u_0$. Along similar lines, a function u can be adopted as a classical solution of the Cauchy problem (5.2.20)–(5.2.21) on the semi-axis $t \geq 0$ if u is a classical solution of problem (5.2.1)–(5.2.2) for any $T > 0$. Here the difference between the classical and strong solutions is connected with the obstacle that the classical one may be nondifferentiable at the point $t = 0$ and the governing differential equation fails to be true at the point $t = 0$.

When classifying the problem to be parabolic, there are various approaches to this notion. In some cases problem (5.2.22)–(5.2.23) is known as a parabolic one if any weak solution of this problem turns out to be classical. In what follows this notion will be used in another sense, the meaning of which is that

(5.2.24) *any weak solution of the abstract Cauchy problem* (5.2.22)–(5.2.23) *is an analytic function on the semi-axis* $t > 0$.

Common practice involves the symbol $V(t)$ for the semigroup generated by the operator A. Within this notation, it is straightforward to verify that condition (5.2.24) is equivalent to the property that for each $x \in X$ the function $V(t)\,x$ is analytic on the semi-axis $t > 0$.

Condition (5.2.24) can be formulated in terms of the **resolvent** of the operator A. We proceed as usual. Let $S(\varphi, \omega)$ be a sector of the complex plane, that is,

$$S(\varphi,\omega) = \{\, \lambda \in \mathbf{C}\colon \, |\arg(\lambda - \omega)| < \varphi,\ \lambda \neq \omega \,\}, \tag{5.2.25}$$

where ω is a real number and $0 < \varphi < \pi$. As we have mentioned above, there are various ways of defining the problem to be parabolic. In view of this, it would be more convenient to involve in subsequent assertions the notion of analytic semigroup. By this we mean that the semigroup $V(t)$ generated by the operator A is **analytic** if and only if condition (5.2.24) holds true.

Theorem 5.2.8 *Let X be a Banach space and A be a closed linear operator, whose domain is dense in X. If the operator A generates a strongly continuous analytic semigroup in the space X, then there are constants $\omega \in \mathbf{R}$, $\varphi \in (\pi/2, \pi)$ and $C > 0$ such that the sector specified by* (5.2.25) *is contained in the resolvent set $\rho(A)$ and the estimate*

$$\| R(\lambda, A) \| \leq \frac{C}{|\lambda - \omega|} \tag{5.2.26}$$

is valid for each $\lambda \in S(\varphi, \omega)$. Conversely, if estimate (5.2.26) *is valid in a certain half-plane* $\operatorname{Re}\lambda > \omega$, *then the operator A generates a strongly continuous analytic semigroup in the space X.*

Denoting, as usual, by $V(t)$ the strongly continuous analytic semigroup generated by the operator A, we claim that the operator $A\,V(t)$ belongs to the space $\mathcal{L}(X)$ for each $t > 0$. Moreover, the estimate

$$\| A\,V(t) \| \leq \frac{M}{t}, \qquad t \to 0, \tag{5.2.27}$$

is valid in some neighborhood of the point $t = 0$. It is worth noting here that estimate (5.2.27) may be true with constant $M < 1/e$, where e refers to the base of natural logarithms. In that case the operator A appears to be bounded. Let us consider the nonhomogeneous Cauchy problem (5.2.20)–(5.2.21) and confine ourselves to classical solutions only. Such a trick allows to involve the concept of analytic semigroup, thereby making the premises of Theorem 5.2.3 less restrictive. All this enables us to obtain the following result.

Theorem 5.2.9 *Let X be a Banach space and the operator A generate in X a strongly continuous analytic semigroup $V(t)$. If $u_0 \in X$ and $f \in C^\alpha([0, T]; X)$, $0 < \alpha < 1$, then a classical solution u of the Cauchy problem* (5.2.1)–(5.2.2) *exists and is unique. Moreover, the function u is given by formula* (5.2.9).

By minor modifications formulae (5.2.10) and (5.2.11) related to $u'(t)$ and $A\,u(t)$ become

$$u'(t) = A\,V(t)\,u_0 + V(t)\,f(t) + \int_0^t A\,V(t-s)\,\big[f(s) - f(t)\big]\,ds\,, \tag{5.2.28}$$

$$A\,u(t) = A\,V(t)\,u_0 + \big(V(t) - I\big)\,f(t) + \int_0^t A\,V(t-s)\,\big[f(s) - f(t)\big]\,ds\,, \tag{5.2.29}$$

which are valid for each $u_0 \in X$ and any function f satisfying Hölder's condition on the segment $[0, T]$. The integrals in relations (5.2.28)–(5.2.29) should be regarded as improper ones. Due to estimate (5.2.27) the singularities of the expressions to be integrated in (5.2.28)–(5.2.29) are summable for $s = t$ as long as the function f is of Hölder's type. Therefore, the integral terms in (5.2.28)–(152.29) are continuous in the norm of the space X on the whole segment $[0, T]$.

When the semigroup $V(t)$ is analytic, one can restate the existence of **strong solutions**, thereby making the corresponding assertion less restrictive.

Theorem 5.2.10 *Let X be a Banach space and the operator A generate a strongly continuous analytic semigroup $V(t)$ in the space X. If $u_0 \in \mathcal{D}(A)$ and $f \in C^\alpha([0, T]; X)$, $0 < \alpha < 1$, then a strong solution u of the Cauchy* *problem* (5.2.1)–(5.2.2) *exists and is unique. Moreover, the function u is given by formula* (5.2.9).

5.3 Linear differential equations of the second order in Banach spaces

The rapid development of the theory of the Cauchy problem for differential equations of the second order during the last two decades stems from the necessity of solving the new scientific and technical problems and is presented in Fattorini (1969a,b, 1985), Ivanov et al. (1995), Kisynski (1972), Kurepa (1982), Lutz (1982), Sova (1966, 1975, 1977), Travis and Webb (1978), Vasiliev et al. (1990). In this section we consider direct problems for abstract differential equations of the second order in a Banach space X. Let A be a closed linear operator in the space X with a dense domain. At present we have at our disposal two elements u_0, $u_1 \in X$ and a function

$$f\colon\ [0, T] \mapsto X$$

and set up the **Cauchy problem** for an abstract equation in which it is required to find the function $u(t)$ from the set of relations

$$u''(t) = A\,u(t) + f(t)\,, \quad 0 < t < T\,, \tag{5.3.1}$$

$$u(0) = u_0\,, \qquad u'(0) = u_1\,. \tag{5.3.2}$$

With regard to problem (5.3.1)–(5.3.2) we confine ourselves to the cases of weak and strong solutions only. A distribution u with values in the space $\mathcal{D}(A)$ is said to be a **weak solution** of the Cauchy problem (5.3.1)–(5.3.2) if it satisfies (5.3.1) in the sense of the equality between elements of the space $\mathcal{D}'\big((0,T);\, X\big)$. Also, it is required that the function u should fall into the category of regular functions from the space $C^1\big([0,\, T];\, X\big)$ and satisfy the initial conditions (5.3.2) as a continuously differentiable function. In this context, it is worth noting that the weak solution u as an element only of the space $C^1\big(\,[0,\, T];\, X\,\big)$ does not necessarily possess a standard second derivative or take the values from the space $\mathcal{D}(A)$. We say that a function u gives a **strong solution** of problem (5.3.1)–(5.3.2) when it belongs to the class

$$C^2([0,\, T];\, X\,) \bigcap C([0,\, T];\, \mathcal{D}(A))$$

and is still subject to both relations (5.3.1)–(5.3.2) in a standard sense. Once treated as a distribution, each strong solution of the Cauchy problem at hand is always a weak solution of the same problem. The conditions

$u_0 \in \mathcal{D}(A)$ and $f \in C([0, T]; X)$ are necessary for a strong solution to exist. However, these conditions are insufficient.

Our subsequent studies are mainly devoted to strong solutions. For this reason by a "solution" we mean in the sequel a strong solution unless otherwise is explicitly stated.

As preliminaries to the solution of the original problem, the object of investigation is the Cauchy problem for the **homogeneous equation**

$$u''(t) = A\,u(t)\,, \quad 0 < t < T\,, \tag{5.3.3}$$

$$u(0) = u_0\,, \qquad u'(0) = u_1\,. \tag{5.3.4}$$

We say that the Cauchy problem (5.3.3)–(5.3.4) is **uniformly well-posed** if the following conditions hold:

(1) *there exists a dense subspace D of the space X such that for all elements u_0, $u_1 \in D$ a strong solution of problem* (5.3.3)–(5.3.4) *exists and is unique;*

(2) *if a sequence of strong solutions $u_n(t)$ to equation* (5.3.3) *is such that $u_n(0) \to 0$ and $u_n'(0) \to 0$ as $n \to \infty$ in the norm of the space X, then $u_n(t) \to 0$ as $n \to \infty$ in the norm of the space $C([0, T]; X)$.*

Necessary and sufficient conditions for the homogeneous Cauchy problem concerned to be uniformly well-posed are established in the following assertion.

Theorem 5.3.1 *The Cauchy problem* (5.3.3)–(5.3.4) *is uniformly well-posed if and only if there are constants M and ω such that for each $\lambda > \omega$ the value λ^2 is contained in the resolvent set $\rho(A)$ of the operator A and for the same value λ the estimate is valid:*

$$\left\| \frac{d^n}{d\,\lambda^n}\left(\lambda\, R(\lambda^2, A)\right) \right\| \le \frac{M\, n\,!}{(\lambda - \omega)^{n+1}}\,, \qquad n = 0, 1, 2, \ldots\,. \tag{5.3.5}$$

Before proceeding to the second order equations, it is reasonable to introduce the concept of strongly continuous cosine function which will be of crucial importance in the sequel. An operator function $C(t)$ defined for all real t with values in the space $\mathcal{L}(X)$ is called a **strongly continuous cosine function** if it possesses the following properties:

(1) $C(0) = I$;

(2) $C(t + s) + C(t - s) = 2\,C(t)\,C(s)$ *for all t, $s \in \mathbf{R}$;*

(3) *for any fixed $x \in X$ the function $C(t)\,x$ is continuous with respect to t in the norm of the space X for all $t \in \mathbf{R}$.*

Strongly continuous cosine functions may be of help in the further derivation of explicit expressions for the Cauchy problem solutions in terms of the input data. However, the relationships between the Cauchy problem solutions and the cosine functions for the second order equations are rather complicated than those for the first order equations and semigroups. Several new notions are aimed to refine the character of these relationships.

Every strongly continuous cosine function $C(t)$ can be associated with the corresponding **sine function** $S(t)$, which is specified by means of the integral

$$S(t) = \int_0^t C(s)\, ds. \tag{5.3.6}$$

It is worth noting here that the integral in (5.3.6) is understood in a strong sense. The very definition implies that the strong derivative of any sine function coincides with the associated cosine function. We are interested in learning more about the space

$$E = \{\, x \in X\colon\ C(t)\, x \in C^1(\mathbf{R}) \}, \tag{5.3.7}$$

containing all the elements of the space X for which the cosine function is strongly differentiable. The space E so defined becomes a Banach space with associated norm

$$\|\, x \,\|_E = \|\, x \,\| + \sup_{0 \le t \le 1} \|\, C'(t)\, x \,\| . \tag{5.3.8}$$

Each uniformly well-posed Cauchy problem of the type (5.3.3)–(5.3.4) can be put in correspondence with a suitable strongly continuous cosine function $C(t)$, whose use permits us to write a solution u of problem (5.3.3)–(5.3.4) as follows:

$$u(t) = C(t)\, u_0 + S(t)\, u_1 , \tag{5.3.9}$$

where $S(t)$ refers to the sine function given by formula (5.3.6). In particular, for $u_0 \in E$ and $u_1 \in X$ we specify a weak solution by appeal to (5.3.9). The condition $u_0 \in E$ is necessary and sufficient for the existence of a weak solution of problem (5.3.3)–(5.3.4). The same problem has a strong solution if and only if $u_0 \in \mathcal{D}(A)$ and $u_1 \in E$ and, in so doing,

$$u'(t) = A\, S(t)\, u_0 + C(t)\, u_1 ,$$

$$u''(t) = C(t)\, A\, u_0 + A\, S(t)\, u_1 .$$

The operators $C(t)$ and $S(t)$ both are commuting with the operator A for any real t. The operator $S(t)$ carries X into E and E into $\mathcal{D}(A)$, thus causing the inclusion

$$A\,S(t) \in \mathcal{L}(E,X)\,.$$

It is worth bearing in mind here that the cosine function $C(t)$ is uniquely determined by the operator A known as the **generator of the cosine function**. The converse is certainly true. More precisely, for any strongly continuous cosine function $C(t)$ there exists a closed linear operator A such that the domain $\mathcal{D}(A)$ is dense in the space X and the related Cauchy problem (5.3.3)–(5.3.4) is uniformly well-posed. Also, formula (5.3.9) is valid and the generator A of the cosine function $C(t)$ can uniquely be recovered.

One can derive several useful formulae, making it possible to express the cosine function $C(t)$ via its generator A and vice versa. The operator A is the strong second order derivative of the cosine function $C(t)$ at zero:

$$A\,x = \left[C(t)\,x \right]'' \Big|_{t=0}$$

with the domain

$$\mathcal{D}(A) = \left\{ x \in X\colon\ C(t)\,x \in C^2(\mathbf{R}) \right\}.$$

In turn, the cosine function $C(t)$ is representable by

$$C(t)\,x = \lim_{k\to\infty} \frac{(-1)^k}{k\,!} \left(\frac{k}{t} \right)^{k+1} \left[\frac{d^k}{d\,\lambda^k} \left(\lambda\, R(\lambda^2,\,A)\,x \right) \right] \Bigg|_{\lambda=k/t} .$$

For any strongly continuous cosine function $C(t)$ there are constants $M \geq 1$ and $\omega \geq 0$ such that for all real t

$$\| C(t) \| \leq M \exp\left(\omega\,|\,t\,| \right). \tag{5.3.10}$$

Combination of definition (5.3.6) of the cosine function and estimate (5.3.10) gives the inequality

$$\| S(t) \| \leq M\,|\,t\,| \exp\left(\omega\,|\,t\,| \right). \tag{5.3.11}$$

For $\lambda > \omega$ the operator functions $C(t)$, $S(t)$ and the **resolvent** $R(\lambda, A)$ of the operator A are related as follows:

$$\lambda\, R(\lambda^2,\,A) = \int_0^\infty C(t)\, e^{-\lambda\,t}\, dt\,,$$

$$R(\lambda^2,\,A) = \int_0^\infty S(t)\, e^{-\lambda\,t}\, dt\,.$$

Here both integrals are understood in a strong sense.

A simple observation that equation (5.3.3) is **autonomous** could be useful in the sequel. Since problem (5.3.3)–(5.3.4) is uniformly well-posed, the Cauchy problem

$$
\begin{aligned}
u''(t) &= A\,u(t)\,, \qquad a < t < b\,,\\
u(t_0) &= u_0\,, \qquad u'(t_0) = u_1\,,
\end{aligned}
$$

is also uniformly well-posed for all numbers a, b, t_0 with $a \le t_0 \le b$. The function

$$u(t) = C(t - t_0)\,u_0 + S(t - t_0)\,u_1$$

suits us perfectly in studying this problem.

Let us come back to the nonhomogeneous Cauchy problem (5.3.1)–(5.3.2) for which the following result is obtained.

Theorem 5.3.2 *Let problem* (5.3.3)–(5.3.4) *be uniformly well-posed. One assumes, in addition, that* $u_0 \in E$, $u_1 \in X$ *and* $f \in C([0, T]; X)$. *Then a weak solution* u *of the Cauchy problem* (5.3.1)–(5.3.2) *exists, is unique and takes the form*

$$u(t) = C(t)\,u_0 + S(t)\,u_1 + \int_0^t S(t-s)\,f(s)\,ds\,, \tag{5.3.12}$$

making it possible to express its derivative by

$$u'(t) = A\,S(t)\,u_0 + C(t)\,u_1 + \int_0^t C(t-s)\,f(s)\,ds\,. \tag{5.3.13}$$

Conditions $u_0 \in E$ and $f \in C\big([0, T]; X\big)$ are necessary for the existence of a strong solution. As far as any strong solution becomes a weak solution, formulae (5.3.12)–(5.3.13) continue to hold for strong solutions. Therefore, the existence of a strong solution is ensured by the continuous differentiability of the integral term on the right-hand side of (5.3.13). Let us stress that in this case the continuity of the function f is insufficient. The results we cite below allow to impose the conditions under which the Cauchy problem at hand is solvable in a strong sense.

Theorem 5.3.3 *Let the Cauchy problem* (5.3.3)-(5.3.4) *be uniformly well-posed. One assumes, in addition, that* $u_0 \in \mathcal{D}(A)$, $u_1 \in E$ *and*

$$f \in C^1([0,T]; X) + C([0,T]; \mathcal{D}(A)).$$

Then a strong solution u *of the Cauchy problem* (5.3.1)-(5.3.2) *exists, is unique and is given by formula* (5.3.12).

As stated above, formula (5.3.13) is established for any strong solution. Some progress will be achieved once we involve in the further development the functions u'' and $A\,u$. If $f = f_1 + f_2$ with the members $f_1 \in C^1([0,T]; X)$ and $f_2 \in C([0,T]; \mathcal{D}(A))$, then

$$\begin{aligned} u''(t) &= C(t)\,[\,A\,u_0 + f_1(0)] + A\,S(t)\,u_1 \\ &\quad + \int_0^t C(t-s)\,f_1'(s)\,ds \\ &\quad + \int_0^t S(t-s)\,A\,f_2(s)\,ds + f_2(t)\,, \end{aligned} \tag{5.3.14}$$

$$\begin{aligned} A\,u(t) &= C(t)\,[\,A\,u_0 + f_1(0)] + A\,S(t)\,u_1 \\ &\quad + \int_0^t C(t-s)\,f_1'(s)\,ds \\ &\quad + \int_0^t S(t-s)\,A\,f_2(s)\,ds - f_1(t)\,. \end{aligned} \tag{5.3.15}$$

By integrating by parts in (5.3.13) we are led to an alternative form of the first derivative

$$\begin{aligned} u'(t) &= C(t)\,u_1 + S(t)\,[\,A\,u_0 + f_1(0)] \\ &\quad + \int_0^t C(t-s)\,f_2(s)\,ds \\ &\quad + \int_0^t S(t-s)\,f_1'(s)\,ds\,. \end{aligned} \tag{5.3.16}$$

With the aid of (5.3.14)–(5.3.16) one can find out when a solution of the Cauchy problem in view becomes more smooth. Indeed, let $f \in C^1([0, T]; X)$, $u_0 \in \mathcal{D}(A)$ and $u_1 \in E$. Substitution $v = u'$ allows to find by formula (5.3.16) that

$$v(t) = C(t)\, v_0 + S(t)\, v_1 + \int_0^t S(t-s)\, g(s)\, ds\,, \tag{5.3.17}$$

where $v_0 = u_1$, $v_1 = A\, u_0 + f(0)$ and $g(t) = f'(t)$. With the inclusions $v_0 \in E$ and $g \in C([0, T]; X)$ in view, formula (5.3.17) serves to motivate that the function v is a weak solution of the Cauchy problem

$$\begin{cases} v''(t) = A\, v(t) + g(t)\,, & 0 \le t \le T\,, \\ v(0) = v_0\,, & v'(0) = v_1\,. \end{cases} \tag{5.3.18}$$

Applying Theorem 5.3.3 to problem (5.3.18) yields the conditions under which a solution u of the Cauchy problem (5.3.1)–(5.3.2) attains the extra smoothness.

Theorem 5.3.4 *Let the Cauchy problem* (5.3.3)–(5.3.4) *be uniformly well-posed and* $u_0 \in \mathcal{D}(A)$, $u_1 \in \mathcal{D}(A)$,

$$A\, u_0 + f(0) \in E\,, \qquad f \in C^2([0, T]; X)\,.$$

Then for a solution u of the Cauchy problem (5.3.1)–(5.3.2) *the inclusion occurs:*

$$u \in C^3([0, T]; X)\,.$$

Following the same procedure it is possible to establish some conditions under which the solution of the Cauchy problem becomes as smooth as we like. Under such an approach we are able to find out when the solution in question possesses the extra "spatial" smoothness. Indeed, allowing $u_0, u_1 \in \mathcal{D}(A)$ and $f \in C([0, T]; \mathcal{D}(A))$ and substituting $w = A\, u$, we involve (5.3.15), whose use permits us to find that

$$w(t) = C(t)\, w_0 + S(t)\, w_1 + \int_0^t S(t-s)\, h(s)\, ds\,,$$

where $w_0 = A\, u_0$, $w_1 = A\, u_1$ and $h(t) = A\, f(t)$. With the relation $A\, u_0 \in E$ in view, the function w gives a weak solution of the Cauchy problem

$$\begin{cases} w''(t) = A\, w(t) + h(t)\,, & 0 < t < T\,, \\ w(0) = w_0\,, & w'(0) = w_1\,. \end{cases} \tag{5.3.19}$$

Under the conditions $w_0 \in \mathcal{D}(A)$, $w_1 \in E$ and $h \in C([0, T]; \mathcal{D}(A))$ imposed above Theorem 5.3.3 applies equally well to the Cauchy problem (5.3.1)–(5.3.2). All this enables us to obtain the following result.

Theorem 5.3.5 *Let the Cauchy problem* (5.3.3)–(5.3.4) *be uniformly well-posed. One assumes, in addition, that* $u_0 \in \mathcal{D}(A^2)$, $u_1 \in \mathcal{D}(A)$, $A u_1 \in E$ *and* $f \in \mathcal{C}([0,T]; \mathcal{D}(A^2))$. *Then a solution* u *of the Cauchy problem* (5.3.1)–(5.3.2) *satisfies the condition*

$$u \in \mathcal{C}([0,T]; \mathcal{D}(A^2)) .$$

The same framework is much applicable in trying to increase the "spatial" smoothness of a solution as high as we like. The main idea behind approach is to move in both directions, thereby providing proper guidelines for conditions of the "mixed" smoothness.

One way of proceeding is to reduce the equation of the second order to the system of equations of the first order. Having substituted $v(t) = u'(t)$, as usual, we may attempt the Cauchy problem (5.3.1)–(5.3.2) in the form

$$\begin{aligned} u'(t) &= v(t)\,, & 0 < t < T\,, \\ v'(t) &= A\,u(t) + f(t)\,, & 0 < t < T\,, \\ u(0) &= u_0\,, & v(0) = u_1\,, \end{aligned}$$

which can be treated from a formal point of view as the Cauchy problem for the first order equation

$$(5.3.20) \qquad \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad 0 < t < T\,,$$

$$(5.3.21) \qquad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Before giving further motivations, a few questions need certain clarification. The first is concerned with a suitably chosen Banach space $\mathcal{X}$ in which the Cauchy problem (5.3.20)–(5.3.21) is completely posed. This question has a unique answer in the case when the natural correspondence between problem (5.3.1)–(5.3.2) and problem (5.3.20)–(5.3.21) is needed. Indeed, given $f = 0$, observe that problem (5.3.1)–(5.3.2) is weakly solvable for any $u_0 \in E$ and $u_1 \in X$. On the other hand, on account of Theorem 5.2.2 the first order equation has a weak solution under any input data. The well-founded choice of the space $\mathcal{X}$ immediately follows from the foregoing:

$$\mathcal{X} = E \times X .$$

The second question needs investigation in connection with the domain of the operator

$$(5.3.22) \qquad \mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}.$$

In that case strong solutions may be of help in achieving the final aim. It is worth mentioning here that the Cauchy problem (5.3.1)–(5.3.2) with $f = 0$ has a strong solution for any $u_0 \in \mathcal{D}(A)$ and $u_1 \in E$. One more **Cauchy problem**

$$(5.3.23) \qquad \begin{cases} w'(t) = \mathcal{A}\, w(t), & 0 < t < T, \\ w(0) = w_0, \end{cases}$$

complements our studies. We know that this problem, in turn, possesses a strong solution for any $w_0 \in \mathcal{D}(\mathcal{A})$. With this relation in view, the problem statement necessitates imposing the extra restriction

$$(5.3.24) \qquad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times E.$$

The main difficulty here lies in the comparison of the uniform well-posedness of problem (5.3.3)–(5.3.4) with the same property of problem (5.3.23). A case in point is as follows. No convergence to zero in the space $C([0, T]; X)$ for the first derivative of the solution is required in the definition of the uniform well-posedness of the Cauchy problem for the second order equations even if the initial data elements vanish. At the same time this property is needed in the statement of problem (5.3.23). This is due to the fact that its solution admits the form

$$w(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}.$$

Nevertheless, the following assertion is valid and makes our exposition more transparent.

Theorem 5.3.6 *Let the Cauchy problem* (5.3.3)–(5.3.4) *be uniformly well-posed and conditions* (5.3.7)–(5.3.8) *hold. Then the Cauchy problem* (5.3.23) *is uniformly well-posed if we agree to consider the space* $\mathcal{X} = E \times X$ *and* (5.3.24) *as the domain of the operator* $\mathcal{A}$ *of the structure* (5.3.22).

From Theorem 5.3.6 it follows that if the operator A generates a strongly continuous cosine function $C(t)$, then the operator $\mathcal{A}$ generates a strongly continuous semigroup $V(t)$. In dealing with the sine function $S(t)$

associated with $C(t)$ it is plain to derive the **explicit expression** for the semigroup $V(t)$:

$$V(t) = \begin{pmatrix} C(t) & S(t) \\ A\,S(t) & C(t) \end{pmatrix}. \tag{5.3.25}$$

As a matter of fact, the semigroup $V(t)$ becomes a group if we accept formula (5.3.25) for all negative values t, too. In this case the function $V(t)$ obeys the **group property**, amounting, for all real values t and s, to the relation

$$V(t+s) = V(t)\,V(s).$$

The group property just considered provides that for any real t the operator $V(t)$ has a bounded inverse $V(t)^{-1}$. By the same token,

$$V(t)^{-1} = V(-t).$$

We will have more to say about relationship between the resolvents of the operators A and $\mathcal{A}$:

$$(\lambda I - \mathcal{A})^{-1} = \begin{pmatrix} \lambda(\lambda^2 I - A)^{-1} & (\lambda^2 I - A)^{-1} \\ A(\lambda^2 I - A)^{-1} & \lambda(\lambda^2 I - A)^{-1} \end{pmatrix}. \tag{5.3.26}$$

The assertion of Theorem 5.3.6 can be inverted in the following sense. Suppose that a Banach space E_1 is continuously embedded into the space X and $\mathcal{D}(A)$ is continuously embedded into the space E_1. In the Banach space $\mathcal{X} = E_1 \times X$ we consider the operator (5.3.22) with the domain

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times E_1.$$

If the operator $\mathcal{A}$ generates a strongly continuous group, then the operator A generates a strongly continuous cosine function and the space E_1 coincides with the space E arising from relations (5.3.7)–(5.3.8) up to equivalent norms on that space.

The next step is to consider the **Cauchy problem** for the **nonlinear equation**

$$u''(t) = A\,u(t) + f(t, u(t), u'(t)), \quad 0 < t < T, \tag{5.3.27}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{5.3.28}$$

This problem needs more a detailed exploration, since its solution is not defined, in general, on the whole segment $[0, T]$. It may happen that the Cauchy problem (5.3.27)–(5.3.28) is solvable only on a certain segment

$[0,\, T^*]$ with $0 < T^* \le T$. Denote by $\bar{S}(y, R)$ a closed ball in the space X of radius $R > 0$ with center y and introduce the set

$$U \;=\; [0, T] \,\times\, S(u_0,\, R) \,\times\, \bar{S}(u_1,\, R)\,.$$

One assumes, in addition, that the ingredient f meets the requirement labeled by (5.3.29) for the reader's convenience:

> *the function f is continuous on the set U and satisfies thereon the Lipschitz condition with respect to the second and third arguments or, more specifically, there is a constant $L > 0$ such that at all points (t, u, v), $(t, w, z) \in U$ one has*

$$\| f(t, u, v) - f(t, w, z)\| \;\le\; L\left(\| u - w \| + \| v - z \|\right). \tag{5.3.29}$$

Observe that the continuity of the function f permits one to define weak and strong solutions of the Cauchy problem (5.3.27)–(5.3.28) in a natural way. Indeed, for any function $u \in C^1\big([0, T^*]; X\big)$ subject to the condition $\big(t, u(t), u'(t)\big) \in U$, valid at each point $t \in [0, T^*]$, the superposition $f\big(t,\, u(t),\, u'(t)\big)$ is continuous on the segment $[0,\, T^*]$ and belongs to the space $\mathcal{D}'\big((0,\, T^*);\, X\,\big)$, thereby justifying a possibility of applying the previous concepts of weak and strong solutions to the case of the nonlinear equation (5.3.27).

Of special interest is a weak solution of the Cauchy problem (5.3.27)–(5.3.28) provided that the homogeneous Cauchy problem (5.3.3)–(5.3.4) is uniformly well-posed and $u_0 \in E$. With relation (5.3.12) in view, the nonlinear problem (5.3.27)–(5.3.28) is equivalent to the **integral equation**

$$u(t) \;=\; C(t)\, u_0 + S(t)\, u_1 + \int_0^t S(t-s)\, f\big(s,\, u(s),\, u'(s)\big)\; ds\,, \tag{5.3.30}$$

when operating in the class of continuously differentiable functions. With the aid of (5.3.13) it is easy to recast this equation as the system of integral equations for the functions $u(t)$ and $v(t) = u'(t)$:

$$\begin{cases} u(t) \;=\; C(t)\, u_0 + S(t)\, u_1 + \int_0^t S(t-s)\, f\big(s,\, u(s),\, v(s)\big)\; ds\,, \\[2ex] v(t) \;=\; A\; S(t)\, u_0 + C(t)\, u_1 + \int_0^t C(t-s)\, f\big(s,\, u(s),\, v(s)\big)\; ds\,. \end{cases} \tag{5.3.31}$$

It is worth bearing in mind here that the system (5.3.31) is viewed in the class of continuous functions. Making the substitutions

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad w_0(t) = \begin{pmatrix} C(t)\, u_0 + S(t)\, u_1 \\ A\, S(t)\, u_0 + C(t)\, u_1 \end{pmatrix},$$

$$F(t, w) = \begin{pmatrix} f(t, u, v) \\ f(t, u, v) \end{pmatrix}, \qquad A(t, s) = \begin{pmatrix} S(t-s) & 0 \\ 0 & C(t-s) \end{pmatrix},$$

we rewrite the system (5.3.31) as the **Volterra integral equation** of the second kind

$$w(t) = w_0(t) + \int_0^t A(t, s)\, F(s, w(s))\; ds\,, \tag{5.3.32}$$

when operating in the class of continuous functions with values in the Banach space $X \times X$. Equation (5.3.32) satisfies all the conditions of Theorem 5.1.19, thus ensuring the local solvability of the system (5.3.31) and the uniqueness of its solution. Because of (5.3.13), the function v refers to the first derivative of the function u and, therefore, equation (5.3.30) is an immediate implication of the system (5.3.31). All this enables us to obtain the following result.

Theorem 5.3.7 *Let the Cauchy problem* (5.3.3)–(5.3.4) *be uniformly well-posed and let $u_0 \in E$ and $u_1 \in X$. One assumes, in addition, that the function f complies with* (5.3.29). *Then there is a value $T^* > 0$ such that the nonlinear Cauchy problem* (5.3.27)–(5.3.28) *has a weak solution u on the segment $[0, T^*]$. Also, if there is no possibility of extending this solution on a more wide segment, then the point $\big(T^*,\, u(T^*),\, u'(T^*)\big)$ belongs to the boundary of the set U. Moreover, any two weak solutions of problem* (5.3.27)–(5.3.28) *will coincide on a common part of the segments of their existence.*

Assume that the second integral equation in the system (5.3.31) is considered with respect to one unknown function v by relating another function u to be continuous and fixed. With these ingredients, the existence of a strong solution of the Cauchy problem (5.3.27)–(5.3.28) amounts to the question of the continuous differentiability of the function v as we have mentioned above. It is possible to overcome this obstacle in just the same way as, for instance, we did in Section 5.2 for the first order equations. For the moment, the function f is supposed to be **Frechet differentiable** at a point $(t_0,\, u_0,\, v_0)$, it being understood that the increment of the function

f can be expressed by

$$\begin{aligned} f(t,u,v) - f(t_0,u_0,v_0) &= f_t(t-t_0) + f_u(u-u_0) + f_v(v-v_0) \\ &\quad + \alpha(t,u,v)\,\big(|t-t_0| + \|u-u_0\| + \|v-v_0\|\big)\,, \end{aligned}$$

where $f_t \in X$; f_u, $f_v \in \mathcal{L}(X)$ and $\alpha(t,u,v) \to 0$ as $(t,u,v) \to (t_0,u_0,v_0)$. Here the element f_t and the operators f_u and f_v stand for the **partial derivatives** of the function f at the point (t_0,u_0,v_0). A function f is said to be **Frechet differentiable** on the set U if it is Frechet differentiable at each point of this set. In what follows we take for granted the collection of conditions labeled by (5.3.33) for the reader's convenience:

(5.3.33) *the function f is Frechet differentiable on the set U, its partial derivatives f_t, f_u, f_v are continuous and satisfy thereon the Lipschitz condition with respect to the third argument.*

Omitting some details and difficulties arising when finding a strong solution of the Cauchy problem in comparison with the work done in Section 5.2 for the first order equations, we cite here only the final results obtained.

Theorem 5.3.8 *Let the Cauchy problem* (5.3.3)–(5.3.4) *be uniformly well-posed and let*

$$u_0 \in \mathcal{D}(A) \quad \text{and} \quad u_1 \in E\,.$$

One assumes, in addition, that the function f is involved in (5.3.29) *and* (5.3.33) *both. Then there is a value $T^* > 0$ such that the nonlinear Cauchy problem* (5.3.27)–(5.3.28) *has on the segment $[0, T^*]$ a strong solution u. Also, if there is no possibility of extending this solution on a more wide segment, then the point $\big(T^*,\, u(T^*),\, u'(T^*)\big)$ belongs to the boundary of the set U. Moreover, any two strong solutions of problem* (5.3.27)–(5.3.28) *will coincide on a common part of the segments of their existence.*

5.4 Differential equations with varying operator coefficients

In this section we deal with direct problems for abstract diffential equations in the case when the operator coefficient depends on the argument t. This part of the theory, being the most difficult one and relating to preliminaries, can serve as the necessary background in advanced theory. In this regard, it is appropriate to mention the books and papers by Amann (1986, 1987), Fattorini (1983), Gil (1987), Henry (1981), Ikawa (1968), Kato (1961, 1970, 1973, 1975b, 1982), Lomovtsev and Yurchuk (1976), Serizawa and Watanabe (1986), Sobolevsky (1961), Sobolevsky and Pogorelenko (1967), Yakubov (1970, 1985), Yosida (1965). The preliminary stage is connected with the **Cauchy problem** in a Banach space X

$$u'(t) = A(t)\,u(t) + f(t)\,, \qquad 0 < t < T\,, \tag{5.4.1}$$

$$u(0) = u_0\,, \tag{5.4.2}$$

where for each $t \in [0, T]$ the operator $A(t)$ is linear and closed and its domain $\mathcal{D}(A(t))$ is dense in the space X. Careful analysis of the Cauchy problem (5.4.1)–(5.4.2) is rather complicated as compared with problem (5.2.1)–(5.2.2). Until now there is no universal or advanced theory for problems of the type (5.4.1)–(5.4.2). For further successful developments of such theory we will be forced to impose several additional conditions, making it possible to distinguish the hyperbolic and parabolic types of equation (5.4.1). The passage from problem (5.2.1)–(5.2.2) to problems with varying operator coefficients necessitates modifying the notion of uniform well-posedness. In preparation for this, we study the **homogeneous Cauchy problem**

$$u'(t) = A(t)\,u(t)\,, \qquad 0 < t < T\,, \tag{5.4.3}$$

$$u(0) = u_0\,. \tag{5.4.4}$$

Recall that special investigations of problem (5.2.3)–(5.2.4) hinge essentially on the property of the translation invariance of the equation (5.2.3) solution. The validity of this property is an immediate implication of the autonomy of equation (5.2.3) and, in turn, implies the uniform well-posedness of the related Cauchy problem on any segment. In particular, the uniform well-posedness of problem (5.2.3)–(5.2.4) provides the same property for another problem

$$\begin{cases} u'(t) = A\,u(t)\,, & s < t < T\,, \\ u(s) = u_0\,, \end{cases}$$

for any $s \in [0, T)$. Additional restrictions are needed in such a setting, since the indicated property fails to be true for nonautonomous equations. Thus, the definition of the uniform well-posedness of the Cauchy problem (5.4.3)–(5.4.4) on the segment $[0, T]$ is accompanied by the solvability of this problem on any segment $[s, T]$ with $s \in [0, T)$. To be more specific, it is required that

(1) *there is a dense subspace D of the space X such that for any value $s \in [0, T)$ and any element $u_0 \in D$ one can find a unique function $u_s(t)$ subject to the following relations: $u_s \in C^1([s, T]; X)$ and for any $t \in [s, T]$*

$$u_s(t) \in \mathcal{D}(A(t)), \qquad u_s'(t) = A(t)\, u_s(t)$$

and $u_s(s) = u_0$.

Another point in our study is concerned with the continuous dependence of a solution upon the input data. Provided condition (1) holds, the input data should include not only the element u_0, but also the parameter s. Especial attention is being paid to the construction of an **evolution operator**

$$V(t, s)\colon\ u_0 \mapsto u_s(t),$$

by means of which the Cauchy problem can be resolved on the segment $[s, T]$. In this direction one more condition is imposed:

(2) *there exists a function $V(t, s)$ taking the values in the space $\mathcal{L}(X)$ and being strongly continuous in the triangle*

$$\Delta = \big\{ (t, s)\colon\ 0 \le s \le T,\ s \le t \le T \big\}$$

such that the Cauchy problem

$$\begin{cases} u'(t) = A(t)\, u(t), & s < t < T, \\ u(s) = u_0, \end{cases}$$

is solved by the function $u(t) = V(t, s)\, u_0$.

In accordance with what has been said, the Cauchy problem (5.4.3)–(5.4.4) is **uniformly well-posed** if both conditions (1)–(2) are satisfied. If the operator coefficient $A(t)$ does not depend on t, the definiton of uniform well-posedness coincides with that given in Section 5.2 and in this case the semigroup $V(t)$ generated by A and the evolution operator $V(t, s)$ just considered are related by

$$V(t, s) = V(t - s).$$

If the Cauchy problem (5.4.3)–(5.4.4) is uniformly well-posed, then the evolution operator $V(t, s)$ is uniformly bounded. That is to say, there is a constant $M > 0$ such that

$$\| V(t, s) \| \le M$$

for all $(t, s) \in \Delta$. Moreover, the operator function $V(t, s)$ possesses the properties similar to those established earlier for semigroups:

$$V(s, s) = I, \qquad 0 \le s \le T, \tag{5.4.5}$$

$$V(t, r)\, V(r, s) = V(t, s), \quad 0 \le s \le r \le t \le T. \tag{5.4.6}$$

By the definition of evolution operator, for any $u_0 \in D$ the function

$$u(t) = V(t, s)\, u_0 \tag{5.4.7}$$

is continuously differentiable on the segment $[s, T]$ and solves equation (5.4.3) on the same segment $[s, T]$. Each such function is called a **strong solution** of (5.4.3). Observe that an arbitrarily chosen element u_0 of the space X does not necessarily lie within the set D. From such reasoning it seems clear that the function u defined by (5.4.7) will be less smooth. One might expect its continuity and no more. For this reason the function u so constructed is said to be a **weak solution** to equation (5.4.3).

As stated in Section 5.2, strongly continuous semigroups are in a one-to-one correspondence with uniformly well-posed Cauchy problems for autonomous equations. In this view, it is reasonable to raise the same question with respect to equations for the time-dependent operator coefficient. We call $V(t, s)$ an **evolution family** if $V(t, s)$ is strongly continuous in Δ and satisfies both conditions (5.4.5)–(5.4.6). Also, we are somewhat uncertain: could any evolution family $V(t, s)$ be adopted as an evolution operator for a uniformly well-posed problem of the type (5.4.3)–(5.4.4)? Unfortunately, the answer is no even in the case of a finite dimension. In dealing with the basic space $X = \mathbf{R}$ and a continuous positive function f being nowhere differentiable, observe that the function

$$V(t, s) = f(t)/f(s)$$

constitutes what is called an evolution family. However, there is no uniformly well-posed Cauchy problem being in a proper correspondence with $V(t, s)$.

Let us compare the concepts of well-posedness for the cases of a constant operator coefficient and a varying one. With regard to problem (5.4.3)–(5.4.4) the transition stage is of artificial character. However, we should bear in mind here that this notion is of less importance for problem (5.4.3)–(5.4.4) as compared with problem (5.2.3)–(5.2.4). To decide for yourself whether a particular Cauchy problem with a constant operator coefficient is uniformly well-posed, a first step is to check the fulfilment of the conditions of Theorem 5.2.1. A key role of this property is connected with a number of corollaries on solvability of nonhomogeneous and nonlinear equations involved in further reasoning. As to the case of a variable coefficient, the situation changes drastically. Of crucial importance is now the unique solvability of the Cauchy problem (5.4.1)–(5.4.2). Furthermore, the construction of the corresponding evolution operator $V(t, s)$ is a preliminary step only and in any event should be followed by proving the solvability of the nonhomogeneous equation in some or other senses. What is more, careful analysis of the evolution operator is carried out by the methods depending essentially on the type of the governing equation.

The first stage is devoted to the **hyperbolic type** recognition for an abstract differential equation. There are several frameworks for this concept, but we confine ourselves to two widespread approaches which are frequently encountered in theory and practice and are best suited for deeper study of hyperbolic differential equations of the second order as well as of symmetric hyperbolic systems of differential equations of the first order.

The first approach is much applicable and appears useful not only for hyperbolic equations of the second order, but also for equations of parabolic type and Schrödinger equation. However, more recently, the contemporary interpretation of parabolic equations owes a debt to the introduction of another concept based on analytic semigroups, whose use permits us to obtain more advanced results.

Consider a family of norms $\| \cdot \|_t$, $0 \le t \le T$, on the Banach space X, each being equivalent to the norm of the space X. The same symbol $\| \cdot \|_t$ will stand for the operator norm induced by this family on the space $\mathcal{L}(X)$. In what follows we take for granted that

(H1) *the domain $\mathcal{D}(A(t))$ of the operator $A(t)$ does not depend on t, that is,*

$$\mathcal{D}(A(t)) = \mathcal{D};$$

(H2) *there is a number $a > 0$ such that for each $t \in [0, T]$ all of the real numbers λ, satisfying the condition $|\lambda| > a$, is contained in the resolvent set of the operator $A(t)$ and the estimate is valid:*

$$\| (\lambda I - A(t))^{-1} \|_t \le \frac{1}{|\lambda| - a};$$

(H3) *there exist $s \in [0, T]$ and λ with $|\lambda| > a$ such that the operator function*

$$B(t) = (\lambda I - A(t)) (\lambda I - A(s))^{-1}$$

is continuously differentiable on the segment $[0, T]$ in the norm of the space $\mathcal{L}(X)$;

(H4) *one can find a nondecreasing function* $\omega(t)$ *such that for all* $t, s \in [0, T]$, $t > s$, *and each* $x \in X$

$$\Big| \, \| x \|_t - \| x \|_s \Big| \leq \big(\omega(t) - \omega(s)\big) \, \| x \|$$

and, moreover, there is a constant $\delta > 0$ *such that for all* $\in [0, T]$ *and each* $x \in X$

$$\| x \|_t \geq \delta \| x \| .$$

We claim that under these agreements the Cauchy problem (5.4.3)–(5.4.4) is **uniformly well-posed**. A more deep result is revealed in the following statement.

Theorem 5.4.1 *Let conditions* $(H1)$–$(H4)$ *be fulfilled. Then the following assertions are true:*

(a) *the Cauchy problem* (5.4.3)–(5.4.4) *is uniformly well-posed;*

(b) *if* $u_0 \in D$ *and*

$$f \in C([0, T]; X), \qquad A(t) f(t) \in C([0, T]; X),$$

then a solution u *of the Cauchy problem* (5.4.1)–(5.4.2) *exists and is unique in the class of functions*

$$u \in C^1([0, T]; X), \qquad A(t) u(t) \in C([0, T]; X).$$

Also, this solution is representable by the formula

$$u(t) = V(t, 0) \, u_0 + \int_0^t V(t, s) \, f(s) \, ds , \tag{5.4.8}$$

where $V(t, s)$ *refers to the evolution operator of the homogeneous Cauchy problem* (5.4.3)–(5.4.4).

The solution u ensured by item (b) of Theorem 5.4.1 is called a **strong solution** of the Cauchy problem (5.4.1)–(5.4.2). Here Theorem 5.4.1 serves as a basis for decision-making about the uniform well-posedness of the homogeneous Cauchy problem (5.4.3)–(5.4.4). Therefore, under the same conditions there exists an evolution operator associated with (5.4.3)–(5.4.4), so that the function u expressed formally by (5.4.8) is continuous for any element $u_0 \in X$ and any function $f \in C([0, T]; X)$. This provides enough reason to define the function u as a weak solution of the Cauchy problem (5.4.1)–(5.4.2) via representation (5.4.8).

As we have mentioned above, the second concept provides proper guidelines for the hyperbolic type recognition with regard to an abstract equation and covers the case of symmetric hyperbolic systems of first order differential equations. Further treatment of the abstract equation (5.4.1) in a similar manner is caused by the assumption that the domain of the operator $A(t)$ depends on t and, in view of this, the notion of stability of the operator function $A(t)$ becomes important and rather urgent. We say that the function $A(t)$ is **stable** if

(S1) *there exist a pair of constants ω and $M > 0$ such that for each $\lambda > \omega$ and any finite set of points $\{ t_i \}_{i=1}^{k}$, satisfying the condition $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$, the estimate is valid:*

$$\left\| (\lambda I - A(t_k))^{-1} (\lambda I - A(t_{k-1}))^{-1} \cdots (\lambda I - A(t_1))^{-1} \right\| \le \frac{M}{(\lambda - \omega)^k} .$$

In a particular case, when the function $A(t)$ does not depend on t, condition (S1) coincides with condition (5.2.5) from Theorem 5.2.1. As a matter of fact, the following condition allows to distinguish the class of hyperbolic equations:

(S2) *there exist a Banach space X_0 embedded densely and continuously into the space X and an operator function $S(t)$ defined on the segment $[0, T]$ and taking the values in the space $\mathcal{L}(X_0, X)$ such that $S(t)$ is strongly continuously differentiable on the segment $[0, T]$ and for each $t \in [0, T]$*

$$S(t)^{-1} \in \mathcal{L}(X, X_0), \qquad X_0 \subset \mathcal{D}(A(t))$$

and the operator function

$$A \in \mathcal{C}([0, T]; \mathcal{L}(X_0, X)) .$$

Moreover, the relationship $S(t)\, A(t)\, S(t)^{-1} = A(t) + R(t)$ takes place for each $t \in [0, T]$, where the operator R belongs to the class $\mathcal{L}(X)$ and the operator function $R(t)$ is strongly continuous on the segment $[0, T]$.

Recall that condition (S2) is aimed at covering the abstract equations corresponding to symmetric hyperbolic systems of first order differental equations. For this, the operator $S(t)$ is treated as an abstract counterpart of the Calderon–Zygmund singular integral operator.

Theorem 5.4.2 *Let conditions* $(S1)$*-*$(S2)$ *hold. Then the following assertions are true:*

(a) *the Cauchy problem* (5.4.3)-(5.4.4) *is uniformly well-posed;*

(b) *if* $u_0 \in X_0$ *and* $f \in C([0, T]; X_0)$, *then a strong solution* u *of the Cauchy problem* (5.4.1)-(5.4.2) *exists and is unique. Moreover, this solution is given by formula* (5.4.8).

We should take into account once again that the element u_0 may occupy, in general, an arbitrary place in the space X and, in view of this, does not necessarily belong to the subspace X_0. Just for this reason formula (5.4.8) gives, under the conditions of Theorem 5.4.2 in combination with $f \in C([0, T]; X)$, only a continuous function being, by definition, a **weak solution** of the Cauchy problem (5.4.1)-(5.4.2).

We proceed to the next case as usual. This amounts to the further treatment of (5.4.1) as an equation of **parabolic type** under the set of constraints

(P1) *the domain* $\mathcal{D}(A(t)) = \mathcal{D}$ *of the operator* A *does not depend on the variable* t *and, in addition, there is a real number* ω *and a positive constant* C *such that for each* $t \in [0, T]$ *the half-plane* $\operatorname{Re}\lambda > \omega$ *is contained in the resolvent set of the operator* $A(t)$. *Also, for any* λ *with* $Re\,\lambda > \omega$ *and each* $t \in [0, T]$ *the estimate holds:*

$$\| R(\lambda, A(t))\| \leq \frac{C}{|\lambda - \omega|} .$$

The condition so formulated provides a natural generalization of the **parabolic type** definition relating to equation (5.2.26) in the case of a constant operator coefficient. Condition (P1) is in common usage and covers plenty of widespread applications such as the Dirichlet problem for a partial differential equation of parabolic type in a space-time cylinder over a bounded domain in the space $\mathbf{R}^n$ and many others. However, while studying Neumann's problem or some other types of boundary conditions and thereby involving equation (5.4.1) in which the domain of the operator coefficient $A(t)$ depends on t, condition (P1) is more stronger and does not fit our purposes. Recent years have seen the publication of numerous papers whose results permit one to overcome the difficulty involved and there are other conditions ensuring the well-posedness of the Cauchy problem for the case of an abstract parabolic equation with varying operator coefficients even if the domain of the operator $A(t)$ depends on the variable t. The conditions mentioned above find a wide range of applications and, in particular, allow to consider partial differential equations of parabolic type with various boundary conditions (see Amann (1986, 1987)). A casual acquaintance with properties of analytic semogroups in **interpolation spaces** is needed in applications of these results to the current framework. But they are beyond the scope of this book.

If you wish to explore this more deeply, you might find it helpful to study condition (P1) on your own. Provided condition (P1) holds, the uniform well-posedness of the Cauchy problem at hand is ensured by the restriction that the operator coefficient is of **Hölder's type**, meaning that

(P2) *there exist a complex number* λ *with* $\operatorname{Re}\lambda > \lambda_0$, *a real constant* $L > 0$ *and a real value* $\beta \in (0, 1]$ *such that for all* $t, s, \tau \in [0, T]$ *the estimate is valid:*

$$\Big\| \big[A(t) - A(s) \big] R(\lambda, A(\tau)) \Big\| \leq L \, |t - s|^{\beta}.$$

Likewise, a function $u \in C([0, T]; X)$ is called a **classical solution** to equation (5.4.1) if

(i) *u is continuously differentiable on the half-open interval* $(0, T]$;

(ii) *for each* $t \in (0, T]$ *the inclusion* $u(t) \in \mathcal{D}(A(t))$ *occurs*;

(iii) *this function solves for* $t \in (0, T]$ *equation* (5.4.1) *subject to the initial condition* $u(0) = u_0$.

Theorem 5.4.3 *Let conditions* (P1)–(P2) *of the present section hold. Then the following assertions are true:*

(a) *the Cauchy problem* (5.4.3)–(5.4.4) *is uniformly well-posed;*

(b) *if* $u_0 \in X$ *and* $f \in C^{\alpha}([0, T]; X)$, $0 < \alpha < 1$, *then a classical solution* u *of the Cauchy problem* (5.4.1)–(5.4.2) *exists, is unique and is given by formula* (5.4.8);

(c) *if, in addition to item* (b), $u_0 \in \mathcal{D}$, *then formula* (5.4.8) *gives a strong solution of the Cauchy problem* (5.4.1)–(5.4.2).

Subsequent studies place special emphasis on the question of solvability of the Cauchy problem for a **quasilinear equation of parabolic type**, whose statement is as follows:

$$u' = A(t, u)\, u + f(t, u)\,, \tag{5.4.9}$$

$$u(0) = u_0\,, \tag{5.4.10}$$

where the operator function $A(t, u)$ takes the values in a set of closed linear operators whose domains are dense. Of special interest is the local solvability of the Cauchy problem at hand under the assumption that the there is a neighborhood of the point $(0,\ u_0)$ of proper type

$$U_\varepsilon = \{ (t,\ u) \colon\ 0 \le t \le \varepsilon,\ \| u - u_0 \| \le \varepsilon \} . \tag{5.4.11}$$

The operator functions $A(t, u)$ and $f(t, u)$ are defined in the specified neighborhood. Suppose also that the operator $A_0 = A(0,\ u_0)$ generates a strongly continuous analytic semigroup or, what amounts to the same things on account of Theorem 5.2.8, there is a half-plane $\operatorname{Re}\lambda > \omega$ in which the **resolvent** of the operator A_0 obeys the estimate

$$\| R(\lambda,\ A_0) \| \le \frac{C}{|\lambda - \omega|} . \tag{5.4.12}$$

One assumes, in addition, that the point $(0,\ u_0)$ has some neighborhood U_ε in which the domain of the operator $A(t, u)$ does not depend on the variable t. This means that for each $(t,\ u) \in U_\varepsilon$

$$\mathcal{D}(A(t, u)) = \mathcal{D} . \tag{5.4.13}$$

When the operator function $A(t, u)$ is smooth enough, the parabolic type of equation (5.4.9) shall remain in full force in some neighborhood of the point $(0,\ u_0)$. It is required that the function u is such that there exist a complex value λ with $\operatorname{Re}\lambda > \omega$, a real number $\beta \in (0,\ 1]$ and a constant $L > 0$ such that for all (t, u), $(s, v) \in U_\varepsilon$ the estimate is valid:

$$\Big\| [A(t, u) - A(s, v)]\ R(\lambda,\ A_0) \Big\| \le L \left(|t - s|^\beta + \| u - v \| \right) . \tag{5.4.14}$$

In this line, a similar condition is imposed on the function f saying that for all (t, u), $(s, v) \in U_\varepsilon$

$$\| f(t, u) - f(s, v) \| \le L \left(|t - s|^\beta + \| u - v \| \right) . \tag{5.4.15}$$

Conditions (5.4.12)–(5.4.15) allow to prove the local solvability by means of the **method of "fixed" coefficients**. As usual, this amounts to reducing equation (5.4.9) related to a fixed function $v(t)$ to the following one:

$$u'(t) = A\big(t, v(t)\big)\, u(t) + f\big(t,\ v(t)\big) , \tag{5.4.16}$$

where v stands in place of u among the arguments of the functions A and f. Holding, for instance, the function v fixed in (5.4.16), we arrive at the linear differential equation related to the function u. If follows from the foregoing that in this case the operator coefficient depends solely on t and admits the form

$$A_v(t) = A\big(t, v(t)\big),$$

while the corresponding nonhomogeneous term becomes

$$f_v(t) = f\big(t, v(t)\big).$$

Adopting similar ideas and joining with the initial condition, we might set up the linear Cauchy problem

$$\text{(5.4.17)} \qquad \begin{cases} u'(t) = A_v(t)\,u(t) + f_v(t), \\ u(0) = u_0 . \end{cases}$$

When the function v happens to be of Hölder's type, it is plain to show that the equation involved in problem (5.4.17) satisfies both conditions (P1)–(P2) and, moreover, can be written as

$$\text{(5.4.18)} \qquad u(t) = V_v(t,0)\,u_0 + \int_0^t V_v(t,s)\,f_v(s)\,ds,$$

where the evolution operator $V_v(t,s)$ corresponds to the operator coefficient $A_v(t)$. Observe that relation (5.4.18) is equivalent, in a certain sense, to the Cauchy problem (5.4.17). On the other hand, equations (5.4.9) and (5.4.17) coincide for $u = v$, making it possible to treat the right-hand side of relation (5.4.18) as an operator carrying every function v into a function u. With these ingredients, the Cauchy problem (5.4.9)–(5.4.10) amounts to the problem of determining a fixed point of the operator (5.4.18). In the appropriate functional spaces any operator so defined satisfies the contraction mapping principle (Theorem 5.1.91), by means of which we establish the unique local solvability of the quasilinear Cauchy problem which interests us.

Theorem 5.4.4 *Let conditions* (5.4.12)–(5.4.15) *hold and* $u_0 \in \mathcal{D}$. *Then there exists* $\varepsilon > 0$ *such that a strong solution* u *of the Cauchy problem* (5.4.9)–(5.4.10) *exists and is unique on the segment* $[0, \varepsilon]$.

5.5 Boundary value problems for elliptic differential equations of the second order

We give below the relevant prerequisities from the theory of linear differential equations in Banach spaces. The books and articles by Balakrishnan (1960), V. Gorbachuk and M. Gorbachuk (1984), Ivanov et al. (1995), Krein and Laptev (1962, 1966a,b), Laptev (1968), Sobolevsky (1968), Trenogin (1966) are devoted to this subject. In this section we consider direct boundary value problems for abstract elliptic differential equations of the second order. Let X be a Banach space and A be a closed linear operator, whose domain is dense in the space X. The **elliptic differential equation** of the second order

$$u''(t) = A\,u(t) + f(t)\,, \qquad 0 < t < T\,, \tag{5.5.1}$$

will be of special investigations in the sequel. There are several approaches to the concept of elliptic type with regard to equation (5.5.1). We accept here the notion involving the positivity of the operator A. By definition, the operator A is said to be **positive** if the real half-line $\lambda \le 0$ enters the resolvent set of the operator A and, in addition, there is a constant $C > 0$ such that for all numbers $\lambda \ge 0$ the estimate

$$\left\| \left(A + \lambda I \right)^{-1} \right\| \le \frac{C}{\lambda + 1} \tag{5.5.2}$$

is valid. Any positive operator A possesses the positive **square root** $A^{1/2}$, meaning that the positive operator $A^{1/2}$ being squared equals A. For the purposes of the present section we develop the scheme of introducing the square root of the operator A. One way of proceeding is to initiate the construction of the operator

$$A^{-1/2} = \frac{1}{\pi} \int\limits_0^\infty \frac{1}{\sqrt{\lambda}} \left(A + \lambda I \right)^{-1} d\lambda\,. \tag{5.5.3}$$

Estimate (5.5.2) serves to motivate that formula (5.5.3) specifies a bounded operator, so that

$$\| A^{-1/2} \| \le C\,,$$

where C is the same constant as in (5.5.2). We note in passing that the operator A is invertible. Due to this fact we are now in a position to introduce

$$A^{1/2} = \left(A^{-1/2} \right)^{-1}.$$

Being the inverse of a bounded operator, the operator $A^{1/2}$ becomes closed. Moreover, for each $x \in \mathcal{D}(A)$

$$A^{1/2} x = \frac{1}{\pi} \int\limits_0^\infty \frac{1}{\sqrt{\lambda}} \left(A + \lambda I \right)^{-1} A\,x\; d\lambda\,. \tag{5.5.4}$$

When the operator A happens to be unbounded, the domain of the operator $A^{1/2}$ is more broader than that of the operator A. In this case one can prove that the operator $A^{1/2}$ will coincide with the closure of the operator defined by the formula on the right-hand side of (5.5.4). One thing is worth noting here. Being concerned with a positive operator A, one can always define each **real power** of this operator in such a way that the **group property**

$$A^{\alpha+\beta} = A^{\alpha} A^{\beta}, \qquad A^0 = I,$$

will be in full force. The operator $A^{1/2}$ plays a key role in later discussions of equation (5.5.1) and possesses some properties that are more principal than its positivity. True, it is to be shown that the operator $-A^{1/2}$ generates a strongly continuous analytic semigroup $V(t)$. Solutions to equation (5.5.1) can be expressed in terms of the semigroup $V(t)$.

Of special interest are classical solutions to equation (5.5.1) with some modification. A function $u \in C^1([0, T]; X)$ is said to be a **classical solution** to equation (5.5.1) if this function is twice continuously differentiable on the interval $(0, T)$, for each $t \in (0, T)$ goes along with the inclusion

$$u(t) \in \mathcal{D}(A)$$

and obeys equation (5.5.1) on the same interval $(0, T)$.

In what follows we take for granted that the function f involved satisfies either

$$\| f(t) - f(s)\| \leq L\, |t-s|^{\alpha}, \qquad 0 < \alpha \leq 1, \tag{5.5.5}$$

or

$$f \in C([0, T]; \mathcal{D}(A^{1/2}))\,. \tag{5.5.6}$$

A simple observation may be useful as further developments occur.

Theorem 5.5.1 *Let the operator A be positive and the function f satisfy either* (5.5.5) *or* (5.5.6). *Then for a classical solution u of equation* (5.5.1) *there are two elements $u_1, u_2 \in X$ such that the function u is representable by*

$$u(t) = V(t)\, A^{-1/2}\, u_1 + V(T-t)\, A^{-1/2}\, u_2 - \frac{1}{2} \int_0^T V(|t-s|)\, A^{-1/2} f(s)\, ds\,, \tag{5.5.7}$$

where $V(t)$ refers to the semigroup generated by the operator $-A^{1/2}$. Conversely, each pair of elements u_1 and u_2 given by formula (5.5.7) *is a classical solution to equation* (5.5.1). *Moreover, a classical solution u to equation* (5.5.1) *complies with the inclusion*

$$A^{1/2}u \in \mathcal{C}([0,T];X)\,. \tag{5.5.8}$$

Two boundary conditions will appear in what follows, whose aims and scope are to determine the elements u_1 and u_2 needed in formula (5.5.7) with further passage to the new variables

$$L_1(u) = \alpha_{11}\, u(0) + \alpha_{12}\, u'(0) + \beta_{11}\, u(T) + \beta_{12}\, u'(T)\,,$$

$$L_2(u) = \alpha_{21}\, u(0) + \alpha_{22}\, u'(0) + \beta_{21}\, u(T) + \beta_{22}\, u'(T)\,.$$

The **boundary value problem** we have considered so far consists of finding a classical solution to equation (5.5.1) supplied by the boundary conditions

$$L_1(u) = f_1\,, \qquad L_2(u) = f_2\,, \tag{5.5.9}$$

where $f_1,\, f_2 \in X$. Of great importance are three particular cases given below. The **Cauchy problem** is connected with the values

$$\alpha_{11} = 1\,, \qquad \alpha_{22} = 1$$

and

$$\alpha_{12} = \beta_{11} = \beta_{12} = \alpha_{21} = \beta_{21} = \beta_{22} = 0\,.$$

The **Dirichlet problem** will be completely posed once we accept

$$\alpha_{11} = 1\,, \qquad \beta_{21} = 1$$

and

$$\alpha_{12} = \beta_{11} = \beta_{12} = \alpha_{21} = \alpha_{22} = \beta_{22} = 0\,.$$

Finally, when the equalities

$$\alpha_{12} = \beta_{22} = 1$$

and

$$\alpha_{11} = \beta_{11} = \beta_{12} = \alpha_{21} = \alpha_{22} = \beta_{21} = 0$$

are put together, there arises the **Neumann problem**.

Theorem 5.5.2 *Let the operator A be positive, the function*

$$f \in C([0,T]; X)$$

and the function u be defined by equality (5.2.7) *with $V(t)$ denoting the semigroup generated by the operator $-A^{1/2}$. Then*

$$u \in C^1([0,T]; X)$$

and

$$u'(t) = -V(t)\,u_1 + V(T-t)\,u_2 + \frac{1}{2}\int_0^t V(t-s)\,f(s)\,ds - \frac{1}{2}\int_t^T V(s-t)\,f(s)\,ds\,. \tag{5.5.10}$$

With the aid of relations (5.5.7) and (5.5.10) it is plain to reduce the boundary conditions (5.5.9) to the system of equations

$$\begin{cases} D_{11}\,u_1 + D_{12}\,u_2 = h_1\,, \\ D_{21}\,u_1 + D_{22}\,u_2 = h_2\,, \end{cases} \tag{5.5.11}$$

where

$$\begin{cases} D_{11} = \alpha_{11}\,A^{-1/2} - \alpha_{12}\,I + \beta_{11}\,V(T)\,A^{-1/2} - \beta_{12}\,V(T)\,, \\ D_{12} = \alpha_{11}\,V(T)\,A^{-1/2} + \alpha_{12}\,V(T) + \beta_{11}\,A^{-1/2} + \beta_{12}\,I\,, \\ D_{21} = \alpha_{21}\,A^{-1/2} - \alpha_{22}\,I + \beta_{21}\,V(T)\,A^{-1/2} - \beta_{22}\,V(T)\,, \\ D_{22} = \alpha_{21}\,V(T)\,A^{-1/2} + \alpha_{22}\,V(T) + \beta_{21}\,A^{-1/2} + \beta_{22}\,I\,, \end{cases} \tag{5.5.12}$$

$$\begin{cases} h_1 = f_1 + \int_0^T K_1(s)\,f(s)\,ds\,, \\ h_2 = f_2 + \int_0^T K_2(s)\,f(s)\,ds\,, \end{cases} \tag{5.5.13}$$

$$(5.5.14)\qquad \begin{cases} K_1(t) \ = \ \frac{1}{2}\,\alpha_{11}\,V(t)\,A^{-1/2} + \frac{1}{2}\,\alpha_{12}\,V(t) \\ \qquad\qquad +\frac{1}{2}\,\beta_{11}\,V(T-t)\,A^{-1/2} - \frac{1}{2}\,\beta_{12}\,V(T-t)\,, \\ \\ K_2(t) \ = \ \frac{1}{2}\,\alpha_{21}\,V(t)\,A^{-1/2} + \frac{1}{2}\,\alpha_{22}\,V(t) \\ \qquad\qquad +\frac{1}{2}\,\beta_{21}\,V(T-t)\,A^{-1/2} - \frac{1}{2}\,\beta_{22}\,V(T-t)\,. \end{cases}$$

When relations (5.5.11) are put together, there arises a system of equations with the bounded and commuting operator coefficients. The system thus obtained can be resolved by means of **Cramer's method**. A key role here is played by the determinant

$$(5.5.15)\qquad D = \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}\,,$$

which is called the **characteristic determinant** of the boundary value problem (5.5.1), (5.5.9).

As a first step towards the solution of problem (5.5.1), (5.5.9), the main idea is connected with the homogeneous boundary value problem

$$(5.5.16)\qquad \begin{cases} u''(t) \ = \ A\,u(t)\,, & 0 \leq t \leq T\,, \\ L_1(u) = 0\,, & L_2(u) = 0\,, \end{cases}$$

because a solution of problem (5.5.1), (5.5.9) is determined up to a solution of problem (5.5.16). Therefore, to decide for yourself whether a solution of the nonhomogeneous problem (5.5.1), (5.5.9) is unique, a next step is to analyze a possibility of the occurrence of a nontrivial solution to the homogeneous problem (5.5.16). The following theorem supplies the answer to this question.

Theorem 5.5.3 *The boundary value problem* (5.5.16) *has a nontrivial solution if and only if zero is an eigenvalue of the characteristic determinant* (5.5.15).

Therefore, a solution of problem (5.5.1), (5.5.9) is unique if and only if there exists the inverse (not necessarily bounded) operator D^{-1}.

Of importance is the case when

$$(5.5.17)\qquad D^{-1} \in \mathcal{L}(X)\,,$$

which assures that the system (5.5.11) is uniquely solvable for any h_1, h_2 and its solution can be written in simplified form as follows:

$$u_1 = D^{-1}\left(D_{22}\,h_1 - D_{12}\,h_2 \right)\,,$$

$$u_2 = D^{-1}\left(D_{11}\,h_2 - D_{21}\,h_1 \right)\,.$$

Putting these together with (5.5.7) we derive the final expression for the solution in question:

$$u(t) = S_1(t)\, f_1 + S_2(t)\, f_2 + \int_0^T G(t,s)\, f(s)\; ds\,, \tag{5.5.18}$$

where

$$S_1(t) = D^{-1} A^{-1/2} \left[V(t)\, D_{22} - V(T-t)\, D_{21} \right], \tag{5.5.19}$$

$$S_2(t) = -D^{-1} A^{-1/2} \left[V(t)\, D_{12} + V(T-t)\, D_{11} \right], \tag{5.5.20}$$

(5.5.21)

$$\begin{aligned} G(t,s) = & D^{-1} A^{-1/2} \left[V(t) \left(D_{22}\, K_1(s) - D_{12}(s)\, K_2(s) \right) + V(T-t) \right. \\ & \left. \times \left(D_{11}\, K_2(s) - D_{21}\, K_1(s) \right) \right] - \tfrac{1}{2}\, V(|\,t-s\,|)\, A^{-1/2}. \end{aligned}$$

The function $G(t,s)$ is referred to as the **Green function** of the boundary value problem (5.5.1), (5.5.9). Preceding manipulations are summarized in the following statement.

Theorem 5.5.4 *Let the operator A be positive and the operator D involved in* (5.5.15) *be in line with* (5.5.17). *One assumes, in addition, that f_1, $f_2 \in X$ and the function f satisfies either* (5.5.5) *or* (5.5.6). *Then a classical solution u of the boundary value problem* (5.5.1), (5.5.9) *exists, is unique and is given by formula* (5.5.18).

The Neumann boundary conditions

$$u'(0) = f_1\,, \qquad u'(T) = f_2 \tag{5.5.22}$$

complement the further development and can serve as one useful exercise to motivate what is done. Under such a formalization the characteristic determinant is of the form

$$D = V(2T) - I\,.$$

Recall that the operator A is positive, that is, satisfies condition (5.5.2). In view of this, there exists a positive number ε such that the spectrum of the operator $-A^{1/2}$ is located in the half-plane $\operatorname{Re}\lambda \le -\varepsilon$. This is due to the fact that for any $t > 0$ the **spectral radius** of the operator $V(t)$ is less than 1. From such reasoning it seems clear that the operator $V(2T) - I$ is invertible, so that the inclusion

$$\left(V(2T) - I \right)^{-1} \in \mathcal{L}(X)$$

occurs. In this case the Neumann boundary value problem concerned complies with (5.5.17), thereby confirming the following statement.

Theorem 5.5.5 *Let the operator A be positive and the function f satisfy either* (5.5.5) *or* (5.5.6). *Then for any elements f_1, $f_2 \in X$ a classical solution u of the Neumann boundary value problem* (5.5.1), (5.5.22) *exists and is unique.*

It is also worthwhile to warn the reader against the following widespread but wrong reasoning. It would be erroneous to think that in the current framework condition (5.5.17) is the unique possible and cannot be omitted or replaced by the others. Even for the problem with the Dirichlet boundary conditions

$$u(0) = f_1 , \qquad u(T) = f_2 , \tag{5.5.23}$$

condition (5.5.17) fails to hold. In this view, it is reasonable to regard boundary conditions distinguishing via representation (5.5.18) as **regular** and **nonregular** ones. This can be always done in the case when the operator D has the inverse (not necessarily bounded). The only inconvenience is caused by the obstacle that the operators S_1, S_2 and $G(t,s)$ are unbounded. We do not touch here all possible types of boundary conditions and confine ourselves to the Dirichlet boundary data (5.5.23) only. The characteristic determinant of the Dirichlet problem admits the form

$$D = \big(I - V(2T)\big)\, A^{-1},$$

whence it follows that the operator

$$D^{-1} = \big(I - V(2T)\big)^{-1} A$$

is unbounded and has the domain $\mathcal{D}(A)$ imposed at the very beginning. Therefore, the system of equations (5.5.11) followed by the relations

$$\begin{cases} D\, u_1 = D_{22}\, h_1 - D_{12}\, h_2 , \\ D\, u_2 = D_{11}\, h_2 - D_{21}\, h_1 , \end{cases} \tag{5.5.24}$$

cannot be resolved for any f_1, $f_2 \in X$. Indeed, condition (5.5.8) holds true for a classical solution of equation (5.5.1). The meaning of this is that f_1, f_2 are not arbitrary elements and should be suitably chosen from the manifold $\mathcal{D}(A^{1/2})$. This condition is necessary for the Dirichlet problem (5.5.1), (5.5.23) to be solvable and appears to be sufficient too. Indeed, in the case of the Dirichlet boundary conditions formulae (5.5.14) become

$$K_1(t) = \tfrac{1}{2}\, V(t)\, A^{-1/2},$$

$$K_2(t) = \tfrac{1}{2}\, V(T-t)\, A^{-1/2}.$$

From relations (5.5.13) it follows that the elements h_1, $h_2 \in \mathcal{D}(A^{1/2})$ for any f_1, $f_2 \in \mathcal{D}(A^{1/2})$, so that

$$A^{1/2}h_1 = A^{1/2} f_1 + \frac{1}{2} \int_0^T V(s)\, f(s)\, ds\,,$$

$$A^{1/2}h_2 = A^{1/2} f_2 + \frac{1}{2} \int_0^T V(T-s)\, f(s)\, ds\,,$$

implying on the basis of relations (5.5.12) that

$$D_{11} = A^{-1/2}, \qquad D_{12} = V(T)\, A^{-1/2},$$

$$D_{21} = V(T)\, A^{-1/2}, \qquad D_{22} = A^{-1/2}.$$

All this enables us to deduce that the elements on the right-hand side of (5.5.24) belong to the manifold $\mathcal{D}(A)$ and

$$\begin{aligned} A\left(D_{22}\, h_1 - D_{12}\, h_2\right) &= A\left(A^{-1/2}h_1 - V(T)\, A^{-1/2}h_2\right) \\ &= A^{1/2}h_1 - V(T)\, A^{1/2}h_2\,, \\ A\left(D_{11}\, h_2 - D_{21}\, h_1\right) &= A\left(A^{-1/2}h_2 - V(T)\, A^{-1/2}h_1\right) \\ &= A^{1/2}h_2 - V(T)\, A^{1/2}h_1\,. \end{aligned}$$

Consequently, equations (5.5.24) are solvable for any f_1, $f_2 \in \mathcal{D}(A^{1/2})$ and

$$u_1 = \left(I - V(2T)^{-1}\right)\left(A^{1/2}h_1 - V(T)\, A^{1/2}h_2\right),$$

$$u_2 = \left(I - V(2T)^{-1}\right)\left(A^{1/2}h_2 - V(T)\, A^{1/2}h_1\right).$$

Putting these together with (5.5.7) we establish representation (5.5.18) with the members

$$S_1(t) = \left(I - V(2T)\right)^{-1}\left(V(t) - V(2T-t)\right),$$

$$S_2(t) = -\left(I - V(2T)\right)^{-1}\left(V(T+t) + V(T-t)\right),$$

$$\begin{aligned} G(t,s) = \tfrac{1}{2}\left(I - V(2T)\right)^{-1} A^{-1/2}\big(& V(t+s) - V(2T+t-s) \\ & + V(2T-t-s) - V(2T-t+s) - V(|t-s|)\big)\,. \end{aligned}$$

As a final result we get the following assertion.

Theorem 5.5.6 *Let the operator A be positive. If the function f satisfies either* (5.5.5) *or* (5.5.6), *then the Dirichlet problem* (5.5.1), (5.5.23) *is solvable if and only if*

$$f_1, f_2 \in \mathcal{D}(A^{1/2}) . \tag{5.5.25}$$

Moreover, under the constraint (5.5.25) *a classical solution u of the Dirichlet problem* (5.5.1), (5.5.23) *exists and is unique.*

Chapter 6

Abstract Inverse Problems for First Order Equations and Their Applications in Mathematical Physics

6.1 Equations of mathematical physics and abstract problems

The method of abstract differential equations provides proper guidelines for solving various problems with partial differential equations involved. Under the approved interpretation a partial differential equation is treated as an ordinary differential equation in a Banach space. We give below one possible example. Let Ω be a bounded domain in the space $\mathbf{R}^n$, whose boundary is sufficiently smooth. The initial boundary value problem for the heat conduction equation can add interest and aid in understanding. Its statement is as follows:

$$
(6.1.1) \qquad \left\{\begin{array}{ll} u_t = \Delta\, u + f(x,t)\,, & (x,\, t) \in G\,, \\ u(x,0) = \varphi(x)\,, & x \in \Omega\,, \\ u(x,t)\Big|_{\partial\,\Omega \times [0,\,T]} = 0\,, & \end{array}\right.
$$

where $G = \Omega \times [0, T]$ is a cylindrical domain. When adopting $X = L_2(\Omega)$ as a basic Banach space, we introduce in the space X a linear (unbounded) operator $A = \Delta$ with the domain $\mathcal{D}(A) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and call it the **Laplace operator**. The function $u(x,t)$ is viewed as an **abstract function** $u(t)$ of the variable t with values in the space X. Along similar lines, the function $f(x,t)$ regards as a function with values in the space X, while the function $\varphi(x)$ is an element $\varphi \in X$, making it possible to treat the direct problem (6.1.1) as the Cauchy problem in the Banach space X for the ordinary differential equation

$$(6.1.2) \qquad \begin{cases} u'(t) = A\,u(t) + f(t)\,, & 0 \leq t \leq T\,, \\ u(0) = \varphi\,. \end{cases}$$

Recall that the **boundary conditions** for the function u involved in (6.1.1) are included in the domain of the operator A.

The well-founded choice of the basic space X owes a debt to several important properties among which the well-posedness of the Cauchy problem, a need for differential properties of a solution with respect to a spatial variable and others. Such a setting is much applicable for non-normed and locally convex topological vector spaces which are used in place of the Banach space just considered. Also, in some cases the space X is replaced by a scale of Banach spaces. Within a wide range of applications which do arise in the sequel, the space X is presupposed to be Banach unless otherwise is explicitly stated.

The main idea behind a natural approach to problems (6.1.1)–(6.1.2) is that all of the basic relations should occur in a pointwise sense. However, there is some difference between problems (6.1.1) and (6.1.2). The essense of the matter is that the operation of differentiation with respect to t is performed in a dissimilar sense for both cases. In particular; the t-derivative arising in problem (6.1.2) is to be understood as a limit of the corresponding difference relation in the norm of the space X. When solutions of both problems are treated as distributions, the dissimilarity between them will disappear. A generalized solution of problem (6.1.2) can uniquely be associated with a distribution from the class $\mathcal{D}'(G)$ subject to (6.1.1) in a certain sense. Because of this, the exploration of problem (6.1.2) is somewhat different from that of problem (6.1.1) by means of special methods, whose use permits us to reveal some properties peculiar to the regularity of generalized solutions to the heat conduction equation in the class $\mathcal{D}'(G)$. For the purposes of the present chapter it would be sufficient to operate only with continuous solutions of problem (6.1.2). A rigorous definition of such solutions will appear in the next section.

Before proceeding to a common setting, we would like to discuss some statements of inverse problems and their abstract analogs. Recent years have seen plenty of publications in this field with concern of the main mathematical physics equations. We refer the reader to Beals and Protopescu (1987), Birman and Solomyak (1987), Bykhovsky (1957), Duvant and Lions (1972), Fujita and Kato (1964), Guirand (1976), Hejtmanek (1970, 1984), Henry (1981), Ikawa (1968), Jorgens (1968), Kato (1975a,b), Kato and Fujita (1962), Lax and Phillips (1960), Lehner and Wing (1955), Lekkerkerker and Kaper (1986), Massey (1972), Mizohata (1959a,b, 1977), Montagnini (1979), Phillips (1957, 1959), Richtmyer (1978, 1981), Ribaric (1973), Sanchez-Palencia (1980), Shikhov (1967, 1973), Sobolevsky (1961), Temam (1979), Vidav (1968, 1970), Voight (1984) and Yakubov (1985). Especial attention in the subsequent studies is being paid to some properties peculiar for some operators by means of which the subsidiary information is provided. There exist two typical situations, where in the first the function f as a member of equation (6.1.1) is of the structure

$$f(x,t) = \Phi(x,t)\, p(t)\,, \tag{6.1.3}$$

where the function Φ is known in advance and the unknown function p is sought. In trying to recover the function p we have to absorb some additional information. One way of proceeding is to describe the solution behavior at a fixed point of the domain Ω by the relation

$$u(x_0,t) = \psi(t)\,, \qquad 0 \le t \le T\,, \tag{6.1.4}$$

where ψ is some known function defined on the segment $[0, T]$. Other ideas with abstract forms are connected with the following setting. When working in the space $Y = \mathbf{R}$, we assume that the functions ψ and p fall into the category of abstract functions of the variable t with values in the space Y. The symbol $\Phi(t)$ designates, as usual, the operator of multiplication of an element $p \in Y$ by the function $\Phi(x,t)$ being viewed as a function of the variable x with a fixed value t. If the function Φ is sufficiently smooth (for example, under the premise that for any fixed argument $t \in [0, T]$ the function $\Phi(x,t)$ as a function of the variable x belongs to the space $L_2(\Omega)$), then $\Phi(t)$ acts from the space Y into the space X. When this is the case, we may attempt relation (6.1.3) in the abstract form

$$f(t) = \Phi(t)\, p(t)\,, \tag{6.1.5}$$

where the operator $\Phi(t) \in \mathcal{L}(Y, X)$ for any $t \in [0, T]$.

In the space $X = L_2(\Omega)$ the operator B acting in accordance with the rule

$$B\, u = u(x_0)$$

is defined on the set of all continuous in Ω functions and is aimed to derive an abstract form of relation (6.1.4). On that space the operator B becomes a linear functional from X into Y, so that relation (6.1.4) can be rewritten as

$$(6.1.6) \qquad B\,u(t) = \psi(t)\,, \qquad 0 \le t \le T\,.$$

When equation (6.1.2) is adopted as a relation satisfied in a pointwise sense, we take for granted that for any fixed value $t \in [0,\, T]$ the element $u(t)$ belongs to the space $\mathcal{D}(A) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. Embedding theorems for Sobolev's spaces yield $W_2^2(\Omega) \subset C(\bar{\Omega})$ for any bounded three-dimensional domain Ω and, therefore, the operator B is defined on any solution to equation (6.1.2). However, this operator fails to be bounded and even closed. On the other hand, if the manifold $\mathcal{D}(A)$ is equipped with the graph norm, that is, with the $W_2^2(\Omega)$-norm, then the operator B acting from $\mathcal{D}(A)$ into Y becomes bounded:

$$(6.1.7) \qquad B \in \mathcal{L}(\mathcal{D}(A),\, Y)\,.$$

Additional information may be prescribed in an integral form as well. The following example helps motivate what is done. The subsidiary information here is that

$$(6.1.8) \qquad \int\limits_{\Omega} u(x,t)\, w(x)\; dx = \psi(t)\,, \qquad 0 \le t \le T\,.$$

Relation (6.1.8) admits an interesting physical interpretation as a result of measuring the temperature u by a perfect sensor of finite size and, in view of this, performs a certain averaging over the domain Ω. Observe that (6.1.8) takes now the form (6.1.6) with the operator B incorporated:

$$B\,u = \int\limits_{\Omega} u(x)\, w(x)\; dx\,.$$

Holding $w \in L_2(\Omega)$ fixed we will show that

$$(6.1.9) \qquad B \in \mathcal{L}(X,\, Y)\,.$$

Indeed, when (6.1.8) is the outcome of measuring the temperature by a real sensor, there is a good reason to accept that the function ω is smooth and finite in the domain Ω. In the case of a sufficiently smooth boundary of the domain Ω the Laplace operator is self-adjoint and

$$\mathcal{D}(A^*) = W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega)\,.$$

Let now $\omega \in \mathcal{D}(A^*)$. Then for any element $u \in \mathcal{D}(A)$ we have

$$B\,A\,u = \big(A\,u,\, w\big)_{L_2(\Omega)} = \big(u,\, A^* w\big)_{L_2(\Omega)} = \int\limits_{\Omega} u(x)\,\Delta\, w(x)\; dx$$

and both operators B and $B\,A$ are bounded. Moreover, the closure of $B\,A$, which is defined by means of the relation

$$\overline{B\,A}\,u = \int\limits_{\Omega} u(x)\,\Delta\, w(x)\; dx\,,$$

complies with the inclusion

$$\overline{B\,A} \in \mathcal{L}(\,X,\,Y\,)\,. \tag{6.1.10}$$

This serves to motivate that the operator B possesses a certain **smoothing property**. In what follows, within the framework of the abstract inverse problem (6.1.2), (6.1.5)–(6.1.6), conditions (6.1.7), (6.1.9)–(6.1.10) are supposed to be true.

Of special interest is one modeling problem in which the function f involved in equation (6.1.1) is representable by

$$f(x,t) = \Phi(x,t)\; p(x)\,, \tag{6.1.11}$$

where the function Φ is known in advance and the coefficient p is sought. Additional information is needed to recover this coefficient and is provided in such a setting by the condition of **final overdetermination**

$$u(x,T) = \psi(x)\,, \qquad x \in \Omega\,. \tag{6.1.12}$$

The abstract statement of the inverse problem concerned should cause no difficulty. We proceed as usual. This amounts to further treatment of the function p as an unknown element of the Banach space $X = L_2(\Omega)$. With this correspondence established, relation (6.1.11) admits the form

$$f(t) = \Phi(t)\,p\,, \tag{6.1.13}$$

where $\Phi(t)$ regards to the same operator performing the multiplication by the function $\Phi(x,t)$ being viewed as a function of the variable x for a fixed value t. If for any fixed value t the function $\Phi(x,t)$ is measurable and essentially bounded as a function of the variable x, then $\Phi(t) \in \mathcal{L}(X)$

for any fixed value $t \in [0, T]$. What is more, condition (6.1.12) can be rewritten as

$$u(T) = \psi , \tag{6.1.14}$$

where ψ is a known element of the space X.

It is necessary to indicate a certain closeness between the statements of the inverse problems (6.1.1), (6.1.3)–(6.1.4) and (6.1.1), (6.1.11)–(6.1.12). In the first one the unknown coefficient of the source term depends solely on t and in this case the subsidiary information is also provided by a function depending on t. In the second one the same principle is acceptable for action: the unknown part of the source and the function built into the overdetermination condition depend only on one and the same variable x. Such a closeness emerged in the abstract analogs of the inverse problems posed above. This is especially true for (6.1.2), (6.1.5)–(6.1.6) and (6.1.2), (6.1.13)–(6.1.14). In the first setting being concerned with a function of the variable t with values in the space Y we are trying to find a function of the variable t with values in the space Y. In another abstract inverse problem an element $p \in X$ is unknown and the subsidiary information is that ψ is an element of the same space X.

There are no grounds to conclude that the principle formulated above is not sufficiently universal to cover on the same footing all possible statements of inverse problems arising in theory and practice. In dealing with problems of another nature (being not inverse) one can encounter other statements in which the unknown parameter and the subsidiary information are connected with functions of several variables. Such settings find a wide range of applications, but do not fit our purposes within the uniform abstract framework and need special investigations. Just for this reason we focus our attention here on such statements of inverse problems, whose constructions not only provide one with insight into what is going on in general from a uniform viewpoint, but also permit one to get quite complete answers to several principal questions of the theory.

6.2 The linear inverse problem with smoothing overdetermination: the basic elements of the theory

Let X and Y be Banach spaces. We have at our disposal a closed linear operator A in the space X, whose domain is dense. Under the natural premises $\Phi \in C([0, T]; \mathcal{L}(Y, X))$, $F \in C([0, T]; X)$, $B \in \mathcal{L}(X, Y)$, $\psi \in C([0, T]; Y)$, $u_0 \in X$ we consider the **inverse problem** of finding a pair of the functions $u \in C([0, T]; X)$ and $p \in C([0, T]; Y)$ from the set of relations

$$u'(t) = A\,u(t) + f(t)\,, \qquad 0 \le t \le T\,, \tag{6.2.1}$$

$$u(0) = u_0\,, \tag{6.2.2}$$

$$f(t) = \Phi(t)\,p(t) + F(t)\,, \qquad 0 \le t \le T\,, \tag{6.2.3}$$

$$B\,u(t) = \psi(t)\,, \qquad 0 \le t \le T\,. \tag{6.2.4}$$

Since only continuous solutions are considered, equation (6.2.1) is to be understood in a sense of distributions. Recall that the the operator A is closed. Consequently, the domain of A becomes a Banach space in the graph norm, making it possible to deal with distributions taking the values from the manifold $\mathcal{D}(A)$. The meaning of relation (6.2.1) is that

$$u \in \mathcal{D}'((0, T), \mathcal{D}(A))$$

and for any function $\varphi \in \mathcal{D}(0, T)$ we should have

$$-\langle u, \varphi' \rangle = A\,\langle u, \varphi \rangle + \langle f, \varphi \rangle\,.$$

Because the function u is continuous from the segment $[0, T]$ into the space X, the remaining relations (6.2.2)–(6.2.4) are meaningful.

At first glance the well-posedness of the Cauchy problem (6.2.1)–(6.2.2) needs certain clarification. We assume that the operator A generates a strongly continuous semigroup $V(t)$, that is, the class of operators in $\mathcal{L}(X)$ which are defined for $t \ge 0$ and satisfy the following conditions:

(1°) *for any $x \in X$ the function $V(t)\,x$ is continuous for all $t \ge 0$ in the space X;*

(2°) $V(0) = I$;

(3°) $V(t+s) = V(t)\,V(s)$ *for any t, $s \ge 0$.*

The operator A is just a strong derivative at zero of the semigroup it generates and, in so doing,

$$\mathcal{D}(A) = \left\{ x \in X\colon\ \exists \lim_{t \to 0} \frac{V(t) - V(0)}{t}\, x \right\},$$

$$A\,x = \lim_{t \to 0} \frac{V(t) - V(0)}{t}\, x\,.$$

For the operator A to generate a strongly continuous semigroup the Hille–Yosida condition is necessary and sufficient, saying that there is a whole ray $\lambda > \lambda_0$ contained in the resolvent set of the operator A and there is a constant $M > 0$ such that on that ray its resolvent $R(\lambda, A)$ obeys for any positive integer n the estimate

$$\| R(\lambda, A)^n \| \le \frac{M}{(\lambda - \lambda_0)^n} . \tag{6.2.5}$$

This condition holds true if and only if the Cauchy problem (6.2.1)–(6.2.2) is uniformly well-posed in the class of strong solutions

$$u \in C^1([0, T]; X) \bigcap C([0, T]; \mathcal{D}(A)) .$$

The Hille–Yosida condition is also sufficient for this problem to be well-posed in the class of distributions. For more a detailed exposition of the well-posedness of the Cauchy problem and the relevant properties of operator semigroups we recommend to see Balakrishnan (1976), Clément et al. (1987), Dunford and Schwartz (1971a,b,c), Fattorini (1983), Goldstein (1985), Henry (1981), Hille and Phillips (1957), Kato (1966), Krein (1967), Mizohata (1977), Pazy (1983), Trenogin (1980), Yosida (1965). It should be noted here that under the conditions imposed above the Cauchy problem (6.2.1)–(6.2.2) has a **continuous solution** $u \in C([0, T]; X)$ for any function $f \in C([0, T]; X)$ and any element $u_0 \in X$, this solution is unique in the indicated class of functions and is given by the formula

$$u(t) = V(t)\, u_0 + \int_0^t V(t-s)\, f(s)\, ds . \tag{6.2.6}$$

If, in addition, $f \in C^1([0, T]; X) + C([0, T]; \mathcal{D}(A))$ and $u_0 \in \mathcal{D}(A)$, then formula (6.2.6) specifies a **strong solution**

$$u \in C^1([0, T]; X) \bigcap C([0, T]; \mathcal{D}(A)) .$$

The assumptions to follow are concerned with some properties of the operator B, which is in charge of surplus information in the form of overdetermination. This operator is supposed to possess a **smoothing effect**, meaning

$$B\,,\ \overline{BA} \in \mathcal{L}(X, Y) . \tag{6.2.7}$$

Theorem 6.2.1 *Let the closed linear operator A with a dense domain generate a strongly continuous semigroup in the space X, condition* (6.2.7) *hold and* $\Phi \in \mathcal{C}([0,\,T];\,\mathcal{L}(Y,X))$, $F \in \mathcal{C}([0,\,T];\,X)$, $u_0 \in X$ *and* $\psi \in \mathcal{C}^1([0,\,T];\,Y)$. *If for any* $t \in [0,\,T]$ *the operator* $B\,\Phi(t)$ *is invertible,*

$$\big(B\,\Phi\big)^{-1} \in \mathcal{C}([0,\,T];\,\mathcal{L}(Y))$$

and the compatibility condition $B\,u_0 = \psi(0)$ *holds, then a solution* $u,\,p$ *of the inverse problem* (6.2.1)–(6.2.4) *exists and is unique in the class of functions*

$$u \in \mathcal{C}([0,\,T];\,X)\,, \qquad\qquad p \in \mathcal{C}([0,\,T];\,Y)\,.$$

Proof Formula (6.2.6) implies that a solution of the Cauchy problem (6.2.1)–(6.2.2) can be expressed by

$$u(t) \;=\; V(t)\,u_0 + \int\limits_0^t V(t-s)\,\Phi(s)\,p(s)\;ds + \int\limits_0^t V(t-s)\,F(s)\;ds\,.$$

Therefore, condition (6.2.4) is equivalent to the following equation:

$$B\left(V(t)\,u_0 + \int\limits_0^t V(t-s)\,\Phi(s)\,p(s)\;ds + \int\limits_0^t V(t-s)\,F(s)\;ds\right) = \psi(t)\,, \qquad 0 \le t \le T\,. \tag{6.2.8}$$

The equation thus obtained contains only one unknown function p and the question of existence and uniqueness of a solution of the inverse problem (6.2.1)–(6.2.4) amounts to the question of existence and uniqueness of a continuous solution to equation (6.2.8). At the next stage we are going to show that equation (6.2.8) can be differentiated. The following lemmas will justify the correctness of this operation.

Lemma 6.2.1 *For any* $u_0 \in X$ *the function* $g(t) = B\,V(T)\,u_0$ *is continuously differentiable on the segment* $[0,\,T]$ *and*

$$g'(t) \;=\; \overline{B\,A}\;V(t)\,u_0\,.$$

Proof Holding $\lambda \in \rho(A)$ fixed and setting

$$u_1 = \big(A - \lambda\,I\big)^{-1}\,u_0\,,$$

we thus have $u_1 \in \mathcal{D}(A)$ and

$$\big(V(t)\,u_1\big)' = V(t)\,A\,u_1 = A\,V(t)\,u_1\,.$$

Furthermore, it is straightforward to verify that

$$g(t) = B\,V(t)\,\big(A - \lambda\,I\big)\,u_1 = B\,A\,V(t)\,u_1 - \lambda\,B\,V(t)\,u_1$$

and, since both operators $B\,A$ and B are bounded, find that

$$\begin{aligned} g'(t) &= \overline{B\,A}\,\big(V(t)\,u_1\big)' - \lambda\,B\,\big(V(t)\,u_1\big)' \\ &= \overline{B\,A}\;V(t)\,A\,u_1 - \lambda\,B\,V(t)\,A\,u_1 \\ &= \overline{B\,A}\;V(t)\,\big(A - \lambda\,I\big)\,u_1 \\ &= \overline{B\,A}\;V(t)\,u_0\,. \end{aligned}$$

■

Lemma 6.2.2 *On the segment* $[0,\,T]$ *the function*

$$g(t) = B\int_0^t V(t-s)\,f(s)\;ds$$

is continuously differentiable for any continuous in X *function* f *and*

$$g'(t) = \overline{B\,A}\int_0^t V(t-s)\,f(s)\;ds + B\,f(t)\,.$$

Proof By analogy with the above lemma the proof is simple to follow. Holding $\lambda \in \rho(A)$ fixed and setting, by definition,

$$h(t) = \big(A - \lambda\,I\big)^{-1} f(t)\,,$$

we deduce that

$$\begin{aligned} A\,h(t) &= \big(\big(A - \lambda\,I\big) + \lambda\,I\big)\,\big(A - \lambda\,I\big)^{-1} f(t) \\ &= f(t) + \big(A - \lambda\,I\big)^{-1} f(t)\,, \end{aligned}$$

so that

$$h\,,\ A\,h\ \in\ \mathcal{C}([0,\,T];\,X)\,. \tag{6.2.9}$$

On the other hand,

$$\begin{aligned} g(t) &= B\int_0^t V(t-s)\,(\,A-\lambda\,I\,)\,h(s)\,ds \\ &= (\,B\,A-\lambda\,B\,)\int_0^t V(t-s)\,h(s)\,ds\,. \end{aligned}$$

From condition (6.2.9) it follows that

$$\Big(\int_0^t V(t-s)\,h(s)\,ds\Big)' = \int_0^t V(t-s)\,A\,h(s)\,ds + h(t)$$

and, due to the boundedness of both operators $B\,A$ and B,

$$g'(t) = (\overline{B\,A}-\lambda\,B)\Big(\int_0^t V(t-s)\,A\,h(s)\,ds + h(t)\Big)\,.$$

Since $B\,A\subset\overline{B\,A}$, we find that

$$g'(t) = \overline{B\,A}\int_0^t V(t-s)\,(\,A-\lambda\,I\,)\,h(s)\,ds + (\,\overline{B\,A}-\lambda\,B\,)\,h(t)\,.$$

From the definition of the function h it seems clear that

$$(\,A-\lambda\,I\,)\,h(t) = f(t)$$

or, what amounts to the same,

$$A\,h(t) = \lambda\,h(t) + f(t)\,.$$

With the relation $h(t)\in\mathcal{D}(A)$, $t\in[0,\,T]$, in view, we deduce that

$$\overline{B\,A}\;h(t) = B\,A\,h(t)$$

and

$$\big(\overline{BA} - \lambda B\big)\, h(t) = B\,A\,h(t) - \lambda\,B\,h(t)$$
$$= B\big(\lambda\,h(t) + f(t)\big) - \lambda\,B\,h(t)$$
$$= B\,f(t)\,.$$

In this line, we get

$$g'(t) = \overline{BA} \int\limits_0^t V(t-s)\, f(s)\; ds + B\,f(t)$$

and the second lemma is completely proved.■

Returning to the proof of Theorem 6.2.1 observe that the left-hand side of equation (6.2.8) is continuously differentiable on account of the preceding lemmas. The compatibility condition $B\,u_0 = \psi(0)$ ensures the equivalence between (6.2.8) and its differential implication taking for now the form

$$\overline{BA}\; V(t)\, u_0 + \overline{BA} \int\limits_0^t V(t-s)\, \Phi(s)\, p(s)\; ds + B\,\Phi(t)\, p(t)$$

$$+ \overline{BA} \int\limits_0^t V(t-s)\, F(s)\; ds + B\,F(t) = \psi'(t)$$

and relying on the formulae derived in proving Lemmas 6.2.1–6.2.2. By assumption, the operator $B\,\Phi(t)$ is invertible. Hence, multiplying the preceding equation by $\big(B\,\Phi(t)\big)^{-1}$ from the left yields the **Volterra integral equation** of the second kind

$$p(t) = p_0(t) + \int\limits_0^t K(t,s)\, p(s)\; ds\,, \tag{6.2.10}$$

where

$$p_0(t) = \big(B\,\Phi(t)\big)^{-1} \Big(\psi'(t) - \overline{BA}\, V(t)\, u_0 - \overline{BA} \int\limits_0^t V(t-s)\, F(s)\; ds - B\,F(t)\Big)\,, \tag{6.2.11}$$

$$K(t,s) = -\big(B\,\Phi(t)\big)^{-1}\, \overline{BA}\; V(t-s)\, \Phi(s)\,. \tag{6.2.12}$$

From the conditions of the theorem and relations (6.2.11)–(6.2.12) we deduce that the function p_0 is continuous on the segment $[0, T]$ and the operator kernel $K(t, s)$ is strongly continuous for $0 \le s \le t \le T$. These properties are sufficient for the existence and uniqueness of the solution to the integral equation (6.2.10) in the class of continuous functions in light of results of Section 5.1 and thereby the theorem is completely proved. ■

Having established the unique solvability of the inverse problem (6.2.1)–(6.2.4) we should decide for ourselves whether the solution in question is continuously dependent on the available input data.

Theorem 6.2.2 *Under the conditions of Theorem* 6.2.1 *there exists a positive constant* $M = M(A, B, \Phi, T)$ *such that a solution* u, p *of the inverse problem* (6.2.1)–(6.2.4) *satisfies the estimates*

$$\| u \|_{C([0,T]; X)} \le M \Big(\| u_0 \|_X + \| \psi \|_{C^1([0,T]; Y)} + \| F \|_{C([0,T]; X)} \Big) ,$$

$$\| p \|_{C([0,T]; Y)} \le M \Big(\| u_0 \|_X + \| \psi \|_{C^1([0,T]; Y)} + \| F \|_{C([0,T]; X)} \Big) .$$

Proof To derive the second inequality one can involve the integral equation (6.2.10), whose solution does obey the estimate

$$\| p \|_{C([0,T]; Y)} \le M_1 \, \| p_0 \|_{C([0,T]; Y)} ,$$

where the constant M_1 depends only on the kernel of this equation, that is, on A, B, Φ and T. On the other hand, relation (6.2.11) implies that

$$\| p_0 \|_{C([0,T]; Y)} \le M_2 \Big(\| u_0 \|_X + \| \psi \|_{C^1([0,T]; Y)} + \| F \|_{C([0,T]; X)} \Big) ,$$

which justifies the desired estimate for the function p. Here the constant M_2 depends only on A, B, Φ and T. With regard to the function u formula (6.2.6) applies equally well to the decomposition

$$f(t) = \Phi(t) \, p(t) + F(t) ,$$

yielding

$$\| u \|_{C([0,T]; X)} \le M_3 \Big(\| u_0 \|_X + \| f \|_{C([0,T]; X)} \Big) ,$$

where the constant M_3 depends only on A and T. Since

$$\| f \|_{C([0,T]; X)} \le M_4 \Big(\| p \|_{C([0,T]; Y)} + \| F \|_{C([0,T]; X)} \Big)$$

with constant M_4 depending solely on Φ and T, the desired estimate for the function p is a simple implication of the preceding inequalities and the estimate for the function u we have established at the initial stage of our study. This completes the proof of the theorem.■

It is interesting to know when equation (6.2.1) is satisfied in a pointwise manner. In this regard, there arises the problem of finding out when the solution of the inverse problem concerned satisfies the condition

$$u \in C^1([0,T];X) \bigcap C([0,T];\mathcal{D}(A)).$$

Because the function f involved in the right-hand side of equation (6.2.1) is continuous, the belonging of u to the space $C([0,T];\mathcal{D}(A))$ will be proved if we succeed in showing that $u \in C^1([0,T];X)$. One well-known fact from semigroup theory may be of help in achieving the final aim: if the function $f \in C^1([0,T];X)$ and the element $u_0 \in \mathcal{D}(A)$, then the function u specified by formula (6.2.6) is continuously differentiable on the segment $[0,T]$ in the norm of the space X, so that

$$u'(t) = V(t)\,(A\,u_0 + f(0)) + \int_0^t V(t-s)\,f'(s)\,ds. \tag{6.2.13}$$

If f admits the form

$$f(t) = \Phi(t)\,p(t) + F(t)$$

with the members

$$\Phi \in C^1([0,T];\mathcal{L}(Y,X)), \qquad F \in C^1([0,T];X),$$

then the continuous differentiability of the function p would be sufficient for the function f to be continuously differentiable. Just for this reason a positive answer to the preceding question amounts to proving the continuous differentiability of a solution to the integral equation (6.2.10).

Theorem 6.2.3 *Let the closed linear operator A with a dense domain be the generator of a strongly continuous semigroup in the space X, condition* (6.2.7) *hold, $\Phi \in C^1([0,T];\mathcal{L}(Y,X))$, $F \in C^1([0,T];X)$, $u_0 \in \mathcal{D}(A)$, $\psi \in C^2([0,T];Y)$, for any $t \in [0,T]$ the operator $B\,\Phi(t)$ be invertible and $(B\,\Phi)^{-1} \in C^1([0,T];\mathcal{L}(Y))$. If the compatibility condition $B\,u_0 = \psi(0)$ is fulfilled, then a solution of the inverse problem* (6.2.1)–(6.2.4) *exists and is unique in the class of functions*

$$u \in C^1([0,T];X) \bigcap C([0,T];\mathcal{D}(A)), \qquad p \in C^1([0,T];Y).$$

Proof As stated above, it is sufficient to prove that under the premises of the theorem a solution to the integral equation (6.2.10) is continuously differentiable. This can be done using equation (6.2.10) together with the equation for the derivative $q(t) = p'(t)$. In trying to justify the correctness of differentiating the integral term we take into account the explicit form of the kernel $K(t,s)$ and formula (6.2.13), implying that

$$\left(\int_0^t V(t-s)\,\Phi(s)\,p(s)\,ds \right)' = V(t)\,\Phi(0)\,p(0)$$

$$+ \int_0^t V(t-s)\,\Phi'(s)\,p(s)\,ds + \int_0^t V(t-s)\,\Phi(s)\,p'(s)\,ds\,.$$

One thing is worth noting here. The value $p(0)$ can be found from equation (6.2.10) related to $t = 0$. With the aid of (6.2.11) we deduce that

$$p(0) = p_0(0) = \big(B\,\Phi(0) \big)^{-1} \Big(\psi'(0) - \overline{BA}\,u_0 - B\,F(0) \Big)\,.$$

By formal differentiating of equation (6.2.10) and minor manipulations with the resulting expressions we are led to the relation

$$p'(t) = q_0(t) + \int_0^t K_1(t,s)\,p(s)\,ds + \int_0^t K(t,s)\,p'(s)\,ds\,, \tag{6.2.14}$$

where

$$q_0(t) = p_0'(t) + R(t)\,\overline{BA}\,V(t)\,\Phi(0)\,p_0(0)\,, \tag{6.2.15}$$

$$\begin{aligned} K_1(t,s) &= R'(t)\,\overline{BA}\,V(t-s)\,\Phi(s) \\ &\quad + R(t)\,\overline{BA}\,V(t-s)\,\Phi'(s)\,, \end{aligned} \tag{6.2.16}$$

$$R(t) = -\big(B\,\Phi(t) \big)^{-1}. \tag{6.2.17}$$

By virtue of (6.2.15)–(6.2.17) the continuity of the function q_0 and the strong continuity of the operator kernel $K_1(t,s)$ for $0 \le s \le t \le T$ are stipulated by the premises of the theorem. The strong continuity of the kernel $K(t,s)$ defined by (6.2.12) has been already established and has been taken into account in the current situation.

Let us consider the system of integral equations

$$p(t) = p_0(t) + \int_0^t K(t,s)\,p(s)\,ds\,,$$

$$q(t) = q_0(t) + \int_0^t K_1(t,s)\,p(s)\,ds + \int_0^t K(t,s)\,q(s)\,ds\,,$$

which has a unique continuous solution. This is due to the continuity of the functions p_0 and q_0 and the strong continuity of the operator kernels $K(t,s)$ and $K_1(t,s)$ (for more detail see Section 5.1). Moreover, this solution can be obtained by means of the **successive approximations**

$$\tilde{p}_{n+1}(t) = p_0(t) + \int_0^t K(t,s)\,\tilde{p}_n(s)\,ds\,, \tag{6.2.18}$$

$$\tilde{q}_{n+1}(t) = q_0(t) + \int_0^t K_1(t,s)\,\tilde{p}_n(s)\,ds + \int_0^t K(t,s)\,\tilde{q}_n(s)\,ds\,, \tag{6.2.19}$$

which converge as $n \to \infty$ to the functions p and q, respectively, uniformly over the segment $[0,\,T]$. In light of the initial approximations $\tilde{p}_0 = 0$ and $\tilde{q}_0 = 0$ it is reasonable to accept

$$\tilde{q}_0 = \tilde{p}_0'\,.$$

Assuming that there exists a positive integer n, for which the equality $\tilde{q}_n = \tilde{p}_n'$ is true, we will show by differentiating (6.2.18) that this equality continues to hold for the next subscript. In giving it all the tricks and tacks remain unchanged as in the derivation of equation (6.2.14). The outcome of this is

$$\tilde{p}_{n+1}'(t) = q_0(t) + \int_0^t K_1(t,s)\,\tilde{p}_n(s)\,ds + \int_0^t K(t,s)\,\tilde{p}_n'(s)\,ds\,.$$

Comparison with (6.2.19) shows that $\tilde{q}_{n+1} = \tilde{p}_{n+1}'$. We deduce by induction on n that the equality

$$\tilde{q}_n = \tilde{p}_n'$$

is valid for any positive integer n.

Therefore, the sequence $\tilde{p}_n$ converges as $n \to \infty$ to the function p uniformly over the segment $[0, T]$. At the same time, the sequence $\tilde{p}'_n$ converges uniformly to the function q. From such reasoning it seems clear that p is continuously differentiable and $p' = q$, thereby completing the proof of the theorem.■

The next step of our study is connected with various estimates for the derivatives of the inverse problem solution similar to those obtained in Theorem 6.2.2 and the existing dependence between its smoothness and the smoothness of the input data. The questions at issue can be resolved on the same footing if the derivatives of the inverse problem (6.2.1)–(6.2.4) solution (if any) will be viewed as the continuous solutions of the same problem but with other input data known as the "**problem in variations**". This approach is much applicable in solving nonlinear problems, since the emerging "problem in variations" appears to be linear. However, this approach provides proper guidelines for deeper study of many things relating to linear problems as well.

Theorem 6.2.4 *Let the conditions of Theorem* 6.2.3 *hold, a pair of the functions u, p solve the inverse problem* (6.2.1)–(6.2.4),

$$w_0 = A\,u_0 + f(0)\,, \qquad G = \Phi' p + F'$$

and $\varphi = \psi'$. Then the functions $w = u'$ and $q = p'$ give a continuous solution of the inverse problem

$$w'(t) = A\,w(t) + g(t)\,, \qquad 0 \le t \le T\,, \tag{6.2.20}$$

$$w(0) = w_0\,, \tag{6.2.21}$$

$$g(t) = \Phi(t)\,q(t) + G(t)\,, \quad 0 \le t \le T\,, \tag{6.2.22}$$

$$B\,w(t) = \varphi(t)\,, \qquad 0 \le t \le T\,. \tag{6.2.23}$$

Proof Since the function p is continuously differentiable, the function

$$f = \Phi\,p + F \in C^1([0, T];\, X)\,.$$

Because of this fact, we might rely on formula (6.2.13) as further developments occur. Substituting $g = f'$ and retaining the notations of Theorem 6.2.4, we recast (6.2.13) as

$$w(t) = V(t)\,w_0 + \int_0^t V(t-s)\,g(s)\,ds\,.$$

Comparison of the resulting equality with (6.2.6) provides support for the view that the function w is just a continuous solution of the Cauchy problem (6.2.20)–(6.2.21). The validity of (6.2.22) becomes obvious on the basis of the relations

$$g = f' = \big(\Phi\, p + F\big)' = \Phi' p + \Phi\, p' + F' = \Phi\, p' + \big(\Phi' p + F'\big) = \Phi\, q + G\,.$$

Now formula (6.2.23) is an immediate implication of (6.2.4) due to the boundedness of the operator B and can be obtained by differentiating (6.2.4) with respect to t. This leads to the desired assertion. ■

Corollary 6.2.1 *If the conditions of Theorem* 6.2.1 *hold, then there exists a constant* $M = M(A, B, \Phi, T)$ *such that a solution* u, p *of the inverse problem* (6.2.1)–(6.2.4) *satisfies the estimates*

$$\| u \|_{C^1([0,T];\, X)} \le M \Big(\| u_0\|_{\mathcal{D}(A)} + \| \psi \|_{C^2([0,\,T];\, Y)} + \| F \|_{C^1([0,T];\, X)}\Big),$$

$$\| u \|_{C([0,T];\, \mathcal{D}(A))} \le M \Big(\| u_0\|_{\mathcal{D}(A)} + \| \psi \|_{C^2([0,\,T];\, Y)} + \| F \|_{C^1([0,T];\, X)}\Big),$$

$$\| p \|_{C^1([0,T];\, Y)} \le M \Big(\| u_0\|_{\mathcal{D}(A)} + \| \psi \|_{C^2([0,\,T];\, Y)} + \| F \|_{C^1([0,T];\, X)}\Big).$$

Proof The derivation of these estimates is simple to follow and is based on Theorem 6.2.2 with regard to the inverse problem (6.2.20)–(6.2.23), according to which there exists a constant M_1 such that

$$\| u' \|_{C^1([0,T];\, X)} \le M_1 \Big(\| w_0\|_X + \| \varphi \|_{C^1([0,T];\, Y)} + \| G \|_{C([0,T];\, X)}\Big),$$

$$\| p' \|_{C([0,T];\, Y)} \le M_1 \Big(\| w_0\|_X + \| \varphi \|_{C^1([0,T];\, Y)} + \| G \|_{C([0,T];\, X)}\Big).$$

Arguing as in dealing with equation (6.2.14) we set $p(0) = p_0(0)$, where the function p_0 is defined by (6.2.11). Since

$$w_0 = A\, u_0 + f(0) = A\, u_0 + \Phi(0)\, p_0(0) + F(0)\,,$$

formula (6.2.11) implies the estimate

$$\| w_0 \|_X \le M_2 \Big(\| u_0\|_{\mathcal{D}(A)} + \| \psi \|_{C^1([0,\,T];\, Y)} + \| F \|_{C([0,T];\, X)}\Big).$$

Recall that $G = \Phi' p + F'$. Using the estimate established in Theorem 6.2.2 for the function p we deduce that

$$\| G \|_{C([0,T];\, X)} \le M_3 \Big(\| u_0\|_X + \| \psi \|_{C^1([0,\,T];\, Y)} + \| F \|_{C^1([0,T];\, X)}\Big),$$

$$\| \varphi \|_{C^1([0,T];\, Y)} = \| \psi' \|_{C^1([0,\,T];\, Y)} \le \| \psi \|_{C^2([0,T];\, Y)},$$

thereby justifying the first and third estimates of Corollary 6.2.1. As can readily be observed, the relation

$$A\,u(t) \;=\; u'(t) - \Phi(t)\,p(t) - F(t)$$

holds true on the strength of (6.2.1) and (6.2.3), so that the second inequality arising from Corollary 6.2.24 becomes evident by successively applying the preceding estimates for u' and p to the right-hand side of (6.2.24). This proves the assertion of the corollary. ■

The result obtained in Theorem 6.2.4 may be of help in revealing the dependence between the smoothness of the inverse problem solution and the smoothness of the input data. Being a solution of the inverse problem (6.2.20)–(6.2.23), the functions w and q give the derivatives of the corresponding solution of the inverse problem (6.2.1)–(6.2.4). With this in mind, the conditions of continuous differentiability of the inverse problem (6.2.20)–(6.2.23) solution turn into the conditions of double continuous differentiability of the inverse problem (6.2.1)–(6.2.4) solution.

In trying to obtain these conditions we make use of Theorem 6.2.3 with regard to the inverse problem (6.2.20)–(6.2.23). Careful analysis of its premises shows that it would be sufficient to achieve $\Phi \in \mathcal{C}^2([0,T];\, \mathcal{L}(Y,X))$, $F \in \mathcal{C}^2([0,T];X)$, $A\,u_0 + f(0) \in \mathcal{D}(A)$, $\psi \in \mathcal{C}^3([0,T];Y)$, $B\,u_0 = \psi(0)$ and

$$B\,(A\,u_0 + f(0)) \;=\; \psi'(0)\,.$$

This type of situation is covered by the following assertion.

Corollary 6.2.2 *Let the closed linear operator A with a dense domain be the generator of a strongly continuous semigroup in the space X, condition* (6.2.7) *hold and $\Phi \in \mathcal{C}^2([0,T];\mathcal{L}(Y,X))$, $F \in \mathcal{C}^2([0,T];X)$, $u_0 \in \mathcal{D}(A)$, $A\,u_0 + f(0) \in \mathcal{D}(A)$, $\psi \in \mathcal{C}^3([0,T];Y)$, $B\,u_0 = \psi(0)$ and*

$$B\,(A\,u_0 + f(0)) \;=\; \psi'(0)\,.$$

If for any $t \in [0,T]$ the operator $B\,\Phi(t)$ is invertible and

$$\left(B\,\Phi\right)^{-1} \in \mathcal{C}^1\left([0,T];\,\mathcal{L}(Y)\right),$$

then a solution u, p of the inverse problem (6.2.1)–(6.2.4) *exists and is unique in the class of functions*

$$u \in \mathcal{C}^2([0,T];X)\,, \qquad p \in \mathcal{C}^2([0,T];Y)\,.$$

In this context, one thing is worth noting. This is concerned with the value $f(0)$. Although the function f is supposed to be unknown in the statement of the inverse problem (6.2.1)–(6.2.4), the value $f(0)$ can easily be found from equality (6.2.3), implying that

$$f(0) = \Phi(0)\, p(0) + F(0)\,.$$

Combination of (6.2.11) and (6.2.12) gives for $t = 0$ the equality

$$p(0) = p_0(0) = \big(B\, \Phi(0)\big)^{-1} \big(\psi'(0) - \overline{B\, A}\; u_0 - B\, F(0)\big)$$

and, because of its form, the compatibility conditions

$$A\, u_0 + f(0) \in \mathcal{D}(A)\,, \qquad\qquad B\,\big(A\, u_0 + f(0)\big) = \psi'(0)$$

can be rewritten only in terms of the input data of the inverse problem (6.2.1)–(6.2.4). The reader is invited to carry out the appropriate manipulations on his/her own.

In concluding this section it remains to note that in Theorem 6.2.3 and Corollary 6.2.2 the differentiability condition for the function $\big(B\,\Phi\big)^{-1}$ can be replaced by the continuity condition for the same function. This is due to the fact that the differentiability of $\big(B\,\Phi\big)^{-1}$ is an implication of the well-known formula for the derivative of the inverse operator

$$\big[\mathcal{A}^{-1}(t)\big]' = -\mathcal{A}^{-1}\, \mathcal{A}'(t)\, \mathcal{A}^{-1}(t)\,.$$

A final remark is that the procedure of establishing the conditions under which the inverse problem solution becomes more smooth can be conducted in just the same way as we did before. Using Theorem 6.2.4 with respect to the inverse problem (6.2.20)–(6.2.23) we might set up the inverse problem for the second derivatives of the problem (6.2.1)–(6.2.4) solution. The inverse problem thus obtained is covered by Theorem 6.2.3 and so the conditions of triple continuous differentiability of the problem (6.2.1)–(6.2.4) solution are derived without any difficulties. Going further with this procedure we reach a solution as smooth as we like.

6.3 Nonlinear inverse problems with smoothing overdetermination: solvability

In the present section we carry over the results of Section 6.2 to the case of a **semilinear evolution equation**. Given Banach spaces X and Y, let A be a closed linear operator in the space X with a dense domain. One assumes, in addition, that the operator A generates a strongly continuous semigroup $V(t)$ and the operator B satisfies the following inclusions:

$$B\,,\ \overline{BA} \in \mathcal{L}(X, Y)\,. \tag{6.3.1}$$

We are interested in recovering a pair of the functions $u \in C([0, T]; X)$ and $p \in C([0, T]; Y)$ from the system of relations

$$u'(t) = A\,u(t) + f\big(t,\, u(t),\, p(t)\big)\,, \quad 0 \le t \le T\,, \tag{6.3.2}$$

$$u(0) = u_0\,, \tag{6.3.3}$$

$$B\,u(t) = \psi(t)\,, \qquad 0 \le t \le T\,, \tag{6.3.4}$$

where

$$f\colon\ [0, T] \times X \times Y \mapsto X$$

is a continuous function. Omitting some details related to the concept of the generalized solution to equation (6.3.2) we mean by a **continuous solution of the Cauchy problem** (6.3.2)–(6.3.3) a continuous solution to the integral equation

$$u(t) = V(t)\,u_0 + \int\limits_0^t V(t-s)\,f\big(s, u(s), p(s)\big)\,ds\,. \tag{6.3.5}$$

Furthermore, we take for granted that the function f has the structure

$$f(t, u, p) = f_1(t, u) + f_2(t, u, p)\,. \tag{6.3.6}$$

The main idea behind **decomposition** is that the properties of the function f_1 should be somewhat different from those of the function f_2 due to the limitations imposed. Representation (6.3.6) is similar to (6.2.3) involved in the exploration of the linear inverse problem posed completely in the preceding section. Although the character of (6.3.6) is not quite typical for a nonlinear problem, it may be of help in investigating some particular questions at issue.

Let us introduce the following notations:

$$S_X(a, R) = \big\{\, x \in X\colon\ \|\,x - a\,\|_X < R\,\big\}\,,$$

$$S_X(a, R, T) = \big\{\,(t, x)\colon\ 0 \le t \le T,\ x \in S_X(a, R)\big\}\,.$$

Allowing the function ψ to be differentiable we define the value

$$z_0 = \psi'(0) - \overline{BA}\,u_0 - B\,f_1(0, u_0) \tag{6.3.7}$$

and require then the following:

(A) *the equation* $B\,f_2(0, u_0, p) = z_0$ *with respect to* p *has a solution* $p_0 \in Y$ *and this solution is unique in the space* Y;

(B) *there exists a mapping*

$$f_3\colon\ [0,\, T] \times Y \times Y \mapsto Y$$

such that

$$B\,f_2(t, u, p) = f_3(t, B\,u, p)\,;$$

(C) *there exists a number* $R > 0$ *such that for any* $t \in [0,\, T]$ *the mapping* $z = f_3(t, \psi(t), p)$ *as a function of* p *is invertible in the ball* $S_Y(p_0, R)$ *and has the inverse*

$$p = \Phi(t, z)\,; \tag{6.3.8}$$

(D) *there is a number* $R > 0$ *such that both functions* $f_1(t, u)$ *and* $f_2(t, u, p)$ *are continuous with respect to the totality of variables on the manifold* $S_{X\times Y}\big((u_0,\, p_0), R, T\big)$ *and satisfy thereon the Lipschitz condition with respect to* $(u,\, p)$;

(E) *there is a number* $R > 0$ *such that the mapping* (6.3.8) *is continuous with respect to* (t, z) *on the manifold* $S_Y(z_0, R, T)$ *and satisfies thereon the Lipschitz condition in* z.

Theorem 6.3.1 *One assumes that the closed linear operator* A, *whose domain is dense, generates a strongly continuous semigroup in the space* X *and conditions* (6.3.1) *and* (6.3.6) *hold. Let* $u_0 \in X$, $\psi \in C^1\big([0,\, T];\, Y\big)$ *and* $B\,u_0 = \psi(0)$. *Under conditions (A)–(E), where* z_0 *is defined by* (6.3.7), *there exists a value* $T_1 > 0$ *such that on the segment* $[0,\, T_1]$ *the inverse problem has a solution* u, p *and this solution is unique in the class of functions*

$$u \in C\big([0,\, T];\, X\big)\,, \qquad p \in C\big([0,\, T];\, Y\big)\,.$$

Proof The conditions of the theorem imply that for any functions $u \in C\big([0,\, T];\, X\big)$ and $p \in C\big([0,\, T];\, Y\big)$ the function

$$f(t) = f\big(t,\, u(t),\, p(t)\big)$$

is continuous and the Cauchy problem (6.3.2)–(6.3.3) is equivalent to the integral equation

$$u(t) = V(t)\,u_0 + \int_0^t V(t-s)\,f\big(s, u(s), p(s)\big)\,ds\,, \tag{6.3.9}$$

whence we are led by condition (6.3.4) to

$$B\,V(t)\,u_0 + B\int_0^t V(t-s)\,f(s)\,ds = \psi(t)\,, \qquad 0 \le t \le T\,. \tag{6.3.10}$$

Arguing as in proving Theorem 6.2.1 it is straightforward to verify that by virtue of Lemmas 6.2.1–6.2.2 along with the compatibility condition relation (6.3.10) is equivalent to its differential implication

$$\begin{aligned} \overline{BA}\;V(t)\,u_0 + \overline{BA}\int_0^t V(t-s)\,f\big(s, u(s), p(s)\big)\,ds & \\ + B\,f\big(t, u(t), p(t)\big) = \psi'(t)\,, \quad 0 \le t \le T\,. & \end{aligned} \tag{6.3.11}$$

Furthermore, set

$$g_0(t) = \psi'(t) - \overline{BA}\,V(t)\,u_0 - B\,f_1\big(t,\, V(t)\,u_0\big)\,.$$

From condition (A) and equation (6.3.11) with $t = 0$ it follows that $p(0) = p_0$. Therefore, conditions (B)–(C) assure us of the existence of a sufficiently small value $T > 0$, for which equation (6.3.11) becomes

$$\begin{aligned} p(t) = \Phi\Big(t,\; & g_0(t) + g_1\,\big(t,\, u(t)\big) \\ & + \int_0^t K(t,s)\,f\big(s,\, u(s),\, p(s)\big)\,ds\Big)\,, \end{aligned} \tag{6.3.12}$$

where

$$g_1(t,u) = -B\,\big(\,f_1(t,u) - f_1(t, V(t)\,u_0\,)\,\big)\,, \qquad K(t,s) = -\overline{BA}\;V(t-s)\,.$$

Relations (6.3.9) and (6.3.12) constitute what is called the system of the **Volterra integro-functional equations** for the functions u and p being equivalent to the inverse problem (6.3.2)–(6.3.4).

At the first stage we are going to show that for a sufficiently small value $T > 0$ the system (6.3.9), (6.3.12) has a unique continuous solution. On account of the inclusion $g_0 \in C([0, T]; Y)$ the function $g_1(t, u)$ is continuous with respect to (t, u) on the manifold $S_X(u_0, R, T)$ and satisfies thereon the Lipschitz condition in u. The kernel $K(t, s)$ is strongly continuous for $0 \le s \le t \le T$. The second step is to replace (6.3.12) by the equation

$$p(t) = \Phi\Bigg(t,\ g_0(t) + g_1\big(t, u_1(t)\big) + \int_0^t K(t, s)\, f\big(s, u(s), p(s)\big)\, ds \Bigg), \tag{6.3.13}$$

where

$$u_1(t) \;=\; V(t)\, u_0 + \int_0^t V(t-s)\, f\big(s, u(s), p(s)\big)\, ds\,. \tag{6.3.14}$$

The couple of the resulting equations (6.3.9) and (6.3.12) is evidently equivalent to the system (6.3.9), (6.3.13). At the next stage we consider the metric space

$$Z = C([0, T];\ S_X(u_0, R) \times S_Y(p_0, R))$$

and introduce the operator

$$G\colon\ (u,\, p) \mapsto (u_1,\, p_1)\,,$$

where u_1 is defined by (6.3.14) and p_1 coincides with the right-hand side of (6.3.13).

The theorem will be proved if we succeed in finding a positive number $T > 0$ for which the operator G has a unique fixed point in the space Z.

The Lipschitz condition for the functions Φ and f_1 implies the inequalities

$$\|\, \Phi(t, z) - \Phi\big(t, g_0(t)\big)\| \;\le\; L\ \|\, z - g_0(t)\|\,, \tag{6.3.15}$$

$$\|\, g_1(t, u_1(t))\| \;\le\; L\ \|\, u_1(t) - V(t)\, u_0\,\|\,. \tag{6.3.16}$$

Let $M_1 = \sup_{0 \le t \le T} \|\, V(t)\|$. With this constant in view, representation (6.3.14) yields

$$\|\, u_1(t) - V(t)\, u_0\,\| \;\le\; M_1 \int_0^t \|\, f\big(s, u(s), p(s)\big)\|\, ds\,. \tag{6.3.17}$$

By the definition of the function p_1 we find with the aid of (6.3.15) that

$$\| p_1(t) - \Phi\big(t, g_0(t)\big)\| \leq L \left\| g_1(t, u_1(t)) + \int_0^t K(t,s)\, f\big(s, u(s), p(s)\big)\, ds \right\| .$$

Estimates (6.3.16) and (6.3.17) together provide sufficient background for the validity of the inequality

$$\| g_1(t,\, u_1(t))\| \leq L\, M_1 \int_0^t \| f\big(s,\, u(s),\, p(s)\big)\|\, ds ,$$

which is followed by

$$\| p_1(t) - \Phi\big(t, g_0(t)\big)\| \leq M_2 \int_0^t \| f\big(s, u(s), p(s)\big)\|\, ds \tag{6.3.18}$$

if we agree to involve the constant

$$M_2 = L^2\, M_1 + L \sup_{0 \leq s \leq t \leq T} \| K(t,s)\| .$$

Comparison of (6.3.7) with the relation

$$g_0(0) = \Psi'(0) - \overline{B\, A}\; u_0 - B\, f_1(0, u_0)$$

shows that $g_0(0) = z_0$. Hence conditions (A)–(C) assure us of the validity of the equality

$$\Phi\big(0,\, g_0(0)\big) = p_0 .$$

Moreover, $V(0)\, u_0 = u_0$. From condition (D) it follows that the function f is bounded on $S_{X\times Y}\big((u_0, p_0), R, T\big)$. Due to this property estimates (6.3.17)–(6.3.18) provide the existence of a sufficiently small number T for which the outcome of mapping by the operator G acts in the space Z.

Let

$$(u_{11},\, p_{11}) = G(u_1,\, p_1)$$

and

$$(u_{22},\, p_{22}) = G(u_2,\, p_2) .$$

The symbol $L(f)$ designates the constant arising from the Lipschitz condition for the function f on the manifold $S_{X\times Y}\big((u_0, p_0), R, T\big)$. With this notation in view, (6.3.9) and (6.3.13) imply that

$$\begin{cases} \| u_{22}(t) - u_{11}(t)\| \leq M_1\, L(f)\, T\ \rho\big((u_2, p_2),\, (u_1, p_1)\big) , \\ \\ \| p_{22}(t) - p_{11}(t)\| \leq M_2\, L(f)\, T\ \rho\big((u_2, p_2),\, (u_1, p_1)\big) , \end{cases} \tag{6.3.19}$$

where ρ refers to a metric on the space Z. Estimates (6.3.19) justifies that there exists a sufficiently small value T, for which G becomes a contraction operator in the space Z. Thus, the desired assertion is an immediate implication of the contraction mapping principle, thereby completing the proof of the theorem. ■

It is interesting to find out when a solution of the inverse problem (6.3.2)–(6.3.4) will be differentiable. To provide the validity of this property, conditions (D) and (E) necessitate making some refinements. It will be sensible to replace them by the following ones:

(D_1) *there exists a number $R > 0$ such that both functions f_1 and f_2 are Frechet differentiable on the manifold $S_{X\times Y}\big((u_0, p_0), R, T\big)$ and their partial derivatives $(f_1)_t$, $(f_1)_u$, $(f_2)_u$, $(f_2)_t$ and $(f_2)_p$ are continuous thereon in the operator norm and satisfy the Lipschitz condition with respect to (u, p);*

(E_1) *there exists a number $R > 0$ such that the function* (6.3.8) *is Frechet differentiable on $S_Y(z_0, R, T)$ and its partial derivatives Φ_t and Φ_z are continuous in the operator norm and satisfy thereon the Lipschitz condition in z.*

Theorem 6.3.2 *Let the closed linear operator A with a dense domain generate a strongly continuous semigroup in the space X and conditions* (6.3.1) *and* (6.3.6) *hold, $u_0 \in \mathcal{D}(A)$, $\psi \in C^2\big([0, T]; Y\big)$ and $B u_0 = \psi(0)$. Under conditions (A)–(C), (D_1) and (E_1), where z_0 is defined by* (6.3.7), *there exists a value $T_1 > 0$ such that on the segment $[0, T_1]$ a solution u, p of the inverse problem* (6.3.2)–(6.3.4) *exists and is unique in the class of functions*

$$u \in C^1\big([0, T_1]; X\big) \bigcap C([0, T_1]; \mathcal{D}(A)), \qquad p \in C^1([0, T_1]; Y).$$

Proof First of all observe that conditions (D_1) and (E_1) imply conditions (D) and (E). Hence by Theorem 6.3.1 the inverse problem (6.3.2)–(6.3.3) has a unique continuous solution for all sufficiently small values T. Since the function $f(t, u(t), p(t))$ is continuous, the continuity of the function $A\,u(t)$ is ensured by relation (6.3.2) because of the continuity of $u'(t)$. Therefore, in order to establish the desired assertion, it suffices to justify the differentiability of the functions u and p both.

Recall that these functions give a continuous solution to the system of the integral equations (6.3.9), (6.3.12), which are currently to be differentiated and supplemented by the relevant equations for the functions $w = u'$ and $q = p'$. By the initial assumptions and relation (6.3.6) there exists a value $R > 0$ such that on the manifold $S_{X\times Y}\big((u_0, p_0), R, T\big)$ the function $f(t, u, p)$ is Frechet differentiable and its partial derivatives f_t, f_u and f_p are continuous and satisfy thereon the Lipschitz condition with respect to (u, p) in the operator norm. Moreover, the function

$$g_0(t) = \psi'(t) - \overline{B\,A}\; V(t)\, u_0 - B\; f_1\big(t,\; V(T)\, u_0\,\big)$$

will be continuously differentiable in t on the space Y and the function

$$g_1(t, u) = -B\,\big(\, f_1(t, u) - f_1(t,\; V(t)\, u_0)\big)$$

will be Frechet differentiable on the manifold $S_X(u_0, R, T)$ and its partial derivatives $g_{1,t}$ and $g_{1,u}$ will be continuous thereon and satisfy the Lipschitz condition with respect to u in the operator norm. Being formally differentiated equations (6.3.9) and (6.3.12) lead to the chains of relations

$$\begin{aligned}
(6.3.20)\qquad w(t) \;&=\; V(t)\,\big(\,A\,u_0 + f(0, u_0, p_0)\big) \\
&\quad + \int\limits_0^t V(t-s)\,\big[\, f_t\big(s, u(s), p(s)\big) \\
&\quad + f_u\big(\, s, u(s), p(s)\big)\; w(s) \\
&\quad + f_p\big(\, s, u(s), p(s)\big)\; q(s)\big]\; ds\,,
\end{aligned}$$

$$\begin{aligned}
(6.3.21)\qquad q(t) \;&=\; \Phi_t\Bigg(\, t, g_0(t) + g_1(t, u(t)) \\
&\quad + \int\limits_0^t K(t,s)\, f(s, u(s), p(s))\; ds\Bigg) \\
&\quad + \Phi_z\Bigg(\, t, g_0(t) +\; g_1(t, u(t)) \\
&\quad + \int\limits_0^t K(t,s)\, f\big(\, s, u(s), p(s)\big)\; ds\Bigg) \\
&\quad \times\; \big(\, g_0'(t) + g_{1,t}(t, u(t)) + g_{1,u}(t, u(t))\, w(t) \\
&\quad + K(t,0)\, f(0, u_0, p_0)\big)
\end{aligned}$$

$$+ \int_0^t K(t,s) \Big[f_t(s, u(s), p(s))$$

$$+ f_u(s, u(s), p(s))\, w(s)$$

$$+ f_p(s, u(s), p(s))\, q(s) \Big] \, ds \, .$$

When the functions $u \in \mathcal{C}([0, T]; X)$ and $p \in \mathcal{C}([0, T]; Y)$ are kept fixed, the system (6.3.20)–(6.3.21) becomes a linear system of the Volterra second order integral equations related to the functions w and q. The nonhomogeneous members of this system are continuous and the kernels are strongly continuous. Whence it follows that there exists a unique continuous solution to the preceding system and it remains to establish the relations $w = u'$ and $q = p'$. With this aim, we agree to consider

$$w_0 = A\, u_0 + f(0, u_0, p_0) \, ,$$

$$q_0 = \Phi_t\big(0, g_0(0)\big) + \Phi_z\big(0, g_0(0)\big)$$

$$\times \; \big(g_0'(0) + g_{1,t}(0, u_0) + g_{1,u}(0, u_0)\; w_0$$

$$+ K(0,0)\, f(0, u_0, p_0)\big) \, .$$

We note in passing that equations (6.3.12) and (6.3.21) are equivalent to the following ones:

$$p(t) = \Phi\Big(t, \, g_0(t) + g_1\big(t, \bar{u}(t)\big) + \int_0^t K(t,s)\, f\big(s, u(s), p(s)\big) \, ds \Big), \tag{6.3.22}$$

(6.3.23)

$$q(t) = \Phi_t \Big(t, \, g_0(t) + g_1\big(t, \bar{u}(t)\big)$$

$$+ \int_0^t K(t,s)\, f\big(s, u(s), p(s)\big) \, ds \Big)$$

$$+ \Phi_z \Big(t, \, g_0(t) + g_1\big(t, \bar{u}(t)\big)$$

$$
\begin{aligned}
&+ \int_0^t K(t,s)\, f\big(s, u(s), p(s)\big)\, ds \Big) \\
&\times \Big(g_0'(t) + g_{1,t}(t, \bar{u}(t)) + g_{1,u}(t, \bar{u}(t))\, \bar{w}(t) \\
&+ K(t,0)\, f(0, u_0, p_0) \\
&+ \int_0^t K(t,s) \big[f_t\big(s, u(s), p(s)\big) \\
&+ f_u\big(s, u(s), p(s)\big)\, w(s) \\
&+ f_p\big(s, u(s), p(s)\big)\, q(s) \big]\, ds \Big) ,
\end{aligned}
$$

where $\bar{u}(t)$ are $\bar{w}(t)$ refer to the right-hand sides of relations (6.3.9) and (6.3.20), respectively.

Let

$$
Z_1 = C\big([0,T];\, S_X(u_0,R) \times S_Y(p_0,R) \times S_X(w_0,R) \times S_Y(q_0,R)\big) .
$$

We deal in that metric space with the operator

$$
G_1\colon\ (u,p,w,g) \mapsto (\bar{u}, \bar{p}, \bar{w}, \bar{g}) ,
$$

where $\bar{p}$ and $\bar{g}$ stand for the right-hand sides of (6.3.22) and (6.3.23), respectively. The operator G_1 so constructed is of the same structure as the operator G arising from the proof of Theorem 6.3.1. Following the scheme of proving the preceding theorem we are now in a position to find a sufficiently small value $T > 0$ for which the operator G_1 becomes contractive on the space Z_1. In this view, it is reasonable to employ the method of successive approximations by means of which a solution of the system of equations (6.3.9), (6.3.20), (6.3.22) and (6.3.23) can be obtained. A final result of such manipulations is as follows:

$$
(6.3.24) \qquad \tilde{u}_{n+1}(t) = V(t)\, u_0 + \int_0^t V(t-s)\, f\big(s, \tilde{u}_n(s), \tilde{p}_n(s)\big)\, ds .
$$

Also, the sequences

$$
(6.3.25) \qquad \tilde{w}_{n+1}(t) = V(t)\, \big(A\, u_0 + f(0, u_0, p_0) \big)
$$

$$+\int_0^t V(t-s)\,\big[\,f_t(s,\tilde{u}_n(s),\tilde{p}_n(s))$$
$$+\ f_u\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\,\tilde{w}_n(s)$$
$$+\ f_p\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\,\tilde{q}_n(s)\big]\ ds\,,$$

$$\tilde{p}_{n+1}(t)\ = \Phi\bigg(\,t, g_0(t)+g_1\big(t,\tilde{u}_{n+1}(t)\big) + \int_0^t K(t,s)\,f\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\ ds\bigg)\,, \tag{6.3.26}$$

$$\begin{aligned}\tilde{q}_{n+1}(t)\ = {}& \Phi_t\bigg(\,t,\ g_0(t)+g_1\big(\,t,\ \tilde{u}_{n+1}(t)\big) + \int_0^t K(t,s)\,f\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\ ds\bigg)\\ &+\Phi_z\bigg(\,t,\ g_0(t)+g_1\big(\,t,\ \tilde{u}_{n+1}(t)\big) + \int_0^t K(t,s)\,f\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\ ds\bigg)\\ &\times\ \bigg(g_0'(t)+g_{1,\,t}\big(t,\tilde{u}_{n+1}(t)\big)\\ &+g_{1,\,u}\big(t,\tilde{u}_{n+1}(t)\big)\,\tilde{w}_{n+1}(t)\\ &+K(t,0)\,f(0,u_0,p_0)\\ &+\int_0^t K(t,s)\,\big[\,f_t\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\\ &+f_u\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\,\tilde{w}_n(s)\\ &+f_p\big(\,s,\tilde{u}_n(s),\tilde{p}_n(s)\big)\,\tilde{q}_n(s)\big]\ ds\bigg)\end{aligned} \tag{6.3.27}$$

will be uniformly convergent on the segment $[0,\ T]$. The initial approximations $\tilde{u}_0(t)$, $\tilde{w}_0(t)$, $\tilde{p}_0(t)$ and $\tilde{q}_0(t)$ may be taken to be zero. Via representations (6.3.24) and (6.3.26) it is not difficult to deduce by induction on n that the functions $\tilde{u}_n$ and $\tilde{p}_n$ are continuously differentiable for all n.

With the initial assumptions in view, careful analysis of (6.3.4) leads to

$$f(t)\ =\ f\big(\,t,\tilde{u}_n(t),\ \tilde{p}_n(t)\big)\ \in\ C^1\big([0,\ T];\ X\,\big)\,,$$

so that the function $\tilde{u}_{n+1}$ satisfies the equation

$$\tilde{u}'_{n+1}(t) - A\,\tilde{u}_{n+1}(t) \;=\; f\big(\,t, \tilde{u}_n(t), \tilde{p}_n(t)\big)\,. \tag{6.3.28}$$

The value $A\,\tilde{u}_{n+1}$ can be found from (6.3.24) by the well-known formula from the **theory of semigroups** (see Fattorini (1983), Kato (1966)).

$$A \int_0^t V(t-s)\, f(s)\; ds \;=\; \int_0^t V(t-s)\, f'(s)\; ds + V(t)\, f(0) - f(t)\,,$$

which is valid for any continuously differentiable function f. If $\tilde{u}_n(0) = u_0$ and $\tilde{p}_n(0) = p_0$, then

$$\begin{aligned} A\,\tilde{u}_{n+1}(t) \;&= V(t)\,\big(\,A\,u_0 + f(0, u_0, p_0)\big) \\ &\quad + \int_0^t V(t-s)\,\big[\,f_t\big(\,s, \tilde{u}_n(s), \tilde{p}_n(s)\big) \\ &\quad + f_u\big(\,s, \tilde{u}_n(s), \tilde{p}_n(s)\big)\, \tilde{u}'_n(s) \\ &\quad + f_p\big(\,s, \tilde{u}_n(s), \tilde{p}_n(s)\big)\, \tilde{p}'_n(s)\big]\; ds \\ &\quad - f\big(\,t, \tilde{u}_n(t), \tilde{p}_n(t)\big)\,. \end{aligned} \tag{6.3.29}$$

Subtracting (6.3.29) from (6.3.25) yields the relation

$$\begin{aligned} \tilde{w}_{n+1}(t) - A\,\tilde{u}_{n+1}(t) \;=\; &\int_0^t V(t-s)\,\big[\,f_u\big(\,s, \tilde{u}_n(s), \tilde{p}_n(s)\big) \\ &\times\; \big(\,\tilde{w}_n(s) - \tilde{u}'_n(s)\big) + f_p\big(\,s, \tilde{u}_n(s), \tilde{p}_n(s)\big) \\ &\times\; \big(\,\tilde{q}_n(s) - \tilde{p}'_n(s)\big]\; ds + f\big(\,t, \tilde{u}_n(t), \tilde{p}_n(t)\big)\,, \end{aligned} \tag{6.3.30}$$

from which (6.3.28) can be subtracted. All this enables us to establish the relationship

(6.3.31)

$$\begin{aligned} \tilde{w}_{n+1}(t) - \tilde{u}'_{n+1}(t) \;=\; &\int_0^t V(t-s)\,\Big[\,f_u\big(\,s, \tilde{u}_n(s), \tilde{p}_n(s)\big) \\ &\times\; \big(\,\tilde{w}_n(s) - \tilde{u}'_n(s)\big) + f_p\big(\,s, \tilde{u}_n(s), \tilde{p}_n(s)\big) \\ &\times\; \big(\,\tilde{q}_n(s) - \tilde{p}'_n(s)\Big]\; ds\,. \end{aligned}$$

Having (6.3.26) differentiated we subtract the resulting expression from (6.3.27) and, after this, arrive at

(6.3.32)

$$\tilde{q}_{n+1}(t) - \tilde{p}'_{n+1}(t) = \Phi_z\Big(t, g_0(t) + g_1(t, \tilde{u}_{n+1}(t))$$

$$+ \int_0^t K(t,s)\, f(\, s, \tilde{u}_n(s), \tilde{p}_n(s))\; ds\Big)$$

$$\times \Big(g_{1,u}(t, \tilde{u}_{n+1}(t))\,\big(\tilde{w}_{n+1}(t) - \tilde{u}'_{n+1}(t)\big)$$

$$+ \int_0^t K(t,s)\,\Big[f_u(s, \tilde{u}_n(s), \tilde{p}_n(s))\,\big(\tilde{w}_n(s) - \tilde{u}'_n(s)\big)$$

$$+ f_p\big(\, s, \tilde{u}_n(s), \tilde{p}_n(s)\big)\,\big(\tilde{q}_n(s) - \tilde{p}'_n(s)\big)\Big]\; ds\Big)\,.$$

With zero initial approximations, the equalities

$$\tilde{w}_0 = \tilde{u}'_0$$

and

$$\tilde{q}_0 = \tilde{p}'_0$$

become valid. With the aid of relations (6.3.31)–(6.3.32) we derive by induction on n the equalities $\tilde{w}_n = \tilde{u}_n$ and $\tilde{q}_n = \tilde{p}'_n$, valid for all $n \in N$, and involve these as further developments occur. Therefore, the sequences $\tilde{u}_n \to u$, $\tilde{u}'_n \to w$, $\tilde{p}_n \to p$ and $\tilde{p}'_n \to q$ as $n \to \infty$ uniformly over the segment $[0, T]$. Finally, the functions u and p are continuously differentiable, thus causing the needed relations $u' = w$ and $p' = q$ and completing the proof of the theorem.■

6.4 Inverse problems with smoothing overdetermination: smoothness of solution

We begin by considering the inverse problem (6.3.2)–(6.3.4) in another version giving a generalization of setting up problem (6.2.1)–(6.2.4) under the agreement that the function f involved in equation (6.3.2) is representable by

$$f(t, u, p) := L_1(t)\, u + L_2(t)\, p + F(t)\,, \tag{6.4.1}$$

where $L_1(t) \in \mathcal{L}(X)$, $L_2(t) \in \mathcal{L}(Y, X)$ and $F(t) \in X$ for every fixed $t \in [0, T]$. In that case the unique solvability is revealed on the whole segment $[0, T]$ as occurred in the inverse problem (6.2.1)–(6.2.4). For the reader's convenience the complete statement of the inverse problem amounts to finding a pair of the functions $u \in C([0, T]; X)$ and $p \in C([0, T]; Y)$ from the system of relations

$$\begin{aligned} u'(t) &= A\, u(t) + L_1(t)\, u(t) \\ &\quad + L_2(t)\, p(t) + F(t)\,, \qquad 0 \le t \le T\,, \end{aligned} \tag{6.4.2}$$

$$u(0) = u_0\,, \tag{6.4.3}$$

$$B\, u(t) = \psi(t)\,, \qquad 0 \le t \le T\,, \tag{6.4.4}$$

where the operator B is in line, as before, with the inclusions

$$B\,,\ \overline{B\,A} \in \mathcal{L}(X, Y) \tag{6.4.5}$$

and equation (6.4.2) is to be understood in a sense of distributions.

Theorem 6.4.1 *One assumes that the closed linear operator A, whose domain is dense, generates a strongly continuous semigroup in the space X and condition* (6.4.5) *holds. Let*

$$L_1 \in C([0, T]; \mathcal{L}(X))\,, \qquad L_2 \in C([0, T]; \mathcal{L}(Y, X))\,,$$

$$F \in C([0, T]; X)\,, \qquad u_0 \in X\,, \qquad \psi \in C^1([0, T]; Y)\,.$$

When $t \in [0, T]$ is kept fixed, the operator $B\, L_2(t)$ is supposed to be invertible and

$$(B\, L_2)^{-1} \in C([0, T]; \mathcal{L}(Y))\,.$$

If the compatibility condition $B\, u_0 = \psi(0)$ is fulfilled, then a solution u, p of the inverse problem (6.4.2)–(6.4.4) *exists and is unique in the class of functions*

$$u \in C([0, T]; X)\,, \qquad p \in C([0, T]; Y)\,.$$

Proof Before we undertake the proof, let us stress that conditions (A)–(E) of Section 6.3 remain valid if we accept the following representations:

$$
\begin{aligned}
f_1(t,u) &= L_1(t)\,u + F(t)\,, \\
f_2(t,u,p) &= L_2(t)\,p\,, \\
f_3(t,z,p) &= B\,L_2(t)\,p\,, \\
\Phi(t,z) &= \big(B\,L_2(t)\big)^{-1} z\,.
\end{aligned}
$$

Therefore, further treatment of the inverse problem (6.4.2)–(6.4.4) amounts to solving the system of the integral equations (6.3.9), (6.3.12) or the equivalent system (6.3.9), (6.3.13) taking under the conditions of Theorem 6.4.1 the form

$$
u(t) = u_0(t) + \int_0^t K_1(t,s)\,u(s)\,ds + \int_0^t L_1(t,s)\,p(s)\,ds\,, \tag{6.4.6}
$$

$$
p(t) = p_0(t) + \int_0^t K_2(t,s)\,u(s)\,ds + \int_0^t L_2(t,s)\,p(s)\,ds\,, \tag{6.4.7}
$$

where

$$
u_0(t) = V(t)\,u_0 + \int_0^t V(t-s)\,F(s)\,ds\,,
$$

$$
p_0(t) = \big(B\,L_2(t)\big)^{-1}\Big(\psi'(t) - \overline{B\,A}\;V(t)\,u_0 - \overline{B\,A}\int_0^t V(t-s)\,F(s)\,ds - B\,F(t) - B\,L_1(t)\,u_0(t)\Big)\,,
$$

$$
\begin{aligned}
K_1(t,s) &= V(t-s)\,L_1(s)\,,\\
L_1(t,s) &= V(t-s)\,L_2(s)\,,\\
K_2(t,s) &= -\,\big(B\,L_2(t)\big)^{-1}\,\big(\overline{B\,A}\;V(t-s)\\
&\qquad \times\; L_1(s) + B\,L_1(t)\,K_1(t,s)\big)\,,\\
L_2(t,s) &= -\,\big(B\,L_2(t)\big)^{-1}\,\big(\overline{B\,A}\;V(t-s)\\
&\qquad \times\; L_2(s) + B\,L_1(t)\,L_1(t,s)\big)\,.
\end{aligned}
$$

It is worth bearing in mind that the premises of this theorem imply the strong continuity of the operator kernels K_1, L_1, K_2, L_2 for $0 \le s \le t \le T$ as well as the continuity of the functions $u_0(t)$ and $p_0(t)$. We know from Section 5.1 that these properties are sufficient for the **system of the Volterra integral equations** (6.4.6)–(6.4.7) to have a unique solution in the class of continuous functions, thereby completing the proof of the theorem.■

If the input data functions possess the extra **smoothness**, then the same property would be valid for a solution. The analog of Theorem 6.3.2 in the linear case of interest is quoted in the following proposition.

Theorem 6.4.2 *One assumes that the closed linear operator A, whose domain is dense, generates a strongly continuous semigroup in the space X and condition* (6.4.5) *is satisfied. Let*

$$
L_1 \in C^1([0,T];\,\mathcal{L}(X))\,, \qquad L_2 \in C^1([0,T];\,\mathcal{L}(Y,X))\,,
$$
$$
F \in C^1([0,T];\,X)\,, \qquad u_0 \in \mathcal{D}(A)
$$

and $\psi \in C^2([0,T];Y)$. If, for any fixed value $t \in [0,T]$, the operator $B\,L_2(t)$ is invertible,

$$
\big(B\,L_2\big)^{-1} \in C^1([0,T];\,\mathcal{L}(Y))
$$

and the compatibility condition $B\,u_0 = \psi(0)$ holds, then a solution u, p of the inverse problem (6.4.2)–(6.4.4) *exists and is unique in the class of functions*

$$
u \in C^1([0,T];\,X) \bigcap C([0,T];\,\mathcal{D}(A))\,, \qquad p \in C^1([0,T];\,Y)\,.
$$

Proof First of all note that the conditions of the theorem provide the validity of conditions (D_1) and (E_1) as well as reason enough for differentiating the system (6.4.6)–(6.4.7) and extending it by joining with equations (6.3.20) and (6.3.23). The system thus obtained is linear and can be written as

$$u(t) = u_0(t) + \int_0^t \big[U_1(t,s)\, u(s) + U_2(t,s)\, p(s) \big] \, ds\,, \tag{6.4.8}$$

$$p(t) = p_0(t) + \int_0^t \big[P_1(t,s)\, u(s) + P_2(t,s)\, p(s) \big] \, ds\,, \tag{6.4.9}$$

$$\begin{aligned} w(t) = w_0(t) + \int_0^t \big[& W_1(t,s)\, u(s) \\ & + W_2(t,s)\, p(s) + W_3(t,s)\, w(s) \\ & + W_4(t,s)\, q(s) \big] \, ds\,, \end{aligned} \tag{6.4.10}$$

$$\begin{aligned} q(t) = q_0(t) + \int_0^t \big[& Q_1(t,s)\, u(s) + Q_2(t,s)\, p(s) \\ & + Q_3(t,s)\, w(s) + Q_4(t,s)\, q(s) \big] \, ds\,, \end{aligned} \tag{6.4.11}$$

where

$$U_1(t,s) = V(t-s)\, L_1(s)\,,$$

$$U_2(t,s) = V(t-s)\, L_2(s)\,,$$

$$u_0(t) = V(t)\, u_0 + \int_0^t V(t-s)\, F(s) \, ds\,,$$

$$\begin{aligned} P_1(t,s) = - \big(B\, L_1(t) \big)^{-1} \big[& \overline{BA}\; V(t-s)\, L_1(s) \\ & + B\, L_1(t)\, V(t-s)\, L_1(s) \big]\,, \end{aligned}$$

$$P_2(t,s) = - \big(B\, L_2(t) \big)^{-1} \big[\overline{BA}\; V(t-s)\, L_2(s)$$

$$+ B\,L_1(t)\,V(t-s)\,L_2(s)\big]\,,$$

$$p_0(t,s) = \big(B\,L_2(t)\big)^{-1}\big[\psi'(t) - \big(\overline{B\,A}$$

$$+ B\,L_1(t)\big)\,u_0(t) - B\,F(t)\big]\,,$$

$$W_1(t,s) = V(t-s)\,L_1'(s)\,,$$

$$W_2(t,s) = V(t-s)\,L_2'(s)\,,$$

$$W_3(t,s) = V(t-s)\,L_1(s)\,,$$

$$W_4(t,s) = V(t-s)\,L_2(s)\,,$$

$$w_0(t) = V(t)\big[A\,u_0 + L_1(0)\,u_0$$

$$+ L_2(0)\,p_0(0) + F(0)\big]$$

$$+ \int_0^t V(t-s)\,F'(s)\,ds\,,$$

$$R(t) = \big(B\,L_2(t)\big)^{-1},$$

$$S(t) = \overline{B\,A} + B\,L_1(t)\,,$$

$$Q_1(t,s) = -\,R'(t)\,S(t)\,V(t-s)\,L_1(s)$$

$$-\,R(t)\,S(t)\,V(t-s)\,L_1'(s)$$

$$-\,R(t)\,B\,L_1'(t)\,V(t-s)\,L_1(s)\,,$$

$$Q_2(t,s) = -\,R'(t)\,S(t)\,V(t-s)\,L_2(s)$$

$$-\,R(t)\,S(t)\,V(t-s)\,L_2'(s)$$

$$-\,R(t)\,B\,L_1'(t)\,V(t-s)\,L_2(s)\,,$$

$$Q_3(t,s) = -\,R(t)\,S(t)\,V(t-s)\,L_1(s)\,,$$

$$Q_4(t,s) = -\,R(t)\,S(t)\,V(t-s)\,L_2(s)\,,$$

$$
\begin{aligned}
q_0(t) = {} & R'(t) \left[g_0(t) - S(t) \int\limits_0^t V(t-s) \, \dot{F}(s) \, ds \right] \\
& + R(t) \left[g_0'(t) - B \, L_1'(t) \int\limits_0^t V(t-s) \, F(s) \, ds \right] \\
& - R(t) \, S(t) \left[V(t) \big(L_1(0) \, u_0 + L_2(0) \, p_0(0) \right. \\
& \left. + F(0) \big) + \int\limits_0^t V(t-s) \, F'(s) \, ds \right] .
\end{aligned}
$$

In light of the theorem premises the operator kernels and the nonhomogeneous terms of the system of the Volterra integral equations (6.4.8)–(6.4.11) will be continuous and strongly continuous, respectively. This implies the existence and uniqueness of the continuous solution u, p, w, q to the system concerned. For further reasoning one can adopt the concluding arguments from the proof of Theorem 6.3.2, since the system (6.4.8)–(6.4.11) coincides with the system (6.3.9), (6.3.20), (6.3.22)–(6.3.23). As $n \to \infty$, the successive approximations $\tilde{u}_n$, $\tilde{p}_n$, $\tilde{w}_n$, $\tilde{q}_n$ specified by (6.3.24)–(6.3.27) converge to a solution of the system (6.4.8)–(6.4.11) uniformly over the segment $[0, T]$. In proving Theorem 6.3.2 we have established that for all n the equalities

$$
\tilde{w}_n = \tilde{u}_n'
$$

and

$$
\tilde{q}_n = \tilde{p}_n'
$$

hold true, thereby completing the proof of the theorem.■

Before returning to the inverse problem (6.3.2)–(6.3.4), let us assume that all the conditions of Theorem 6.3.2 are satisfied. Hence there exists a segment $[0, T_1]$ on which both functions $w = u'$ and $q = p'$ are continuous and satisfy the integral equation (6.3.20). Further treatment of the integral equation (6.3.20) involves an alternative form of writing

$$
\begin{aligned}
w(t) = {} & V(t) \, w_0 + \int\limits_0^t V(t-s) \left[L_1(s) \, w(s) \right. \\
& \left. + L_2(s) \, q(s) + F(s) \right] ds , \qquad (6.4.12)
\end{aligned}
$$

where

$$
(6.4.13) \qquad \begin{cases} w_0 = A\,u_0 + f(0, u_0, p_0)\,, \\ L_1(t) = f_u\big(t, u(t), p(t)\big)\,, \\ L_2(t) = f_p\big(t, u(t), p(t)\big)\,, \\ F(t) = f_t\big(t, u(t), p(t)\big)\,. \end{cases}
$$

Because the operator B is bounded, the equalities

$$
B\,u'(t) = \big(B\,u(t)\big)' = \psi'(t)
$$

become valid, whose use permits us to establish the relationship

$$
B\,w(t) \;=\; \psi'(t)\,.
$$

On the other hand, with the aid of relation (6.4.12) we deduce that the function w is just a continuous solution of the Cauchy problem

$$
w'(t) \;=\; A\,w(t) + h(t)\,, \qquad\qquad w(0) \;=\; w_0\,,
$$

with $h(t) = L_1(t)\,w(t) + L_2(t)\,q(t) + F(t)$ incorporated. Thus, we arrive at the following assertion.

Corollary 6.4.1 *Under the conditions of Theorem* 6.3.2 *there exists a number* $T_1 > 0$ *such that on the segment* $[0,\, T_1]$ *a solution* u, p *of the inverse problem* (6.3.2)-(6.3.4) *is continuously differentiable and the derivatives* $w = u'$ *and* $q = p'$ *give a continuous solution of the inverse problem*

$$
(6.4.14) \qquad \begin{cases} w'(t) = Aw(t) + L_1(t)w(t) + L_2(t)q(t) + F(t)\,, & 0 \le t \le T_1\,, \\ w(0) = w_0\,, & \\ Bw(t) = \varphi(t)\,, & 0 \le t \le T_1\,, \end{cases}
$$

where w_0, L_1, L_2 *and* F *are given by relations* (5.4.15) *and* $\varphi = \psi'$.

By applying Theorem 6.4.2 to problem (6.4.14) one can derive the conditions under which a solution of (6.3.2)–(6.3.4) becomes twice continuously differentiable. The inverse problem for the second derivatives of this solution can be written in the explicit form in complete agreement with Corollary 6.4.1. In its framework we are able to establish sufficient conditions for the solution smoothness up to any desired order.

We give below one possible example of such a process.

Corollary 6.4.2 *Let under the conditions of Theorem* 6.3.2 *the functions* f_1, f_2 *and* Φ *be twice Frechet differentiable on the manifolds*

$$S_{X\times Y}\big((u_0, p_0), R, T\big)$$

and

$$S_Y(z_0, R, T)\,,$$

respectively. If

$$\psi \in C^3([0, T]; Y)$$

and

$$A\,u_0 + f(0, u_0, p_0) \in \mathcal{D}(A)\,,$$

then there exists a number $T_1 > 0$ *such that on the segment* $[0, T_1]$ *a solution* u, p *of problem* (6.3.2)–(6.3.4) *is twice continuously differentiable and the functions* $w = u'$ *and* $g = p'$ *give a continuously differentiable solution of the inverse problem* (6.4.14).

6.5 Inverse problems with singular overdetermination: semilinear equations with constant operation in the principal part

When working in Banach spaces X and Y, we consider the inverse problem of determining a pair of the functions $u \in C^1([0, T]; X)$ and $p \in C([0, T]; Y)$ from the system of relations

$$u'(t) = A\,u(t) + f\big(t, u(t), p(t)\big)\,, \quad 0 \le t \le T\,, \tag{6.5.1}$$

$$u(0) = u_0\,, \tag{6.5.2}$$

$$B\,u(t) = \psi(t)\,, \qquad 0 \le t \le T\,, \tag{6.5.3}$$

under the following restrictions: an operator A with a dense domain is closed and linear in the space X and generates a strongly continuous semigroup $V(t)$;

$$f\colon [0, T] \times X \times Y \mapsto X\,;$$

a linear operator B is such that the domain $\mathcal{D}(B)$ belongs to the space X and the range belongs to the space Y and, in addition, there is a nonnegative integer m, for which the inclusion

$$B \in \mathcal{L}\big(\mathcal{D}(A^m); Y\big) \tag{6.5.4}$$

occurs, and the function f is treated as a sum

$$f(t,u,p) = f_1(t,u) + f_2(t,u,p)\,. \tag{6.5.5}$$

Retaining the notations of Section 6.3

$$S_X(a,R) = \{\, x \in X\colon \|x-a\|_X < R \,\},$$

$$S_X(a,R,T) = \{\, (t,x)\colon 0 \le t \le T,\ x \in S_X(a,R)\}\,,$$

accepting the inclusions $u_0 \in \mathcal{D}(B\,A)$ and $f_1(0,\,u_0) \in \mathcal{D}(B)$ and allowing the function ψ to be differentiable at zero, we define the element

$$z_0 = \psi'(0) - B\,A\,u_0 - B\,f_1(0,\,u_0) \tag{6.5.6}$$

and, after this, impose the following conditions:

(A) *the equation* $B\,f_2(0,u_0,p) = z_0$ *with respect to* p *has a unique solution* $p_0 \in Y$*;*

(B) *there is a mapping*

$$f_3\colon [0,\,T] \times Y \times Y \mapsto Y$$

such that

$$B\,f_2(t,u,p) = f_3\big(t, B\,u, p\big)\,;$$

(C) *there is a number* $R > 0$ *such that for any* $t \in [0,\,T]$ *the mapping*

$$z = f_3\big(t, \psi(t), p\big)$$

has in the ball $S_Y(p_0, R)$ *the inverse*

$$p = \Phi(t,z)\,; \tag{6.5.7}$$

(D) *there is a number* $R > 0$ *such that for* $k = 0,\,1,\,m{+}1$ *both functions*

$$A^k\,f_1\big(t,\,A^{-k}\,u\big)$$

and

$$A^k\,f_2\big(t,\,A^{-k}\,u,\,p\big)$$

are continuous with respect to (t,u,p) *and satisfy the Lipschitz condition with respect to* (u,p) *on the manifold*

$$S_{X\times Y}\big((A^k\,u_0,\,p_0), R, T\big)\,;$$

(E) *there is a value* $R > 0$ *such that the mapping* Φ *specified by* (6.5.7) *is continuous with respect to* (t,z) *and satisfies the Lipschitz condition in* z *on the manifold* $S_Y(z_0, R, T)$.

These conditions are similar to those imposed in Section 6.3 for the problem with smoothing overdetermination. Having involved these conditions one can state a local unique solvability of the inverse problem with **singular overdetermination**. Before proceeding to careful analysis, we begin by establishing some properties of a solution of the related direct problem.

Lemma 6.5.1 *Let the linear operator A whose domain is dense in the Banach space X generate a strongly continuous semigroup $V(t)$ and the inclusions*

$$u_0 \in \mathcal{D}(A)\,, \qquad \lambda \in \rho(A)$$

occur. If there exists a number $R > 0$ such that both functions $f(t,u)$ and $g(t,u) = \big(A - \lambda I\big)\, f\big(t, (A-\lambda I)^{-1} u\big)$ are continuous with respect to (t,u) and satisfy the Lipschitz condition in u on the manifold $S_X(a,R,T)$, then the Cauchy problem

$$(6.5.8) \qquad \begin{cases} u'(t) = A\,u(t) + f\big(t,\, u(t)\big)\,, & 0 \le t \le T\,, \\ u(0) = u_0\,, \end{cases}$$

in the class of functions

$$u \in C^1\big([0,\, T];\, X\big) \bigcap C\big([0,\, T];\, \mathcal{D}(A)\big)$$

is equivalent to the integral equation

$$(6.5.9) \qquad u(t) = V(t)\,u_0 + \int\limits_0^t V(t-s)\, f(s,\, u(s))\; ds\,, \qquad 0 \le t \le T\,,$$

in the class of functions

$$u \in C\big([0,\, T];\, \mathcal{D}(A)\big)\,.$$

Proof Let u be a solution of the Cauchy problem at hand. Since $f(t,u)$ and $u(t)$ are continuous, the function

$$(6.5.10) \qquad F(t) = f\big(t,\, u(t)\big)$$

is continuous on the segment $[0,\, T]$ in the space X. Then so is the function

$$(6.5.11) \qquad w(t) = \big(A - \lambda\, I\big)\, u(t)\,.$$

This owes a debt to a proper choice of the class of solvability of problem (6.5.8). On the other hand, this property remains valid on the same segment and in the same space for the function

$$A\,F(t) = (A - \lambda I)\, f(t, (A - \lambda I)^{-1}\, w(t)) + \lambda\, f(t, u(t))$$
$$= g(t, w(t)) + \lambda\, f(t, u(t))\,.$$

From the **theory of semigroups** it is clear that the inclusions

$$F(t),\ A\,F(t) \in \mathcal{C}([0,\, T];\, X)$$

and $u_0 \in \mathcal{D}(A)$ ensure that the Cauchy problem

$$\begin{cases} u'(t) \ = \ A\,u(t) + F(t)\,, & 0 \le t \le T\,, \\ u(0) \ = \ u_0\,, \end{cases}$$

is solved by the function

$$u(t) \ = \ V(t)\,u_0 + \int\limits_0^t V(t-s)\,F(s)\;ds\,,$$

which implies (6.5.9) and the inclusion $u \in \mathcal{C}([0,\, T];\, \mathcal{D}(A))$ as far as the function $F(t) = f(t,\, u(t))$ is concerned (see Fattorini (1983) and Mizohata (1977)).

To prove the converse, a function $u \in \mathcal{C}([0,\, T];\, \mathcal{D}(A))$ is adopted as a solution of the integral equation (6.5.9). By introducing the function F with the aid of (6.5.10) we obtain

$$u(t) \ = \ V(t)\,u_0 + \int\limits_0^t V(t-s)\,F(s)\;ds\,. \tag{6.5.12}$$

Since $u \in \mathcal{C}([0,\, T];\, \mathcal{D}(A))$, the function w given by formula (6.5.11) is continuous on the segment $[0,\, T]$ in the space X. Then so is either of the functions $F(t)$ and

$$A\,F(t) \ = \ g(t,\, w(t)) + \lambda\, f(t,\, u(t))\,.$$

Due to the inclusion $u_0 \in \mathcal{D}(A)$ the function $u(t)$ specified by (6.5.12) is continuously differentiable, so that

$$u'(t) \ = \ A\,u(t) + F(t) \tag{6.5.13}$$

for all $t \in [0,\, T]$. Equality (6.5.10) implies that the function u solves equation (6.5.8). The continuity of the derivative of the function follows from (6.5.13), while the equality $u(0) = u_0$ is an immediate implication of (6.5.12) with $t = 0$. This proves the assertion of the lemma. ■

Lemma 6.5.2 *If all the conditions of Lemma* 6.5.1 *hold and*

$$w_0 = (A - \lambda I)\, u_0 \,,$$

then a function

$$u \in C^1([0, T]; X) \bigcap C([0, T]; \mathcal{D}(A))$$

is a solution of the Cauchy problem (6.5.8) *if and only if the pair of the functions* $u(t)$ *and*

$$w(t) = (A - \lambda I)\, u(t)$$

is a continuous solution of the system of integral equations

$$u(t) = V(t)\, u_0 + \int_0^t V(t-s)\, f(s, u(s))\, ds\,, \quad 0 \leq t \leq T\,, \tag{6.5.14}$$

$$w(t) = V(t)\, w_0 + \int_0^t V(t-s)\, g(s, w(s))\, ds\,, \quad 0 \leq t \leq T\,. \tag{6.5.15}$$

Proof Let u be a solution of the Cauchy problem (6.5.8) in the indicated class of functions. Due to Lemma 6.5.1 the function u thus obtained satisfies equation (6.5.14), while the function w defined by (6.5.11) is continuous. By minor manipulations with applying the operator $A - \lambda I$ to both sides of equation (6.5.14) we derive equation (6.5.15).

To prove the converse, it is supposed that a pair of the functions $u, w \in C([0, T]; X)$ satisfies the system (6.5.14)–(6.5.15). Then the inclusion

$$\tilde{u} \in C([0, T]; \mathcal{D}(A))$$

is established for the function

$$\tilde{u}(t) = (A - \lambda I)^{-1}\, w(t)\,.$$

Since $u_0 = (A - \lambda I)^{-1} w_0$ and for any $\tilde{u} \in \mathcal{D}(A)$ the relationship

$$f(t, \tilde{u}) = (A - \lambda I)^{-1}\, g(t, (A - \lambda I)\, \tilde{u})$$

takes place, the outcome of applying the operator $\left(A - \lambda I\right)^{-1}$ to both sides of (6.5.15) is

$$\tilde{u}(t) = V(t)\,u_0 + \int_0^t V(t-s)\,f\big(s,\,\tilde{u}(s)\big)\,ds\,, \qquad 0 \le t \le T\,.$$

This serves to motivate that the functions u and $\tilde{u}$ give solutions to equation (6.5.14) with Lipschitz nonlinearity. From the theory of Volterra integral equations it follows that any solution of this equation in the class of continuous functions is unique on the whole segment of its existence, it being understood that for all values $t \in [0,\,T]$

$$u(t) = \tilde{u}(t)\,,$$

implying that

$$w(t) = \left(A - \lambda I\right) u(t)\,.$$

In that case the inclusion $u \in C\big([0,\,T];\,\mathcal{D}(A)\big)$ occurs, thereby justifying by Lemma 6.5.1 that the function u is a solution of the Cauchy problem (6.5.8) and completing the proof of the lemma.■

Theorem 6.5.1 *If all the conditions of Lemma* 6.5.1 *hold, then there is a value* $T_1 > 0$ *such that problem* (6.5.8) *has a solution*

$$u \in C^1\big([0,T_1];\,X\big) \bigcap C\big([0,T_1];\,\mathcal{D}(A)\big)\,,$$

which is unique on the whole segment of its existence.

Proof The main idea behind proof is connected with joint use of Theorem 6.5.1 and Lemmas 6.5.1–6.5.2, making it possible to reduce the Cauchy problem (6.5.8) to the system of the integral equations (6.5.14)–(6.5.15) in the class of continuous functions. We note in passing that this system breaks down into two separate equations each of which appears to be a Volterra equation of the second kind with a strongly continuous operator kernel with Lipshitz nonlinearity. Because of this fact, these will be solvable if $T > 0$ is small enough. If this happens, the solution so constructed will be unique on the whole segment of its uniqueness, whence the assertion of Theorem 6.5.1 follows immediately. ■

Theorem 6.5.2 *If under the conditions of Lemma* 6.5.1 *there is a non-negative integer* m *such that the inclusion* $u_0 \in \mathcal{D}(A^{m+1})$ *occurs and the function*

$$h(t,u) = \left(A - \lambda I\right)^{m+1} f\big(t,\,(A - \lambda I)^{-m-1}\,u\big)$$

is continuous in (t, u) and satisfies on the manifold $S_X(u_0, R, T)$ the Lipschitz condition in u for some number $R > 0$. Then there is a value $T_1 > 0$ such that a solution u of the Cauchy problem (6.5.8) *belongs to the class of functions*

$$\mathcal{C}^1([0, T_1]; \mathcal{D}(A^m)) \bigcap \mathcal{C}([0, T_1]; \mathcal{D}(A^{m+1})) .$$

Proof By Theorem 6.5.1 the Cauchy problem (6.5.8) is solvable if the value $T > 0$ is small enough. Moreover, by Lemma 6.5.1 this solution satisfies the integral equation (6.5.9). By merely setting $v_0 = A^{m+1} u_0$ we can write down the integral equation

$$v(t) = V(t)\, v_0 + \int_0^t V(t-s)\, h(s, v(s))\, ds . \tag{6.5.16}$$

With the initial assumptions in view, equation (6.5.16) is of the Volterra type with Lipschitz nonlinearity and can be resolved in the class of functions $\mathcal{C}([0, T]; X)$ for a sufficiently small value $T > 0$. A good look at the proof of Theorem 6.5.1 is recommended for further careful analysis. Plain calculations show that the function

$$\tilde{u}(t) = \big(A - \lambda I\big)^{-m-1} v(t)$$

is a continuous solution to equation (6.5.9). In just the same way as in the proof of Theorem 6.5.1 it is not difficult to demonstrate that this solution is unique. Therefore, $\tilde{u}(t) = u(t)$, yielding the relationship

$$u(t) = \big(A - \lambda I\big)^{-m-1} v(t)$$

with the members $v \in \mathcal{C}([0, T]; X)$ and $u \in \mathcal{C}([0, T]; \mathcal{D}(A^{m+1}))$.

In what follows we agree to consider

$$z_0 = \big(A - \lambda I\big)^m u_0 ,$$

$$F(t) = \big(A - \lambda I\big)^m f(t, u(t)) ,$$

$$z(t) = \big(A - \lambda I\big)^m u(t) .$$

Applying the operator $\big(A - \lambda I\big)^m$ to equation (6.5.9) yields that the function $z(t)$ is subject to the following relation:

$$z(t) = V(t)\, z_0 + \int_0^t V(t-s)\, F(s)\, ds , \qquad 0 \le t \le T . \tag{6.5.17}$$

By assumption, the inclusion $z_0 \in \mathcal{D}(A)$ occurs. The relations

$$\begin{aligned} F(t) &= (A - \lambda I)^{-1} h(t, (A - \lambda I)^{m+1} u(t)), \\ A F(t) &= [(A - \lambda I) + \lambda I] \, F(t) \\ &= h(t, (A - \lambda I)^{m+1} u(t)) + \lambda \, F(t) \end{aligned}$$

provide support for the view that the functions $F(t)$ and $A F(t)$ are continuous on the segment $[0, T]$ in the space X. From the theory of semigroups it follows that the function $z(t)$ is continuously differentiable in the space X and satisfies the differential equation

$$z'(t) = A z(t) + F(t) . \tag{6.5.18}$$

The continuous differentiability of the function $z(t)$ serves as a basis for the inclusion $u \in C^1([0, T]; \mathcal{D}(A^m))$, thereby completing the proof of the theorem. ■

Remark By applying the operator $(A - \lambda I)^m$ to equation (6.5.8) we derive the equation

$$(A - \lambda I)^m u'(t) = A z(t) + F(t) . \tag{6.5.19}$$

Comparison of (6.5.18) and (6.5.19) shows that premises of Theorem 6.5.2 assure us of the validity of the relation

$$[(A - \lambda I)^m u(t)]' = (A - \lambda I)^m u'(t) . \tag{6.5.20}$$

Let us come back to the inverse problem (6.5.1)–(6.5.3). When comparing conditions (A)–(E) of the present section with those of Section 6.3, some difference is recognized in connection with condition (D) in which the operator A is required to be invertible. It should be noted that this restriction is not essential in subsequent studies of the inverse problem (6.5.1)–(6.5.3). Indeed, upon substituting

$$u(t) = e^{\lambda t} v(t) \tag{6.5.21}$$

there arises for the new functions v and p the same inverse problem as we obtained for the function u. This amounts to studying the system of relations

$$\begin{cases} v'(t) = A_\lambda v(t) + f_\lambda(t, v(t), p(t)) , & 0 \le t \le T , \\ v(0) = v_0 , & \\ B v(t) = \psi_\lambda(t) , & 0 \le t \le T , \end{cases} \tag{6.5.22}$$

where

$$A_\lambda = A - \lambda I ,$$

$$f_\lambda(t, v, p) = e^{-\lambda t} f\big(t, e^{\lambda t} v, p\big) ,$$

$$v_0 = u_0 ,$$

$$\psi_\lambda(t) = e^{-\lambda t} \psi(t) .$$

By merely choosing $\lambda \in \rho(A)$ we get the inverse problem (6.5.22) in which the operator A_λ involved is invertible. If the operator A generates a strongly continuous semigroup $V(t)$, then the operator A_λ is a generator of the strongly continuous semigroup $V_\lambda(t) = e^{-\lambda t} V(t)$. Substitution (6.5.21) permits us to restate all the conditions (A)–(E) as requested.

Theorem 6.5.3 *Let the closed linear operator A with a dense domain generate a strongly continuous semigroup in the Banach space X, the operator B be in line with* (6.5.4), *conditions (A)–(E) and* (6.5.5) *hold, $u_0 \in \mathcal{D}(A^{m+1})$, $\psi \in C^1\big([0, T]; Y\big)$ and the compatibility condition $B u_0 = \psi(0)$ occur. Then there is a number $T_1 > 0$ such that the inverse problem* (6.5.1)–(6.5.3) *has a solution*

$$u \in C^1\big([0, T_1]; \mathcal{D}(A^m)\big) \bigcap C\big([0, T_1]; \mathcal{D}(A^{m+1})\big) , \qquad p \in C\big([0, T_1]; Y\big)$$

and this solution is unique in the indicated class of functions.

Proof Lemma 6.5.1 implies that for any continuous in Y function p the Cauchy problem (6.5.1)–(6.5.2) is equivalent in the class of functions $u \in C\big([0, T]; \mathcal{D}(A)\big)$ to the integral equation for $0 \le t \le T$

$$u(t) = V(t) u_0 + \int_0^t V(t-s)\ f\big(s, u(s), p(s)\big)\ ds , \tag{6.5.23}$$

where $V(t)$ is a semigroup generated by the operator A. It is worth noting here that the premises of Theorem 6.5.2 for $\lambda = 0$ are ensured by condition (D). Thus, the function $v(t) = A^{m+1} u(t)$ is continuous and satisfies the equation

$$v(t) = V(t) v_0 + \int_0^t V(t-s)\, h\big(s, v(s), p(s)\big)\ ds , \tag{6.5.24}$$

where

$$v_0 = A^{m+1}\, u_0\,, \qquad\qquad h(t, v, p) \;=\; A^{m+1}\, f\big(t, A^{-m-1} v, p\big)$$

and the argument t is small enough. The reader may scrutinize the development of equation (6.5.16) in the proof of Theorem 6.5.2. On account of the preceding remark,

$$\big[A^m\, u(t)\big]' = A^m\, u'(t)\,.$$

Also, we thus have

$$\big(B\, u(t)\big)' = B\, u'(t)$$

if the operator B satisfies condition (6.5.4) and $u(t)$ is a solution of problem (6.5.1)–(6.5.2). Applying the operator B to (6.5.1) yields the relation

$$\psi'(t) \;=\; B\, A\, u(t) + B\, f\big(t, u(t), p(t)\big)\,,$$

which is equivalent to (6.5.3) on the strength of the compatibility condition $B\, u_0 \;=\; \psi(0)$. By virtue of conditions (6.5.5), (B) and (C) the preceding can be rewritten as

$$p(t) \;=\; \Phi\big(t,\, \psi'(t) - B\, A\, u(t) - B\, f_1(t, u(t))\big)\,. \tag{6.5.25}$$

Let us calculate $A\, u(t)$ with the aid of (6.5.23). For later use, it will be sensible to introduce the notations

$$h_1(t, v) \;=\; A^{m+1}\, f_1(t, A^{-m-1} v)\,,$$

$$g_0(t) \;=\; \psi'(t) - B\, V(t)\, A\, u_0 - B\, A^{-m-1} h_1(t, V(t)\, v_0)\,,$$

$$g_1(t, v) \;=\; -B\, A^{-m-1}\big(h_1(t, v) - h_1(t, V(t)\, v_0)\big)\,,$$

$$K(t, s) \;=\; -B\, V(t-s)\, A^{-m}\,.$$

Let $v(t)$ be a solution to equation (6.5.24). With the relation

$$A\, f(s, u(s), p(s)) = A^{-m} h(s, v(s), p(s))$$

in view, we may attempt equality (6.5.25) in the form

$$p(t) \;=\; \Phi\big(t, g_0(t) + g_1(t, v(t)\big) + \int_0^t K(t, s)\, h(s, v(s), p(s))\, ds\,. \tag{6.5.26}$$

From the conditions of the theorem it follows immediately that the function $g_0 \in C([0, T]; Y)$, the function $g_1(t, v)$ is continuous with respect to (t, v) and satisfies on the manifold $S_X(v_0, R, T)$ the Lipschitz condition in v. What is more, the kernel $K(t, s)$ is strongly continuous for $0 \le s \le t \le T$. When relations (6.5.24) and (6.5.26) are put together, the outcome of this is a system of the Volterra integro-differential equations related to the functions $v(t) = A^{m+1} u(t)$ and $p(t)$. The converse is certainly true: if the pair of the functions $v(t)$ and $p(t)$ is a solution of the system concerned, then the functions $u(t) = A^{-m-1} v(t)$ and $p(t)$ give a solution of the inverse problem in the indicated class of functions.

Let us show that the system of equations (6.5.24), (6.5.26) has a unique continuous solution by relating $T > 0$ to be small enough. As a first step towards the proof, we replace equation (6.5.26) by the following one:

$$p(t) = \Phi\big(t, g_0(t) + g_1(t, v_1(t))\big) + \int_0^t K(t, s)\, h(s, v(s), p(s))\, ds, \tag{6.5.27}$$

where

$$v_1(t) = V(t)\, v_0 + \int_0^t V(t-s)\, h(s, v(s), p(s))\, ds. \tag{6.5.28}$$

It is clear that the system of equations (6.5.24), (6.5.26) is equavalent to the system (6.5.24), (6.5.27).

At the next stage we introduce in the metric space

$$Z = C\big([0, T]; S_X(v_0, R) \times S_Y(p_0, R)\big)$$

the operator

$$G\colon (v, p) \mapsto (v_1, p_1),$$

where v_1 is defined by (6.5.28) and p_1 is equal to the right-hand side of (6.5.27). The assertion we must prove is that the operator G has a unique fixed point in the space Z if $T > 0$ is sufficiently small.

From the Lipschitz condition related to the functions Φ and h_1 it follows that

$$\| \Phi(t, v) - \Phi(t, g_0(t)) \| \le L\, \| v - g_0(t) \|, \tag{6.5.29}$$

$$\| g_1(t, v_1(t)) \| \le L\, \| v_1(t) - V(t)\, v_0 \|. \tag{6.5.30}$$

With respect to the constant

$$M = \max_{0 \le s \le t \le T} \left\{ L^2 \| V(t-s) \| , \, L \, \| K(t,s) \| , \, \| V(t-s) \| \right\}$$

we deduce from (6.5.24), (6.5.27) and (6.5.29)–(6.5.30) the estimates

$$\| v_1(t) - V(t) \, v_0 \| \le M \int_0^t \| h(s, v(s), p(s)) \| \, ds \, , \tag{6.5.31}$$

$$\| p_1(t) - \Phi(t, g_0(t)) \| \le M \int_0^t \| h(s, v(s), p(s)) \| \, ds \, . \tag{6.5.32}$$

Since

$$g_0(0) = \psi'(0) - B \, A \, u_0 - B \, f_1(0, u_0) = z_0 \, ,$$

where the element z_0 is defined by (6.5.6), conditions (A)–(C) imply that $\Phi(0, g_0(0)) = p_0$. Moreover,

$$V(0) \, v_0 = v_0$$

and, in view of this, condition (D) ensures that the function $h(t, v, p)$ becomes bounded on the manifold

$$S_{X \times Y} \big((v_0, p_0), R, T \big) \, .$$

This property in combination with estimates (6.5.31)–(6.5.32) assures that the operator G maps the space Z onto itself if $T > 0$ is small enough.

Let $\big(v_{11}, \, p_{11} \big) = G(v_1, \, p_1)$ and $\big(v_{22}, \, p_{22} \big) = G(v_2, \, p_2)$. The symbol $L(h)$ is used for the Lipschitz constant of the function h on the manifold

$$S_{X \times Y} \big((v_0, p_0), R, T \big) \, .$$

With this constant introduced, it is straightforward to verify that (6.5.24) and (6.5.27) together imply that

$$\| v_{22}(t) - v_{11}(t) \| \le M \, L(h) \, T \, \rho\big((v_2, p_2), \, (v_1, p_1) \big) \, , \tag{6.5.33}$$

$$\| p_{22}(t) - p_{11}(t) \| \le M \, L(h) \, T \, \rho\big((v_2, p_2), \, (v_1, p_1) \big) \, , \tag{6.5.34}$$

where ρ refers to a metric on the space Z. Due to estimates (6.5.33)–(6.5.34) the operator G is a contraction operator on the space Z if the value $T > 0$ is sufficiently small.

The assertion of the theorem is now a plain implication of the contraction mapping principle and is completely proved. ■

Consider now the case of a **linear equation** under the approved decomposition of the function f:

$$f(t, u, p) = L_1(t)\, u + L_2(t)\, p + f(t), \tag{6.5.35}$$

where for each fixed value $t \in [0, T]$ the inclusions $L_1(t) \in \mathcal{L}(X)$, $L_2(t) \in \mathcal{L}(Y; X)$ and $f(t) \in X$ occur. In this case a unique solvability of the inverse problem concerned is obtained on the whole segment $[0, T]$.

Theorem 6.5.4 *Let the closed linear operator A with a dense domain generate a strongly continuous semigroup in the Banach space X, the operator B satisfy condition* (6.5.4), *the inclusions $u_0 \in \mathcal{D}(A^{m+1})$ and $\psi \in C^1([0, T]; Y)$ and the compatibility condition $B\, u_0 = \psi(0)$ hold. Let decomposition* (6.5.35) *take place and the operator functions be such that*

$$L_1(t),\ A\, L_1(t)\, A^{-1},\ A^{m+1}\, L_1(t)\, A^{-m-1} \in C([0, T]; \mathcal{L}(X)),$$

$$A^{m+1}\, L_2(t)\, A^{-m-1} \in C([0, T]; \mathcal{L}(Y; X)).$$

If, in addition, the function $A^{m+1} f(t) \in C([0, T]; X)$, the operator $B\, L_2(t)$ is invertible for each $t \in [0, T]$ and

$$(B\, L_2)^{-1} \in C([0, T]; \mathcal{L}(Y)),$$

then the inverse problem (6.5.1)–(6.5.2) *has a solution*

$$u \in C^1([0, T]; \mathcal{D}(A^m)) \cap C([0, T]; \mathcal{D}(A^{m+1})), \qquad p \in C([0, T]; Y)$$

and this solution is unique in the indicated class of functions.

Proof The initial assumptions assure us of the validity of conditions (A)–(E) with the ingredients

$$f_1(t, u) = L_1(t)\, u + f(t),$$

$$f_2(t, u, p) = L_1(t)\, p,$$

$$f_3(t, z, p) = B\, L_2(t)\, p,$$

$$\Phi(t, z) = (B\, L_2(t))^{-1}\, z.$$

All this enables us to involve in the further development Theorem 6.5.3 and reduce the inverse problem at hand to the system of the integral equations

(6.5.24), (6.5.26) taking in the framework of Theorem 6.5.4 the form

$$v(t) = v_1(t) + \int_0^t K_1(t,s)\, v(s)\, ds + \int_0^t L_1(t,s)\, p(s)\, ds\,, \tag{6.5.36}$$

$$p(t) = p_1(t) + \int_0^t K_2(t,s)\, v(s)\, ds + \int_0^t L_2(t,s)\, p(s)\, ds\,, \tag{6.5.37}$$

where

$$v_1(t) = V(t)\, A^{m+1} u_0 + \int_0^t V(t-s)\, A^{m+1} f(s)\, ds\,,$$

$$\begin{aligned} p_1(t) = \big(B\, L_2(t)\big)^{-1} \Big(& \psi'(t) - B\, f(t) \\ & - B\, V(t)\, A\, u_0 - B\, L_1(t)\, V(t)\, u_0 \\ & - B \int_0^t V(t-s)\, A\, f(s)\, ds \\ & - B\, L_1(t) \int_0^t V(t-s)\, f(s)\, ds \Big)\,, \end{aligned}$$

$$K_1(t,s) = V(t-s)\, A^{m+1}\, L_1(s)\, A^{-m-1},$$

$$L_1(t,s) = V(t-s)\, A^{m+1}\, L_2(s)\,,$$

$$\begin{aligned} K_2(t,s) = -\big(B\, L_2(t)\big)^{-1} B \Big(& V(t-s)\, A\, L_1(s) \\ & \times A^{-m-1} + L_1(t)\, V(t-s)\, L_1(s)\, A^{-m-1} \Big)\,, \end{aligned}$$

$$L_2(t,s) = -\left(B\,L_2(t)\right)^{-1} B\Big(V(t-s)\,A\,L_2(s)$$
$$+ L_1(t)\,V(t-s)\,L_2(s)\Big)\,.$$

Note that the continuity of the function $v_1(t)$ and the strong continuity of the kernels $K_1(t,s)$ and $L_1(t,s)$ immediately follow from the conditions of the theorem. In turn, the continuity of the function $p_1(t)$ is an immediate implication of the set of relations

$$\begin{aligned}
B\,f(t) &= \left(B\,A^{-m}\right) A^{-1}\left(A^{m+1}\,f(t)\right),\\
B\,V(t)\,A\,u_0 &= \left(B\,A^{-m}\right) V(t)\left(A^{m+1}\,u_0\right),\\
B\,L_1(t)\,V(t)\,u_0 &= \left(B\,A^{-m}\right) A^{-1}\left(A^{m+1}\,L_1(t)\right.\\
&\quad\times\left. A^{-m-1}\right) V(t)\left(A^{m+1}\,u_0\right),\\
B\,V(t-s)\,A\,f(s) &= \left(B\,A^{-m}\right) V(t-s)\left(A^{m+1}\,f(s)\right),\\
B\,L_1(t)\,V(t-s)\,f(s) &= \left(B\,A^{-m}\right) A^{-1}\left(A^{m+1}\,L_1(t)\,A^{-m-1}\right)\\
&\quad\times\ V(t-s)\left(A^{m+1}\,f(s)\right).
\end{aligned}$$

Likewise, the strong continuity of the kernels $K_2(t,s)$ and $L_2(t,s)$ is established from the set of representations

$$\begin{aligned}
B\,V(t-s)\,A\,L_1(s)\,A^{-m-1} &= \left(B\,A^{-m}\right) V(t-s)\left(A^{m+1}\,L_1(s)\,A^{-m-1}\right),\\
B\,L_1(t)\,V(t-s)\,L_1(s)\,A^{-m-1} &= \left(B\,A^{-m}\right) A^{-1}\left(A^{m+1}\,L_1(t)\,A^{-m-1}\right)\\
&\quad\times\ V(t-s)\left(A^{m+1}\,L_1(s)\,A^{-m-1}\right),\\
B\,V(t-s)\,A\,L_2(s) &= \left(B\,A^{-m}\right) V(t-s)\left(A^{m+1}\,L_2(s)\right),\\
B\,L_1(t)\,V(t-s)\,L_2(s) &= \left(B\,A^{-m}\right) A^{-1}\left(A^{m+1}\,L_1(t)\,A^{-m-1}\right)\\
&\quad\times\ V(t-s)\left(A^{m+1}\,L_2(s)\right).
\end{aligned}$$

Therefore, the system (6.5.36)–(6.5.37) constitutes what is called a linear system of the Volterra integral equations of the second kind with continuous nonhomogeneous terms and strongly continuous kernels. As one might expect, this system has a solution on the whole segment and, moreover, this solution is unique in the class of continuous functions (see Section 5.1). This provides reason enough to conclude that the system (6.5.24), (6.5.26) has a unique continuous solution on the whole segment $[0, T]$ and thereby complete the proof of the theorem.■

Let us highlight a particular case of Theorem 6.5.4 with $m = 0$ in which condition (6.5.4) admits an alternative form

$$B \in \mathcal{L}(X, Y). \tag{6.5.38}$$

This type of situation is covered by the following assertion.

Corollary 6.5.1 *Let the closed linear operator A with a dense domain generate a strongly continuous semigroup in the Banach space X, the operator B be in line with condition* (6.5.38) *and the relations $u_0 \in \mathcal{D}(A)$,*

$$\psi \in C^1([0, T]; Y), \qquad B u_0 = \psi(0)$$

hold. When the decomposition

$$f(t, u, p) = L_1(t)\, u + L_2(t)\, p + f(t)$$

is accepted with

$$L_1,\ A L_1 A^{-1} \in C([0, T]; \mathcal{L}(X)),$$

$$A L_2 \in C([0, T]; \mathcal{L}(Y, X)),$$

$$A f \in C([0, T]; X),$$

the operator $B L_2(t)$ is invertible for each $t \in [0, T]$ and

$$(B L_2)^{-1} \in C([0, T]; \mathcal{L}(Y)),$$

the inverse problem (6.5.1)–(6.5.3) *has a solution*

$$u \in C^1([0, T]; X) \bigcap C([0, T]; \mathcal{D}(A)), \qquad p \in C([0, T]; Y)$$

and this solution is unique in the indicated class of functions.

In trying to apply the results obtained to partial differential equations and, in particular, to the equation of neutron transport some difficulties do arise in verifying the premises of Theorem 6.5.4 or Corollary 6.5.1. We quote below other conditions of solvability of the inverse problem (6.5.1)–(6.5.3) relating to the linear case (6.5.35) and condition (6.5.38).

Theorem 6.5.5 *Let the closed linear operator A with a dense domain generate a strongly continuous semigroup in the Banach space X, the operator B satisfy condition* (6.5.38), *the inclusions $u_0 \in \mathcal{D}(A)$, $\psi \in C^1([0, T]; Y)$ occur and the compatibility condition $B u_0 = \psi(0)$ hold. If the decomposition*

$$f(t, u, p) = L_1(t)\, u + L_2(t)\, p + f(t)$$

takes place with $L_1 \in C^1([0, T]; \mathcal{L}(X))$, $L_2, A L_2 \in C([0, T]; \mathcal{L}(Y, X))$ and the function $f = f_1 + f_2$, where

$$f_1 \in C^1([0, T]; X)$$

and

$$f_2 \in C([0, T]; \mathcal{D}(A)),$$

the operator $B L_2(t)$ is invertible for each $t \in [0, T]$ and

$$(B L_2)^{-1} \in C([0, T]; \mathcal{L}(Y)),$$

then the inverse problem (6.5.1)–(6.5.3) *has a solution*

$$u \in C^1([0, T]; X), \qquad p \in C([0, T]; Y)$$

and this solution is unique in the indicated class of functions.

Proof Being concerned with the functions $u \in C^1([0, T]; X)$ and $p \in C([0, T]; Y)$, we operate in the sequel with the new functions

$$F_1(t) = L_1(t)\, u(t), \qquad F_2(t) = L_2(t)\, p(t).$$

From the conditions of the theorem it seems clear that the inclusions

$$F_1 \in C^1([0, T]; X), \qquad F_2, A F_2 \in C([0, T]; X)$$

occur, due to which relations (6.5.1)–(6.5.2) are equivalent to

$$u(t) = V(t)\, u_0 + \int_0^t V(t-s)\, f(s)\, ds + \int_0^t V(t-s)\, F_1(s)\, ds + \int_0^t V(t-s)\, F_2(s)\, ds. \tag{6.5.39}$$

In what follows we deal with a new function

$$\tilde{u}_0(t) = V(t)\, u_0 + \int_0^t V(t-s)\, f(s)\; ds\,.$$

In the theory of semigroups the inclusion

$$\tilde{u}_0(t) \in \mathcal{C}^1\big([0,\, T];\, X\big) \bigcap \mathcal{C}([0,\, T];\, \mathcal{D}(A))$$

was established for any element $u_0 \in \mathcal{D}(A)$ and any function $f = f_1 + f_2$ with the members $f_1 \in \mathcal{C}^1\big([0,\, T];\, X\big)$ and $f_2 \in \mathcal{C}\big([0,\, T];\, \mathcal{D}(A)\big)$. Likewise, using the well-known formulae of this theory we deduce the relationships

$$\text{(6.5.40)} \quad A \int_0^t V(t-s)\, F_1(s)\; ds = \int_0^t V(t-s)\, F_1'(s)\; ds + V(t)\, F_1(0) - F_1(t)\,,$$

$$\text{(6.5.41)} \quad A \int_0^t V(t-s)\, F_2(s)\; ds = \int_0^t V(t-s)\, A\, F_2(s)\; ds\,.$$

In the derivation of the governing equation for the function p the operator B applies to equation (6.5.1) with further reference to relation (6.5.35). The outcome of this is

$$\text{(6.5.42)} \qquad \psi'(t) = B\, A\, u(t) + B\, L_1(t)\, u(t) + B\, L_2(t)\, p(t) + B\, f(t)\,.$$

Calculating the values $A\, u(t)$ and $B\, L_1(t)\, u(t)$ with the aid of relations (6.5.39)–(6.5.41), substituting the resulting expressions into (6.5.42) and resolving the relevant equations with respect to the function $p(t)$, we find that

$$(6.5.43) \qquad p(t) = \big(B\,L_2(t)\big)^{-1}\Bigg[\psi'(t) - B\,A\,\tilde{u}_0(t)$$
$$- B\left(\int\limits_0^t V(t-s)\,F_1'(s)\,ds + V(t)\,F_1(0) - F_1(t)\right)$$
$$- B\int\limits_0^t V(t-s)\,A\,F_2(s)\,ds - B\,L_1(t)\,\tilde{u}_0(t)$$
$$- B\,L_1(t)\int\limits_0^t V(t-s)\,F_1(s)\,ds$$
$$- B\,L_1(t)\int\limits_0^t V(t-s)\,F_2(s)\,ds - B\,f(t)\Bigg]\,.$$

The next step is to insert in (6.5.43) the explicit formulae, whose use permits us to express the functions F_1 and F_2 via the functions u and p. After that, the nonintegral term

$$-F_1(t) = -L_1(t)\,u(t)$$

should be excluded from further consideration. This can be done using formula (6.5.39). As a final result we get

$$(6.5.44) \qquad p(t) = \tilde{p}_0(t) + \int\limits_0^t K(t,s)\,u(s)\,ds$$
$$+ \int\limits_0^t L(t,s)\,u'(s)\,ds$$
$$+ \int\limits_0^t M(t,s)\,p(s)\,ds\,,$$

where

$$\begin{aligned}
\tilde{p}_0(t) &= \big(B\, L_2(t)\big)^{-1} \Big[\, \psi'(t) - B\, A\, \tilde{u}_0(t) \\
&\qquad - B\, V(t)\, L_1(0)\, u_0 - B\, f(t) \Big] , \\
K(t,s) &= -\, \big(B\, L_2(t)\big)^{-1} B\, V(t-s)\, L_1'(s) , \\
L(t,s) &= -\, \big(B\, L_2(t)\big)^{-1} B\, V(t-s)\, L_1(s) , \\
M(t,s) &= -\, \big(B\, L_2(t)\big)^{-1} B\, V(t-s)\, A\, L_2(s) .
\end{aligned}$$

As equation (6.5.44) contains a derivative of the function u, there is a need for enlarging the system (6.5.39), (6.5.44) by an equation for u', which is obtained by formal differentiating of both sides of (6.5.39). Using the general rules of differentiating

$$\Bigg(\int_0^t V(t-s)\, F_1(s)\; ds \Bigg)' = \int_0^t V(t-s)\, F_1'(s)\; ds + V(t)\, F_1(0) ,$$

$$\Bigg(\int_0^t V(t-s)\, F_2(s)\; ds \Bigg)' = \int_0^t V(t-s)\, A\, F_2(s)\; ds + F_2(t)$$

and excluding the nonintegral term $F_2(t) = L_2(t)\, p(t)$ with the aid of (6.5.44), we arrive at

$$\begin{aligned}
u'(t) &= \tilde{w}_0(t) + \int_0^t K(t,s)\, u(s)\; ds \\
&\qquad + \int_0^t L_1(t,s)\, u'(s)\; ds \\
&\qquad + \int_0^t M_1(t,s)\, p(s)\; ds ,
\end{aligned} \tag{6.5.45}$$

where

$$\begin{aligned}
\tilde{w}_0(t) &= \tilde{u}_0'(t) + V(t)\,L_1(0)\,u_0 + L_2(t)\,\tilde{p}_0(t)\,, \\
K_1(t,s) &= V(t-s)\,L_1'(s) + L_2(t)\,K(t,s)\,, \\
L_1(t,s) &= V(t-s)\,L_1(s) + L_2(t)\,L(t,s)\,, \\
M_1(t,s) &= V(t-s)\,A\,L_2(s) + L_2(t)\,M(t,s)\,.
\end{aligned}$$

In what follows the object of investigation is the system of equations

$$\begin{aligned}
u(t) &= \tilde{u}_0(t) + \int_0^t V(t-s)\,L_1(s)\,u(s)\;ds \\
&\quad + \int_0^t V(t-s)\,L_2(s)\,p(s)\;ds\,,
\end{aligned} \tag{6.5.46}$$

$$\begin{aligned}
p(t) &= \tilde{p}_0(t) + \int_0^t K(t,s)\,u(s)\;ds \\
&\quad + \int_0^t L(t,s)\,w(s)\;ds \\
&\quad + \int_0^t M(t,s)\,p(s)\;ds\,,
\end{aligned} \tag{6.5.47}$$

$$\begin{aligned}
w(t) &= \tilde{w}_0(t) + \int_0^t K_1(t,s)\,u(s)\;ds \\
&\quad + \int_0^t L_1(t,s)\,w(s)\;ds \\
&\quad + \int_0^t M_1(t,s)\,p(s)\;ds\,,
\end{aligned} \tag{6.5.48}$$

which constitute what is called a system of the Volterra integral equations of the second kind with continous nonhomogeneous terms and strongly continuous kernels. This system has a unique continuous solution being a uniform limit of the **successive approximations**

$$\begin{aligned}
u_{n+1}(t) &= \tilde{u}_0(t) + \int_0^t V(t-s)\, L_1(s)\, u_n(s)\, ds \\
&\quad + \int_0^t V(t-s)\, L_2(s)\, p_n(s)\, ds\,, \\
p_{n+1}(t) &= \tilde{p}_0(t) + \int_0^t K(t,s)\, u_n(s)\, ds \\
&\quad + \int_0^t L(t,s)\, w_n(s)\, ds \\
&\quad + \int_0^t M(t,s)\, p_n(s)\, ds\,, \\
w_{n+1}(t) &= \tilde{w}_0(t) + \int_0^t K_1(t,s)\, u_n(s)\, ds \\
&\quad + \int_0^t L_1(t,s)\, w_n(s)\, ds \\
&\quad + \int_0^t M_1(t,s)\, p_n(s)\, ds\,.
\end{aligned}$$

By plain calculations we are led to

(6.5.49)

$$\begin{aligned}
u'_{n+1}(t) &= A\, u_{n+1}(t) + \Big(\tilde{u}'_0(t) - A\, \tilde{u}_0(t) \Big) \\
&\quad + L_1(t)\, u_n(t) + L_2(t)\, p_n(t)\,,
\end{aligned}$$

(6.5.50)

$$\begin{aligned} w_{n+1}(t) &= A\, u_{n+1}(t) + \Big(\tilde{u}_0'(t) - A\, \tilde{u}_0(t) \Big) \\ &\quad + L_1(t)\, u_n(t) + L_2(t)\, p_{n+1}(t) \\ &\quad + \int_0^t V(t-s)\, L_1(s) \big(w_n(s) - u_n'(s) \big)\, ds\,. \end{aligned}$$

Comparison of (6.5.49) and (6.5.50) gives

(6.5.51)
$$\begin{aligned} w_{n+1}(t) - u_{n+1}'(t) &= L_2(t)\, \big(p_{n+1}(t) - p_n(t) \big) \\ &\quad + \int_0^t V(t-s)\, L_1(s) \big(w_n(s) - u_n'(s) \big)\, ds\,. \end{aligned}$$

The sequence $p_n(t)$ converges to the function $p(t)$ as $n \to \infty$ uniformly over the segment $[0,\, T]$, so that the sequence

$$c_n = \sup_{t \in [0,\, T]} \Big\| L_2(t)\, \big(p_{n+1}(t) - p_n(t) \big) \Big\| \longrightarrow \quad \text{as} \quad n \to \infty\,.$$

In conformity with (6.5.51) the norm of the difference

$$r_n(t) = w_n(t) - u_n'(t)$$

can be estimated as follows:

(6.5.52)
$$\| r_{n+1}(t) \| \leq c_n + M \int_0^t \| r_n(s) \|\, ds\,,$$

where M is a positive constant. We claim that the sequence of functions $z_n(t)$ converges uniformly to zero as $n \to \infty$. Indeed, estimate (6.5.52) being iterated yields the inequality

(6.5.53)
$$\begin{aligned} \| r_{k+m}(t) \| &\leq \sum_{s=0}^{m-1} c_{k+m-1-s}\, \frac{(M\, t)^s}{s\,!} \\ &\quad + \frac{(M\, t)^m}{m\,!} \sup_{t \in [0,\, T]} \| r_k(t) \|\,. \end{aligned}$$

Let $\varepsilon > 0$. We choose the subscript k_0 in such a way that for all $k \geq k_0$ the inequality

$$|c_k| < \frac{\varepsilon}{2} e^{-MT}$$

holds. Then

$$\left| \sum_{s=0}^{m-1} c_{k+m-1-s} \frac{(Mt)^s}{s!} \right| < \frac{\varepsilon}{2} e^{-MT} \sum_{s=0}^{\infty} \frac{(MT)^s}{s!} = \frac{\varepsilon}{2} \tag{6.5.54}$$

for all $k \geq k_0$ and $m \geq 1$. Holding a number m_0 fixed so that

$$\frac{(MT)^m}{m!} \sup_{t \in [0,T]} \| r_{k_0}(t) \| < \frac{\varepsilon}{2} \tag{6.5.55}$$

for all $m > m_0$, we provide $n = k_0 + m$, $m > m_0$, as long as $n > k_0 + m_0$. For this reason the bound

$$\| r_n(t) \| = \| r_{k_0+m}(t) \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

is obtained for all values $t \in [0, T]$ on the basis of (6.5.53)–(6.5.55). But this means that the sequence of functions $r_n(t)$ converges to zero uniformly over the segment $[0, T]$.

To complete the proof, we observe that the sequence of functions

$$u_n'(t) = w_n(t) - r_n(t)$$

converges uniformly to $w(t)$ as $n \to \infty$, since the sequence $w_n(t)$ converges to the same function $w(t)$ and the sequence $r_n(t)$ converges uniformly to zero in both cases as $n \to \infty$. For this reason the function $u(t)$ is differentiable, $u'(t) = w(t)$ and, in particular, $u \in C^1([0, T]; X)$. By inserting

$$L_1(s)\, u(s) = F_1(s), \qquad L_2(s)\, p(s) = F_2(s)$$

and involving the explicit formula for $\tilde{u}_0(t)$, equation (6.5.46) is transformed into (6.5.39). Since the function u is continuously differentiable, we thus have

$$F_1 \in C^1([0, T]; X); \qquad F_2,\ A F_2 \in C([0, T]; X).$$

With these inclusions in view, relation (6.5.39) yields (6.5.1)–(6.5.2). Upon substituting u' in place of w involved in (6.5.47) we come to (6.5.44). Arguing in reverse order we see that relation (6.5.44) is followed by (6.5.43), which is equivalent to (6.5.42) on account of (6.5.39). It is worth recalling here that formula (6.5.39) gives a solution of the Cauchy problem (6.5.1)–(6.5.2). Because of (6.5.1), relation (6.5.42) becomes equivalent to the equality

$$B\,u'(t) = \psi'(t)\,,$$

which, in turn, is equivalent to

$$\big(B\,u(t)\big)' = \psi'(t)$$

as far as the operator B is bounded. Further derivation of (6.5.3) is stipulated by the compatibility condition $B\,u_0 = \psi(0)$.

In accordance with what has been said, we conclude that if the functions

$$u \in C^1([0,T];\,X) \bigcap C([0,T];\,\mathcal{D}(A))\,, \qquad p \in C([0,T];\,Y)$$

give a solution of the inverse problem (6.5.1)–(6.5.3), then the triple

$$(u,\ p,\ w = u')$$

is just a continuous solution of the system (6.5.46)–(6.5.48). Vice versa, if a triple of the functions

$$(u,\ p,\ w)$$

is a continuous solution of the system (6.5.46)–(6.5.48), then $w = u'$ and the pair of the relevant functions $(u,\ p)$ solves the inverse problem (6.5.1)–(6.5.3). As stated above, the system of the integral equations (6.5.46)–(6.5.48) has a continuous solution and this solution is unique. So, the desired assertion of Theorem 6.5.5 follows immediately. ■

6.6 Inverse problems with smoothing overdetermination: quasilinear parabolic equations

Unlike the case of constant coefficients, the complete and uniform theory for equations with **variable operator coefficients** is less advanced. In trying to overcome some difficulties involved different approaches to various wide classes of equations are offered. As in the theory of partial differential equations, in recent years the main objects of investigation were the parabolic and hyperbolic types of abstract equations. Various equations of parabolic type will be covered within the framework of the present section. Readers who are interested in obtaining more detailed knowledge of the general theory of abstract parabolic equations are advised to study

Amann (1986, 1987), Fattorini (1983), Henry (1981), Krein (1967), Mizohata (1977), Sobolevsky (1961), Yosida (1965).

Let X and Y be Banach spaces. In what follows by A we mean a mapping carrying out a subset from $[0, T] \times X$ into a class of closed linear operators in the space X. It is supposed, in addition, that

$$f\colon [0, T] \times X \times Y \mapsto X , \qquad B\colon [0, T] \mapsto \mathcal{L}(X, Y)$$

and

$$\psi\colon [0, T] \mapsto Y .$$

In such a setting we are looking for a pair of the functions

$$u \in C^1([0, T]; X), \qquad p \in C([0, T]; Y),$$

satisfying the system of relations

$$u'(t) = A(t, u(t))\, u(t) + f(t, u(t), p(t)) , \quad 0 \le t \le T , \tag{6.6.1}$$

$$u(0) = u_0 , \tag{6.6.2}$$

$$B(t)\, u(t) = \psi(t) , \qquad 0 \le t \le T . \tag{6.6.3}$$

It is appropriate to mention here that the function u satisfies equation (6.6.1) in a pointwise manner, meaning, in particular, that for any $t \in [0, T]$ the element $u(t)$ belongs to the domain of the operator $A(t, u(t))$. Relations (6.6.2) and (6.6.3) together imply a necessary condition for the compatibility of the input data

$$B(0)\, u_0 = \psi(0) . \tag{6.6.4}$$

Equation (6.6.1) is required to be of **parabolic type**, that is,

(P1) *the closed linear operator $A_0 = A(0, u_0)$ has the dense domain $\mathcal{D}$, the half-plane $\operatorname{Re}\lambda \ge 0$ is contained in the resolvent set of the operator A_0 and for any λ with $\operatorname{Re}\lambda \ge 0$ the inequality holds:*

$$\left\| (A_0 - \lambda I)^{-1} \right\| \le \frac{c}{1 + |\lambda|} ;$$

(P2) *there are numbers $\alpha \in (0, 1)$ and $R > 0$ such that for all $u \in X$ with $\| u \| \le R$ and all $t \in [0, T]$ the linear closed operator $A(t, A_0^{-\alpha} u)$ has the domain $\mathcal{D}$ and there exists a constant $\beta \in (0, 1)$ such that for all t, $s \in [0, T]$ and all u, $v \in X$ with $\| u \| \le R$ and $\| v \| \le R$ the estimate is valid:*

$$\left\| \left[A(t,\, A_0^{-\alpha}\, u) - A(s,\, A_0^{-\alpha}\, v) \right] A_0^{-1} \right\| \le c\, \left(|\, t - s\,|^{\beta} + \|\, u - v\,\| \right).$$

The reader can encounter in the modern literature different definitions of belonging to the parabolic class. This is due to the fact that different methods are much applicable in investigating the Cauchy problem (6.6.1)–(6.6.2). Common practice involves the definition of Sobolevsky (1961) and so we adopt this definition in subsequent studies.

We impose the extra restrictions on the input data of the inverse problem concerned:

(P3) $u_0 \in \mathcal{D}, \quad \|\, A_0^{\alpha}\, u_0 \| < R, \quad \psi \in C^{1+\beta}([0,\, T];\, Y)$.

The operator B is subject to the following conditions:

(P4) *for all $t,\, s \in [0,\, T]$ and all $u,\, v \in X$ with $\|\, u\,\| \le R$ and $\|\, v\,\| \le R$ the operator $B(t)\, A\big(\, t,\, A_0^{-\alpha} u\,\big)\, A_0^{-\alpha}$ is bounded, the estimate*

$$\left\| \, B(t) \left[A\big(\, t, A_0^{-\alpha}\, u\,\big) - A\big(\, s, A_0^{-\alpha}\, v\,\big) \right] A_0^{-\alpha} \right\| \le c \Big(|\, t - s\,|^{\beta} + \|\, u - v\,\| \Big) \tag{6.6.5}$$

is true and

$$B \in C^{1+\beta}([0,\, T];\, \mathcal{L}(X, Y))\,. \tag{6.6.6}$$

The boundedness of the operator

$$B(t)\, A\big(\, t,\, A^{-\alpha} u\,\big)\, A_0^{-\alpha}$$

ensures that the operator B possesses a certain **smoothing effect** similar to (6.3.1). However, the requirement imposed above is more weaker than the condition for the operator $B\, A$ to be bounded.

As in Section 6.3 we may attempt the function f in the form

$$f(t, u, p) \;=\; f_1(t, u) + f_2(t, u, p)\,. \tag{6.6.7}$$

The conditions playing here the same role as conditions (A)–(E) in Section 6.3 are as follows:

(P5) *for the element*

$$z_0 \,=\, \psi'(0) + B(0)\, A_0\, u_0 - B(0)\, f_1(0, u_0) - B'(0)\, u_0 \tag{6.6.8}$$

the equation

$$B(0)\, f_2(0, u_0, p) = z_0$$

with respect to p has a unique solution $p_0 \in Y$;

(P6) there exists a mapping f_3: $[0, T] \times Y \times Y \mapsto Y$, for which

$$B(t)\; f_2(t, u, p) = f_3\big(t,\, B(t)\, u,\, p\big)\,;$$

(P7) for any fixed value $t \in [0, T]$ the mapping

$$z = f_3(t, \psi(t), p)$$

has in the ball $S_Y(p_0, R)$ the inverse

$$p = \Phi(t, z)\,; \tag{6.6.9}$$

(P8) for all t, $s \in [0, T]$ and all u, v with $\| u \| \leq R$ and $\| v \| \leq R$ and all p, $q \in Y$ with $\| p - p_0 \| \leq R$ and $\| q - p_0 \| \leq R$ the collection of inequalities hold:

$$\| f_1\big(t, A_0^{-\alpha} u\big) - f_1\big(s, A_0^{-\alpha} v\big) \| \leq c\,\big(|\, t - s\,|^\beta + \| u - v \|\big)\,,$$

$$\| f_2\big(t, u, p\big) - f_2\big(s, v, q\big) \| \leq c\,\Big(|\, t - s\,|^\beta + \| u - v \| + \| p - q \|\Big)\,;$$

(P9) there is a number $R_1 > 0$ such that for all t, $s \in [0, T]$ and all elements z_1, $z_2 \in Y$ with $\| z_1 - z_0 \| \leq R_1$ and $\| z_2 - z_0 \| \leq R_1$ the mapping (6.6.9) obeys the estimate

$$\| \Phi(t, z_1) - \Phi(s, z_2) \| \leq c\,\big(|\, t - s\,|^\beta + \| z_1 - z_2 \|\big)\,.$$

Theorem 6.6.1 *Let conditions* (6.6.7), $(P1)$-$(P9)$ *and* (6.6.4) *hold and*

$$\sigma < \min\{\, 1 - \alpha,\, \beta\,\}\,.$$

Then there exists a number $T_1 > 0$ such that on the segment $[0, T_1]$ a solution u, p of the inverse problem (6.6.1)-(6.6.3) *exists and is unique in the class of functions*

$$u \in C^1([0, T_1];\, X)\,, \qquad p \in C^\sigma([0, T_1];\, Y)\,.$$

Proof Involving the function p and holding the values $T_1 \in (0, T]$ and $K > 0$ fixed, we are interested in problem (6.6.1)-(6.6.3). We denote by $Q = Q(T_1, K, \sigma)$ the set of all functions $v \in C([0, T_1]; X)$ for which $v(0) = A_0^\alpha u_0$ and

$$\| v(t) - v(s) \| \leq K \, | t - s |^\sigma , \qquad \forall t, \, s \in [0, T_1] .$$

The set so constructed is closed and bounded in the space $C([0, T_1]; X)$. The value T_1 is so chosen as to satisfy the inequality $\| v(t) \| \leq R$ for all $v \in Q$ and all $t \in [0, T_1]$. This is certainly true if

$$T_1 < \big((R - \| A_0^\alpha u_0 \|) / K \big)^{1/\sigma} ,$$

since the bound $\| v(0) \| < R$ is stipulated by condition (P3). On the same grounds as before, we define $P = P(T_1, K, \sigma)$ as the set of all functions $p \in C([0, T_1]; Y)$ for which $p(0) = p_0$ and

$$\| p(t) - p(s) \| \leq K \, | t - s |^\sigma , \qquad \forall t, \, s \in [0, T_1] .$$

It is possible to reduce the value T_1 provided that the inequality

$$\| p(t) - p_0 \| < R$$

holds for all $p \in P$ and all $t \in [0, T_1]$. In dealing with the functions $v \in Q$, $p \in P$ we define the operator function

$$A(v; t) = A\big(t, \, A_0^{-\alpha} \, v(t)\big)$$

and the function

$$F(v, p; t) = f\big(t, \, A_0^{-\alpha} \, v(t), p(t)\big) .$$

Due to the boundedness of the operator $A_0^{-\alpha}$ condition (P8) implies that

$$\Big| F(v, p; t) - F(v, p; s) \Big| \leq c \, | t - s |^\sigma , \tag{6.6.10}$$

where the constant c does not depend on $v \in Q$ and $p \in P$ both.

The **Cauchy problem** related to the function u comes second:

$$\begin{cases} u'(t) = A(v; t) \, u(t) + F(v, p; t) , & 0 \leq t \leq T_1 , \\ u(0) = u_0 . \end{cases} \tag{6.6.11}$$

It is well-known from Sobolevsky (1961) that for a sufficiently small value $T_1 > 0$ and any $v \in Q$ the operator function $A(v; t)$ generates an **evolution operator** $U(v; t, s)$ being strongly continuous for all $0 \leq s \leq t \leq T_1$. A solution of problem (6.5.11) in the class $u \in C^1([0, T_1]; X)$ exists and is given by the formula

$$u(t) = U(v;t,0)\,u_0 + \int_0^t U(v;t,s)\,F(v,p;s)\,ds\,. \tag{6.6.12}$$

Furthermore,

$$\left\| A_0^\alpha \left[U(v;t,0) - U(v;s,0) \right] A_0^{-1} \right\| \le c\,|t-s|^{1-\alpha} \tag{6.6.13}$$

and for any function $F \in C([0, T_1]; X)$ the estimate is true:

$$\begin{aligned} \left\| A_0^\alpha \left(\int_0^{t+\Delta t} U(v;t+\Delta t,s)\,F(s)\,ds - \int_0^t U(v;t,s)\,F(s)\,ds \right) \right\| & \\ \le c\,|\Delta t|^{1-\alpha} \left(|\log|\Delta t|| + 1 \right) \max_{[0,T_1]} \|F(t)\|\,. & \end{aligned} \tag{6.6.14}$$

What is more,

$$\| A_0^\alpha\, U(v;t,s) \| \le c\,|t-s|^{-\alpha}. \tag{6.6.15}$$

Here the constant c is independent of $v \in Q$ (for more detail see Sobolevsky (1961)). Recall once again that the same symbol c may stand for different constants in later calculations.

It is also a matter of the general experience that for a sufficiently small value T_1 there exists in the class Q a unique function depending, in general, on p and providing the same solutions to both problems (6.6.1)–(6.6.2) and (6.6.11). It was shown in Sobolevsky (1961) that $u(t) = A_0^{-\alpha}\, v(t)$.

Holding the function v fixed we are going to derive the governing equation for another function p. By virtue of conditions (6.6.4) and (6.6.6) relation (6.6.3) is equivalent to

$$B(t)\,u'(t) + B'(t)\,u(t) = \psi'(t)\,. \tag{6.6.16}$$

In turn, (6.6.1), (6.6.12) and (6.6.7) are followed by

$$\begin{aligned} B(t)\;u'(t) &= B(t)\left(f(t,u(t),p(t)) - A(t,u(t))\,u(t) \right) \\ &= B(t)\,f_1\left(t, A_0^{-\alpha}\,v(t)\right) + B(t)\,f_2\left(t,u(t),p(t)\right) \\ &\quad - B(t)\,A(v;t)\,U(v;t,0)\,u_0 - B(t)\,A(v;t) \\ &\quad \times \int_0^t U(v;t,s)\,F(v,p;s)\,ds\,. \end{aligned} \tag{6.6.17}$$

Likewise, (6.6.12) implies that

$$B'(t)\,u(t) \;=\; B'(t)\left(U(v;t,0)\,u_0 + \int_0^t U(v;t,s)\,F(v,p;s)\,ds\right). \tag{6.6.18}$$

To make our exposition more transparent and write some things in simplified form, it is convenient to introduce the notations

$$\begin{aligned} w(t) \;&=\; A_0^{\alpha}\,U(v;t,0)\,u_0 + A_0^{\alpha}\int_0^t U(v;t,s)\,F(v,p;s)\,ds\,, \\ a(v,t) \;&=\; \psi'(t) - B(t)\;f_1\big(t,\,A_0^{-\alpha}\,v(t)\big)\,, \\ D(v;t) \;&=\; B(t)\;A(v;t)\;A_0^{-\alpha} - B'(t)\;A_0^{-\alpha}\,, \\ z(t) \;&=\; a(v;t) + D(v;t)\;w(t)\,. \end{aligned}$$

Substituting (6.6.17) and (6.6.18) into (6.6.16) yields

$$B(t)\;f_2\big(t,u(t),p(t)\big) \;=\; z(t)\,. \tag{6.6.19}$$

The increment of the function z can be evaluated as follows:

$$\begin{aligned} \|\,z(t)-z(s)\| \le\; &\|\,a(v;t)-a(v;s)\| \\ &+\;\|\,D(v;t)-D(v;s)\|\,\cdot\,\|\,w(t)\,\| \\ &+\;\|\,D(v;s)\|\,\cdot\,\|\,w(t)-w(s)\|\,. \end{aligned} \tag{6.6.20}$$

From the definition of the function $a(v;t)$ it follows that

$$\begin{aligned} \|\,a(v;t)-a(v;s)\|\; \le \|\,\psi'(t)-\psi'(s)\| &+ \|\,B(t)-B(s)\| \\ \times\;\|\,f_1\big(t,\,A_0^{-\alpha}\,v(t)\big)\| &+ \|\,B(s)\| \\ \times\;\|\,f_1\big(t,\,A_0^{-\alpha}\,v(t)\big) &- f_1\big(s,\,A_0^{-\alpha}\,v(s)\big)\|\,. \end{aligned}$$

Since $v\in Q$, the inequality $\|\,v(t)\| < R$ holds true for any $t\in[0,\,T_1]$. Therefore, under condition (P8) there is a constant $M>0$ such that for all $t\in[0,\,T_1]$ and all $v\in Q$ the estimate

$$\|\,f_1(t,\,A_0^{-\alpha}\,v(t))\| \;\le\; M$$

is valid. A proper choice of the constant M (enlarged if necessary) provides the validity of the inequality

$$\| B(t)\| \leq M$$

for all $t \in [0, T_1]$. By conditions (P3), (6.6.6) and (P8),

$$\| a(v;t) - a(v;s)\| \leq c\,|t-s|^{\beta} + c\,M\,|t-s| + c\,M\Big(|t-s|^{\beta} + \| v(t) - v(s)\|\Big)\,. \tag{6.6.21}$$

From the definition of $D(v;t)$, conditions (P4) and (6.6.6) in combination with the boundedness of the operator $A_0^{-\alpha}$ we deduce that

$$\Big\| D(v;t) - D(v;s)\Big\| \leq c\Big(|t-s|^{\beta} + \| v(t) - v(s)\|\Big)\,. \tag{6.6.22}$$

For any $v \in Q$, $p \in P$ the value $F(v,p;0) = f(0,u_0,p_0)$ and (6.6.10) together imply the existence of a constant $M > 0$ such that for any $v \in Q$, $p \in P$ and all $t \in [0, T_1]$

$$\| F(v,p;t)\| \leq M\,.$$

Since $u_0 \in \mathcal{D}$, we are led by relations (6.6.13) and (6.6.14) to

$$\| w(t+\Delta t) - w(t)\| \leq c\,|\Delta t|^{1-\alpha}\,\| A_0\, u_0\| + c\,M\,|\Delta t|^{1-\alpha}\Big(|\log|\Delta t|| + 1\Big)\,. \tag{6.6.23}$$

Keeping $\sigma < 1-\alpha$ and letting $\Delta t \to 0$, it is not difficult to establish the **asymptotic relation**

$$w(t+\Delta t) - w(t) = o(|\Delta t|^{\sigma})\,,$$

which is uniform in $v \in Q$, $p \in P$ and over the segment $t \in [0, T_1]$. Furthermore, by the definition of the function w we establish the relationship

$$w(0) = A_0^{\alpha}\, u_0\,,$$

thereby clarifying that the function w belongs to Q for a sufficiently small number T_1 and any $v \in Q$, $p \in P$.

In the current situation the action of the operator A_0^{α} with respect to relation (6.6.12) leads to the equation for the function

$$v(t) = w(t)\,. \tag{6.6.24}$$

Observe that $z(0) = z_0$, where the element z_0 is defined by (6.6.8). Inserting the value $s = 0$ in (6.6.20) and taking into account (6.6.21)–(6.6.22), we arrive at

$$\| z(t) - z_0 \| \leq R_1 , \qquad \forall\, t \in [0, T_1] ,$$

when T_1 is small enough and both elements v and w belong to Q. By virtue of assumptions (P6)–(P7) equation (6.6.19) can be recast as

$$p(t) = \Phi\big(t, z(t)\big) . \tag{6.6.25}$$

The desired assertion will be proved if we succeed in showing that the system (6.6.24)–(6.6.25) possesses in $Q \times P$ a unique solution for a sufficiently small value T_1. This can be done using the function

$$g(t) = \Phi\big(t, a(w;t) + D(w;t)\, w(t)\big) .$$

It is straightforward to verify that $g(0) = p_0$. Moreover, condition (P9) implies the estimate

(6.6.26)
$$\begin{aligned} \| g(t) - g(s) \| \leq c\ \Big(| t - s |^\beta &+ \| a(w;t) - a(w;s) \| \\ &+ \| D(w;t) - D(w;s) \| \cdot \| w(t) \| \\ &+ \| D(w;s) \| \cdot \| w(t) - w(s) \| \Big) . \end{aligned}$$

Using estimate (6.6.22) behind we deduce that there is a constant $M > 0$, for which the inequality

$$\| D(w,t) \| \leq M$$

holds true for any $w \in Q$ and all $t \in [0, T_1]$. By successively applying inequalities (6.6.21)–(6.6.22) with w standing in place of v and relying on (6.6.26), we find that

$$\| g(t) - g(s) \| \leq c\ \big(| t - s |^\beta + \| w(t) - w(s) \| \big)$$

with a positive constant $c > 0$. Since $\sigma < \beta$, the existence of a sufficiently small value T_1 such that the function g belongs to P is ensured by the preceding inequality in combination with (6.6.23).

At the next stage the metric space $Q \times P$ is equipped with the metric β induced by the norm of the space $C([0, T_1]; X) \times C([0, T_1]; Y)$, making it possible to introduce in this space the operator

$$G\colon\ (v, p) \mapsto (w, g) .$$

As we have mentioned above, the operator G carries out $Q \times P$ into itself if the value T_1 is sufficiently small.

As can readily be observed, the solutions of the system (6.6.24)–(6.6.25) coincide with the fixed points of the operator G.

Let $v_1, v_2 \in Q$; $p_1, p_2 \in P$ and

$$(w_1, g_1) = G(v_1, p_1), \qquad (w_2, p_2) = G(v_2, p_2).$$

Putting these together with the relationships $u_1 = A_0^{-\alpha} w_1$ and $u_2 = A_0^{-\alpha} w_2$ we derive the system

$$\text{(6.6.27)} \qquad \begin{cases} u_1'(t) = A(v_1; t)\, u_1(t) + F(v_1, p_1; t), \\ u_2'(t) = A(v_2; t)\, u_2(t) + F(v_2, p_2; t), \end{cases}$$

supplied by the condition $u_1(0) = u_2(0) = u_0$. Whence it follows that the function $u = u_1 - u_2$ solves the **Cauchy problem**

$$\text{(6.6.28)} \qquad \begin{cases} u'(t) = A(v_1; t)\, u(t) + F(t), \quad 0 \le t \le T_1, \\ u(0) = 0, \end{cases}$$

with $F(t) = \big[A(v_2; t) - A(v_1; t)\big]\, u_2(t) + F(v_1, p_1; t) - F(v_2, p_2; t)$ incorporated.

By condition (P2) and the definition of Q the function

$$A(t) = \big[A(v_2; t) - A(v_1; t)\big]\, A_0^{-1}$$

happens to be of Hölder's type in the space $\mathcal{L}(X)$ with exponent σ and constant not depending on $v_1, v_2 \in Q$.

On the other hand, estimates (1.71) and (2.27) from Sobolevsky (1961) could be useful in the sequel if they are written in terms of problem (6.6.27). With these in view, inequality (6.6.10) clarifies that the function $A_0 u_2$ is of Hölder's type for $t > 0$ and is uniformly bounded with respect to $v_2 \in Q$, $p_2 \in P$, $t \in [0, T_1]$, that is,

$$\text{(6.6.29)} \qquad \big\| A_0\, u_2(t) \big\| \le M.$$

Therefore, a solution of the Cauchy problem (6.6.28) can be represented via the evolution operator (for more detail see Sobolevsky (1961))

$$\text{(6.6.30)} \qquad u(t) = \int_0^t U(v_1; t, s)\, F(s)\, ds.$$

We are led by assumption (P2) to

$$\| A(t)\| \le c\, \| v_2(t) - v_1(t)\| . \tag{6.6.31}$$

In turn, the combination of (6.6.7) and (P8) gives the inequality

$$\| F(v_1, p_1; t) - F(v_2, p_2; t)\| \le c \Big(2\, \| v_2(t) - v_1(t)\| + \| p_2(t) - p_1(t)\| \Big) . \tag{6.6.32}$$

Applying the operator A_0^α to (6.6.30) with further passage to the appropriate norms we deduce with the aid of (6.6.15), (6.6.29) and (6.6.31)–(6.6.32) that

$$\| w_1(t) - w_2(t)\| \le \int_0^t c\, | t - s |^{-\alpha} \Big(c\, M\, \| v_2(s) - v_1(s)\| + c\, (2\, \| v_2(t) - v_1(t)\| + \| p_2(t) - p_1(t)\|) \Big)\, ds .$$

This provides reason enough to conclude that there is a positive constant $c_1 > 0$ such that

$$\| w_1 - w_2\|_{C([0,T_1]; X)} \le c_1\, T_1^{1-\alpha} \rho\big((v_1, p_1), (v_2, p_2)\big) , \tag{6.6.33}$$

where ρ refers to a metric of the space $Q \times P$.

By assumption (P9), the following estimate

$$\| g_1(t) - g_2(t)\| \le c \Big(\| a(w_1; t) - a(w_2; t)\| + \| D(w_1; t) - D(w_2; t)\| \cdot \| w_1(t)\| + \| D(w_2; t)\| \cdot \| w_1(t) - w_2(t)\| \Big)$$

is ensured by the definition of the function g. By virtue of condition (P8) we thus have

$$\| a(w_1; t) - a(w_2; t)\| \le \| B(t)\| \cdot \| w_1(t) - w_2(t)\| ,$$

while condition (P4) implies that

$$\| D(w_1; t) - D(w_2; t)\| \le c\, \| w_1(t) - w_2(t)\| .$$

From the inclusion $w_1 \in Q$ it follows that $\| w_1(t)\| < R$. The estimate

$$\| g_1 - g_2\|_{C([0,T_1]; Y)} \le c_2\, T_1^{1-\alpha} \rho\big((v_1, p_1), (v_2, p_2)\big) \tag{6.6.34}$$

can be justified by the uniform boundedness of the operator functions $D(w_2, t)$ and $B(t)$ and the preceding estimate (6.6.33). Here $c_2 > 0$ is a positive constant. On the basis of estimates (6.6.33)–(6.6.34) we conclude that the operator G is contractive in $Q \times P$ if T_1 is sufficiently small. This completes the proof of the theorem. ■

6.7 Inverse problems with singular overdetermination: semilinear parabolic equations

In this section we consider the inverse problem (6.6.1)–(6.6.3) in the case when equation (6.6.1) is **semilinear**. The appropriate problem statement is as follows:

$$u'(t) = A(t)\, u(t) + f\big(t, u(t), p(t)\big)\,, \quad 0 \le t \le T\,, \tag{6.7.1}$$

$$u(0) = u_0\,, \tag{6.7.2}$$

$$B\, u(t) = \psi(t)\,, \quad 0 \le t \le T\,. \tag{6.7.3}$$

Here the operator B is supposed to be bounded from the space X into the space Y and (6.7.3) is termed a **singular overdetermination**. The presence of (6.7.1) in the class of **parabolic equations** is well-characterized by the following condition:

(PP1) *for any $t \in [0, T]$ the linear operator A is closed in the space X and its domain is dense and does not depend on t, that is,*

$$\mathcal{D}(A(t)) = \mathcal{D}\,.$$

Also, if λ is such that $\operatorname{Re}\lambda \ge 0$, *then for all $t \in [0, T]$ and $\lambda \in \rho(A(t))$*

$$\left\| \big(A(t) - \lambda\, I \big)^{-1} \right\| \le \frac{c}{|\lambda| + 1}\,,$$

where c is a positive constant.

Further treatment of the inverse problem concerned necessitates involving the well-posedness of the **direct linear problem**

$$\begin{cases} u'(t) = A(t)\, u(t) + f(t)\,, & 0 \le t \le T\,, \\ u(0) = u_0\,. \end{cases} \tag{6.7.4}$$

Granted (PP1), a sufficient condition for the direct problem (6.7.4) to be well-posed is as follows:

(PP2) *there exist a constant $c > 0$ and a value $\alpha \in (0, 1]$ such that for all t, s, $\tau \in [0, T]$ the inequality holds:*

$$\left\| \big[A(t) - A(s) \big]\, A^{-1}(\tau) \right\| \le c\, |t - s|^{\alpha}\,.$$

Under conditions (PP1)–(PP2) the direct problem (6.7.4) has a strongly continuous evolution operator in the triangle

$$\Delta(T) = \{(t,s)\colon\ 0 \le s \le T,\ s \le t \le T\}.$$

If each such case $u_0 \in \mathcal{D}$ and the function f is subject to either of the following conditions: $A f \in C([0,T]; X)$ or f is of Hölder's type in the space X, then problem (6.7.4) has a unique solution $u \in C^1([0,T]; X)$, this solution is given by the formula

$$u(t) = U(t,0)\, u_0 + \int_0^t U(t,s)\, f(s)\, ds \tag{6.7.5}$$

and the operator function

$$W_1(t,s) = A(t)\, U(t,s)\, A^{-1}(s)$$

is strongly continuous in $\Delta(T)$. What is more, the function

$$W_{1+\beta}(t,s) = A^{1+\beta}(t)\, U(t,s)\, A^{-1-\beta}(s)$$

will also be strongly continuous in $\Delta(T)$ for $\beta < \alpha$. If one assumes, in addition, that $u_0 \in \mathcal{D}(A^{1+\beta}(0))$ and

$$A^{1+\beta}(t)\, f(t) \in C([0,T]; X),$$

then a solution of problem (6.7.4) will comply with the inclusion

$$A^\beta(0)\, u(t) \in C^1([0,T]; X). \tag{6.7.6}$$

We refer the reader to Sobolevsky (1961).

Let the operator B meet one more requirement similar in form to (5.1.7):

$$B \in \mathcal{L}(\mathcal{D}(A^\beta(0)), Y), \qquad \beta < \alpha, \tag{6.7.7}$$

where the manifold $\mathcal{D}(A^\beta(0))$ is equipped with the norm

$$\|u\|_\beta = \|A^\beta(0)\, u\|.$$

On the same grounds as before, we may attempt the function f in the form

$$f(t,u,p) = f_1(t,u) + f_2(t,u,p). \tag{6.7.8}$$

With the aid of relations (6.7.6)–(6.7.7) it is not difficult to show that the two conditions

$$(6.7.9) \qquad \psi \in C^1([0,T]; Y), \qquad B u_0 = \psi(0)$$

are necessary for the solvability of the inverse problem (6.7.1)–(6.7.3) in the class of functions

$$u \in C^1([0,T]; X), \qquad p \in C([0,T]; Y).$$

By means of the functions $\psi \in C^1([0,T]; Y)$ and $u_0 \in \mathcal{D}(B A(0))$ we define the element

$$(6.7.10) \qquad z_0 = \psi'(0) + B A(0) u_0 - B f_1(0, u_0)$$

and involve it in the additional requirements:

(PP3) *the equation* $B f_2(0, u_0, p) = z_0$ *with respect to* p *has a unique solution* $p_0 \in Y$;

(PP4) *there exists a mapping*

$$f_3\colon [0,T] \times Y \times Y \mapsto Y,$$

for which

$$(6.7.11) \qquad B f_2(t, u, p) = f_3(t, B u, p);$$

(PP5) *there is a number* $R > 0$ *such that for any* $t \in [0,T]$ *the mapping*

$$z = f_3(t, \psi(t), p)$$

has in the ball $S_Y(p_0, R)$ *the inverse*

$$(6.7.12) \qquad p = \Phi(t, z).$$

With the relation $u_0 \in \mathcal{D}(A^{1+\beta}(0))$ in view, the following manifolds

$$S_1(R,T) = \{(t,z,p)\colon 0 \le t \le T, \| z - A(0) u_0 \| < R, \| p - p_0 \| < R\},$$

$$S_0(R,T) = \{(t,z,p)\colon 0 \le t \le T, \| z - u_0 \| < R, \| p - p_0 \| < R\},$$

$$S_{1+\beta}(R,T) = \{(t,z,p)\colon 0 \le t \le T, \| z - A^{1+\beta}(0) u_0 \| < R, \| p - p_0 \| < R\}$$

are aimed to impose the extra smoothness restrictions:

(PP6) *there is a number $R > 0$ such that for $k = 0, 1, \beta + 1$ either of the functions*

$$A^k(t)\, f_1\big(t,\, A^{-k}(t)\, u\big)$$

and

$$A^k(t)\, f_2\big(t,\, A^{-k}(t)\, u,\, p\big)$$

is continuous with respect to the totality of variables and satisfies on the manifold $S_k(R,T)$ the Lipschitz condition with respect to (u,p);

(PP7) *the mapping (6.7.12) is continuous on the set $S_Y(z_0,\ R,T)$ and satisfies thereon the Lipschitz condition in z.*

Theorem 6.7.1 *Let conditions $(PP1)$, $(PP2)$, (6.7.7) and (6.7.9) hold. If $u_0 \in \mathcal{D}\big(A^{1+\beta}(0)\big)$ and conditions (6.7.8), $(PP3)$–$(PP7)$ are valid with the element z_0 given by formula (6.7.10), then there is a value $T_1 > 0$ such that on the segment $[0,\, T_1]$ a solution u, p of the inverse problem (6.7.1)–(6.7.3) exists and is unique in the class of functions*

$$u \in C^1\big([0,\, T_1];\, X\big)\,, \qquad A\, u \in C\big([0,\, T_1];\, X\big)\,, \qquad p \in C\big([0,\, T_1];\, Y\big)\,.$$

Proof By assumption, the inclusions

$$p \in C\big([0,\, T];\, Y\big)$$

and

$$v(t) = A(t)\, u(t) \in C\big([0,\, T];\, X\big)$$

imply that the superposition

$$f_1(t) \;=\; f\big(\, t,\, A^{-1}(t)\, v(t),\, p(t)\big)$$

is continuous. The same remains valid for the function $A(t)\, f_1(t)$. If so, the Cauchy problem (6.7.1)–(6.7.2) will be equivalent to the integral equation

$$u(t) \;=\; U(t,0)\, u_0 + \int\limits_0^t U(t,s)\, f\big(\, s, u(s), p(s)\big)\; ds \tag{6.7.13}$$

in the class of functions $u \in C\big([0,\, T];\, X\big)$ for which the inclusion

$$A\, u \;\in\; C\big([0,\, T];\, X\big)$$

occurs. Indeed, a solution to equation (6.7.1) complies with the condition

$$A^{1+\beta}(t)\,u(t) \in \mathcal{C}([0,T];\,X)$$

if the value T is sufficiently small. To make sure of it, we first perform the substitution $z(t) = A^{1+\beta}(t)\,u(t)$ with

$$z_0(t) = A^{1+\beta}(t)\,U(t,0)\,u_0 = W_{1+\beta}(t,0)\,A^{1+\beta}(0)\,u_0\,,$$

thereby justifying that the function u satisfies the integral equation (6.7.13) if and only if the function z solves the integral equation

$$z(t) = z_0(t) + \int_0^t W_{1+\beta}(t,s)\,g\big(s, z(s), p(s)\big)\,ds\,, \tag{6.7.14}$$

where

$$g(t,z,p) = A^{1+\beta}(t)\,f\big(t,\,A^{-1-\beta}(t)\,z,\,p\big)\,.$$

It is worth noting here that z_0 is continuous on the segment $[0,T]$ and the kernel $W_{1+\beta}$ is strongly continuous on the set $\Delta(T)$. Moreover, by condition (PP6) the function g is continuous and satisfies the Lipschitz condition in z. This serves to motivate that equation (6.7.14) has a unique local solution in the class of continuous functions. Reducing the value T, if necessary, one can accept (6.7.14) to be solvable on the whole segment $[0,T]$. Recall that T may depend on the function p.

Due to the interrelation between solutions to equations (6.7.13) and (6.7.14) the function

$$f(t) = f\big(t,\,u(t),\,p(t)\big)$$

satisfies the equality

$$A^{1+\beta}(t)\,f(t) = A^{1+\beta}(t)\,f\big(t,\,A^{-1-\beta}z(t),\,p(t)\big)$$

and is continuous on the strength of condition (PP6). In view of this, a solution of the Cauchy problem (6.7.1)–(6.7.2) obeys (6.7.6). Because of (6.7.7),

$$\big(B\,u(t)\big)' = B\,u'(t)\,,$$

so that relation (6.7.3) is equivalent to the following one:

$$B\,u'(t) = \psi'(t)\,, \qquad 0 \le t \le T\,,$$

making it possible to reduce the preceding on the basis of (6.7.1) to

$$p(t) = \Phi\big(t, \psi'(t) - B\,A(t)\,u(t) - B\,f_1\big(t,u(t)\big)\big)\,. \tag{6.7.15}$$

Here we have taken into account conditions (6.7.8), (PP4) and (PP5).

By involving formula (6.7.13) and substituting $z(t) = A^{1+\beta}(t)\, u(t)$ we are led to an alternative form of wriiting (6.7.15):

$$(6.7.16)\quad p(t) = \Phi\left(t,\, g_0(t) + g_1(t, z(t)) + \int_0^t K(t,s)\, g\big(s, z(s), p(s)\big)\, ds \right),$$

where

$$\begin{aligned} g_0(t) &= \psi'(t) - B\, A(t)\, U(t,0)\, u_0 - B\, f_1\big(t,\, U(t,0)\, u_0\big), \\ g_1(t,z) &= -B\,\big(f_1(t,\, A^{-1-\beta}(t)\, z) - f_1(t,\, U(t,0)\, u_0)\big), \\ K(t,s) &= -B\, A(t)\, U(t,s)\, A^{-1-\beta}(s), \\ g(t,z,p) &= A^{1+\beta}(t)\, f\big(t, A^{-1-\beta}(t)\, z, p\big). \end{aligned}$$

Since

$$\begin{aligned} B\, A(t)\, U(t,0)\, u_0 &= \big(B\, A^{-\beta}(0)\big)\, \big(A^{\beta}(0)\, A^{-\beta}(t)\big) \\ &\quad \times A^{1+\beta}(t)\, U(t,0)\, u_0 \\ &= \big(B\, A^{-\beta}(0)\big)\, \big(A^{\beta}(0)\, A^{-\beta}(t)\big) \\ &\quad \times W_{1+\beta}(t,0)\, \big(A^{1+\beta}(0)\, u_0\big), \end{aligned}$$

the function $g_0(t) \in C\big([0, T]; Y\big)$. This is due to the fact that the function $A^{\beta}(0)\, A^{-\beta}(t)$ is strongly continuous (see Sobolevsky (1961)). Also, we may attempt the function $g_1(t,z)$ in the form

$$\begin{aligned} g_1(t,z) &= \big(B\, A^{-\beta}(0)\big)\, \big(A^{\beta}(0)\, A^{-\beta}(t)\big)\, A^{-1}(t) \\ &\quad \times \Big(A^{1+\beta}(t)\, f_1\big(t, A^{-1-\beta}(t)\, z\big) \\ &\quad - A^{1+\beta}(t)\, f_1\big(t, A^{-1-\beta}(t) \\ &\quad \times W_{1+\beta}(t,0)\, A^{1+\beta}(0)\, u_0\big)\Big), \end{aligned}$$

implying that the function g_1 is continuous with respect to the totality of variables and satisfies the Lipschitz condition in z. Likewise, the kernel $K(t,s)$ admits the form

$$K(t,s) = -\big(B\, A^{-\beta}(0)\big)\, \big(A^{\beta}(0)\, A^{-\beta}(t)\big)\, W_{1+\beta}(t,s),$$

thereby justifying the strong continuity of $K(t,s)$ in $\Delta(T)$.

Thus, the system of the integral equations (6.7.14), (6.7.16) is derived. Observe that it is of some type similar to that obtained in Section 6.3 for the system (6.3.9), (6.3.12). Hence, exploiting some facts and adopting the well-developed tools of Theorem 6.3.1, it is plain to show that the system (6.7.14), (6.7.16) is solvable, thereby completing the proof of the theorem. ■

Of importance is the **linear case** when

$$f(t,u,p) = L_1(t)\,u + L_2(t)\,p + F(t)\,. \tag{6.7.17}$$

Theorem 6.7.2 *Let conditions (PP1)-(PP2), (6.7.7) and (6.7.9) hold, $u_0 \in \mathcal{D}(A^{1+\beta}(0))$, the operator functions*

$$L_1,\ A\,L_1\,A^{-1}, A^{1+\beta}\,L_1\,A^{-1-\beta} \in C([0,T];\mathcal{L}(X))\,,$$

$$A^{1+\beta}\,L_2 \in C([0,T];\,\mathcal{L}(Y,X))$$

and the function

$$A^{1+\beta}\,F \in C([0,T];\,X)\,.$$

If for any $t \in [0,T]$ the operator $B\,L_2(t)$ is invertible in the space Y and

$$(B\,L_2)^{-1} \in C([0,T];\,\mathcal{L}(Y))\,,$$

then a solution u, p of the inverse problem (6.7.1)-(6.7.3), (6.7.17) exists and is unique in the class of functions

$$u \in C^1([0,T];\,X)\,,\qquad A\,u \in C([0,T];\,X)\,,\qquad p \in C([0,T];\,Y)\,.$$

Proof By virtue of conditions (PP3)-(PP7) with the ingredients

$$f_1(t,u) = L_1(t)\,u + F(t)\,,$$

$$f_2(t,u,p) = L_2(t)\,p\,,$$

$$f_3(t,z,p) = B\,L_2(t)\,p\,,$$

$$\Phi(t,z) = \big(B\,L_2(t)\big)^{-1} z$$

one can recast the system of equations (6.7.14), (6.7.16) as

$$z(t) = z_0(t) + \int_0^t \big(K_1(t,s)\, z(s) + L_1(t,s)\, p(s)\big)\, ds\,, \tag{6.7.18}$$

$$p(t) = p_0(t) + \int_0^t \big(K_2(t,s)\, z(s) + L_2(t,s)\, p(s)\big)\, ds\,, \tag{6.7.19}$$

where

$$\begin{aligned}
z_0(t) &= W_{1+\beta}(t,0)\, A^{1+\beta}(0)\, u_0 \\
&\quad + \int_0^t W_{1+\beta}(t,s)\, A^{1+\beta}(s)\, F(s)\, ds\,, \\
K_1(t,s) &= W_{1+\beta}(t,s)\, \big(A^{1+\beta}(s)\, L_1(s)\, A^{-1-\beta}(s)\big)\,, \\
L_1(t,s) &= W_{1+\beta}(t,s)\, \big(A^{1+\beta}(s)\, L_2(s)\big)\,, \\
p_0(t) &= \big(B\, L_2(t)\big)^{-1} \Big[\psi'(t) - B\, F(t) \\
&\quad - B\, A(t)\, U(t,0)\, u_0 - B\, L_1(t)\, U(t,0)\, u_0 \\
&\quad - B\, A(t) \int_0^t U(t,s)\, F(s)\, ds \\
&\quad - B\, L_1(t) \int_0^t U(t,s)\, F(s)\, ds \Big]\,, \\
K_2(t,s) &= -\, \big(B\, L_2(t)\big)^{-1} \Big[B\, A(t)\, U(t,s)\, L_1(s) \\
&\quad \times A^{-1-\beta}(s) + B\, L_1(t)\, A^{-1-\beta}(t)\, W_{1+\beta}(t,s) \\
&\quad \times A^{1+\beta}(s)\, L_1(s)\, A^{-1-\beta}(s)\Big]\,, \\
L_2(t,s) &= -\, \big(B\, L_2(t)\big)^{-1} \Big[B\, A(t)\, U(t,s)\, L_2(s) \\
&\quad + B\, L_1(t)\, A^{-1-\beta}(t)\, W_{1+\beta}(t,s)\, A^{1+\beta}(s)\, L_2(s)\Big]\,.
\end{aligned}$$

The continuity of the function $z_0(t)$ and the strong continuity of the kernels K_1 and L_1 are a corollary of the initial assumptions, while the continuity of the function $p_0(t)$ follows from the set of relations

$$\begin{aligned}
B\,F(t) &= \left(B\,A^{-\beta}(0)\right)\left(A^{\beta}(0)\,A^{-\beta}(t)\right) \\
&\quad\times A^{-1}(t)\left(A^{1+\beta}(t)\,F(t)\right), \\
B\,A(t)\,U(t,0)\,u_0 &= \left(B\,A^{-\beta}(0)\right)\left(A^{\beta}(0)\,A^{-\beta}(t)\right) \\
&\quad\times W_{1+\beta}(t,0)\left(A^{1+\beta}(0)\,u_0\right), \\
B\,L_1(t)\,U(t,0)\,u_0 &= \left(B\,A^{-\beta}(0)\right)\left(A^{\beta}(0)\,A^{-\beta}(t)\right)A^{-1}(t) \\
&\quad\times \left(A^{1+\beta}(t)\,L_1(t)\,A^{-1-\beta}(t)\right) \\
&\quad\times W_{1+\beta}(t,0)\left(A^{1+\beta}(0)\,u_0\right), \\
B\,A(t)\,U(t,s)\,F(s) &= \left(B\,A^{-\beta}(0)\right)\left(A^{\beta}(0)\,A^{-\beta}(t)\right) \\
&\quad\times W_{1+\beta}(t,s)\left(A^{1+\beta}(s)\,F(s)\right), \\
B\,L_1(t)\,U(t,s)\,F(s) &= \left(B\,A^{-\beta}(0)\right)\left(A^{\beta}(0)\,A^{-\beta}(t)\right)A^{-1}(t) \\
&\quad\times \left(A^{1+\beta}(t)\,L_1(t)\,A^{-1-\beta}(t)\right) \\
&\quad\times W_{1+\beta}(t,s)\left(A^{1+\beta}(s)\,F(s)\right).
\end{aligned}$$

In turn, the strong continuity of the kernels K_2 and L_2 on $\Delta(T)$ is ensured by the identities

$$\begin{aligned}
B\,A(t)\,U(t,s)\,L_1(s)\,A^{-1-\beta}(s) &= \left(B\,A^{-\beta}(0)\right)\left(A^{\beta}(0)\,A^{-\beta}(t)\right)W_{1+\beta}(t,s) \\
&\quad\times \left(A^{1+\beta}(s)\,L_1(s)\,A^{-1-\beta}(s)\right), \\
B\,L_1(t)\,A^{-1-\beta}(t) &= \left(B\,A^{-\beta}(0)\right)\left(A^{\beta}(0)\,A^{-\beta}(t)\right)A^{-1}(t) \\
&\quad\times \left(A^{1+\beta}(t)\,L_1(t)\,A^{-1-\beta}(t)\right).
\end{aligned}$$

The desired assertion is simple to follow, since the system of the Volterra integral equations (6.7.18)–(6.7.19) appears to be linear. ■

6.8 Inverse problems with smoothing overdetermination: semilinear hyperbolic equations

As in Section 6.7 it is required to find the functions $u \in C^1([0, T]; X)$ and $p \in C([0, T]; Y)$ from the set of relations

$$u'(t) = A(t)\,u(t) + f(t, u(t), p(t)), \quad 0 \le t \le T, \tag{6.8.1}$$

$$u(0) = u_0, \tag{6.8.2}$$

$$B\,u(t) = \psi(t), \qquad 0 \le t \le T. \tag{6.8.3}$$

For any $t \in [0, T]$ a linear operator $A(t)$ with a dense domain is supposed to be closed in the space X. The symbol B designates a linear operator acting from X into Y and

$$f\colon [0, T] \times X \times Y \mapsto X .$$

Much progress in investigating the inverse problem (6.8.1)–(6.8.3) is connected with the subsidiary information that the **linear direct problem**

$$u'(t) = A(t)\,u(t) + f(t), \qquad 0 \le t \le T, \tag{6.8.4}$$

$$u(0) = u_0, \tag{6.8.5}$$

is well-posed.

Before proceeding to a more detailed framework, it is worth noting here that the reader can encounter in the modern literature several different definitions and criteria to decide for yourself whether equation (6.8.1) is of hyperbolic type. Below we offer one of the existing approaches to this issue and take for granted that

(H1) *for any $t \in [0, T]$ the operator $A(t)$ is the generator of a strongly continuous semigroup in the space X;*

(H2) *there are constants M and β such that for any $\lambda > \beta$ and any finite collection of points $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$ the inequality holds:*

$$\left\| (A(t_k) - \lambda I)^{-1} (A(t_{k-1}) - \lambda I)^{-1} \cdots (A(t_1) - \lambda I)^{-1} \right\| \le \frac{M}{(\lambda - \beta)^k} .$$

One assumes, in addition, that there exists a Banach space X_0, which can densely and continuously be embedded into the space X. In this view, it is reasonable to confine yourself to particular cases where

(H3) *there exists an operator function* $S(t)$ *defined on the segment* $[0, T]$ *with values in the space* $\mathcal{L}(X_0, X)$; *this function is strongly continuously differentiable on the segment* $[0, T]$ *and for any fixed value* $t \in [0, T]$ *the operator*

$$[S(t)]^{-1} \in \mathcal{L}(X, X_0)$$

and

$$S(t)\, A(t)\, [S(t)]^{-1} = A(t) + R(t),$$

where $R(t) \in \mathcal{L}(X)$ *and the operator function* $R(t)$ *is strongly continuous in the space* X;

(H4) *for any* $t \in [0, T]$ *the space* $X_0 \subset \mathcal{D}(A(t))$ *and the operator function* $A \in \mathcal{C}([0, T]; \mathcal{L}(X_0, X))$.

Under conditions (H1)–(H4) there exists an **evolution operator** $U(s,t)$, which is defined on the set

$$\Delta(T) = \{(t,s):\ 0 \le s \le T,\ s \le t \le T\}$$

and is strongly continuous on $\Delta(T)$ in the norm of the space X. By means of this operator a solution u of problem (6.8.4) is expressed by

$$u(t) = U(t,0)\, u_0 + \int_0^t U(t,s)\, f(s)\, ds\,. \tag{6.8.6}$$

For $u_0 \in X$ and $f \in \mathcal{C}([0, T]; X)$ formula (6.8.6) gives a continuous solution (in a sense of distributions) to equation (6.8.1) subject to condition (6.8.2). Furthermore, with the inclusions $u_0 \in X_0$ and $f \in \mathcal{C}([0, T]; X_0)$ in view, the preceding formula serves for a solution of the Cauchy problem (6.8.1)–(6.8.2) in the class of functions $\mathcal{C}^1([0, T]; X)$ (for more detail see Fattorini (1983), Kato (1970, 1973), Massey (1972)).

The present section devotes to the case when the operator B possesses a **smoothing effect**, meaning

$$B \in \mathcal{L}(X,Y), \qquad \overline{BA} \in \mathcal{C}([0, T]; \mathcal{L}(X,Y))\,. \tag{6.8.7}$$

Now our starting point is the solvability of the inverse problem (6.8.1)–(6.8.3) in the class of continuous functions. Omitting some details, a continuous solution of the inverse problem concerned is to be understood as a pair of the functions

$$u \in C([0, T]; X), \qquad p \in C([0, T]; Y),$$

satisfying the system of relations

$$(6.8.8) \qquad u(t) = U(t,0)\, u_0 + \int_0^t U(t,s)\, f\big(s, u(s), p(s)\big)\, ds\,, \quad 0 \le t \le T,$$

$$(6.8.9) \qquad B\, u(t) = \psi(t)\,, \qquad 0 \le t \le T.$$

The approved decomposition

$$(6.8.10) \qquad f(t,u,p) = f_1(t,u) + f_2(t,u,p)$$

will be used in the sequel. Allowing the element ψ to be differentiable at zero we define the element

$$(6.8.11) \qquad z_0 = \psi'(0) - \overline{B\, A(0)}\, u_0 - B\, f_1(0, u_0)$$

and assume that

(H5) *the equation* $B\, f_2(0, u_0, p) = z_0$ *with respect to* p *has a unique solution* $p_0 \in Y$;

(H6) *there exists a mapping* f_3: $[0, T] \times Y \times Y \mapsto Y$ *such that*

$$(6.8.12) \qquad B\, f_2(t,u,p) = f_3(t, B\, u, p)\,;$$

(H7) *there is a number* $R > 0$ *such that for any* $t \in [0, T]$ *the mapping* $z = f_3\big(t, \psi(t), p\big)$ *has in the ball* $S_Y(p_0, R)$ *the inverse*

$$(6.8.13) \qquad p = \Phi(t, z)\,.$$

In addition to the algebraic conditions (H5)–(H7), we impose the extra smoothness conditions:

(H8) *there is a number* $R > 0$ *such that either of the functions* $f_1(t,u)$ *and* $f_2(t,u,p)$ *is continuous with respect to the totality of variables and satisfies on the manifold* $S_{X\times Y}\big((u_0, p_0), R, T\big)$ *the Lipschitz condition with respect to* (u,p);

(H9) *there is a number* $R > 0$ *such that the mapping* (6.8.13) *is continuous with respect to the totality of variables and satisfies the Lipschitz condition in* z *on the manifold* $S_Y\big(z_0, R, T\big)$, *where the element* z_0 *is defined by* (6.8.11).

Theorem 6.8.1 *Let conditions* (6.8.7), (6.8.10), $(H1)$-$(H9)$ *hold and* $u_0 \in X$, $\psi \in C^1([0,T];Y)$ *and* $B\,u_0 = \psi(0)$. *Then there is a value* $T_1 > 0$ *such that on the segment* $[0, T_1]$ *a solution* u, p *of the inverse problem* (6.8.1)-(6.8.3) *exists and is unique in the class of continuous functions*

$$u \in C([0,T];X), \qquad p \in C([0,T];Y).$$

Proof By the initial assumptions the function

$$f(t) = f(t, u(t), p(t)) \tag{6.8.14}$$

is continuous on $[0,T]$ for arbitrary fixed functions $u \in C([0,T];X)$ and $p \in C([0,T];Y)$. Within notation (6.8.14), the definition of the solution of the inverse problem (6.8.8)-(6.8.9) is involved in the further derivation of the equation

$$\psi(t) = B\,U(t,0)\,u_0 + B\int_0^t U(t,s)\,f(s)\,ds. \tag{6.8.15}$$

In the sequel we shall need yet two assertions to be proved.

Lemma 6.8.1 *For any element* $u_0 \in X$ *and all values* $t \geq s$ *the function*

$$\psi_0(t) = B\,U(t,s)\,u_0$$

is differentiable and

$$\psi_0'(t) = \overline{B\,A(t)}\,U(t,s)\,u_0\,.$$

Proof Recall that the subspace X_0 is dense in the space X. Hence there exists a sequence of elements u_n from the space X_0 converging to an element u_0 in the space X. Set

$$\psi_n(t) = B\,U(t,s)\,u_n$$

and appeal to the well-known estimate for the evolution operator

$$\|U(t,s)\| \leq M\,\exp\big(\beta\,(t-s)\big)\,.$$

The outcome of this is

$$\|\psi_n(t) - \psi_0(t)\| \leq M\,\|B\|\,\exp\big(|\beta|\,T\big)\,\|u_n - u_0\|\,,$$

implying that the sequence of functions ψ_n converges as $n \to \infty$ to the function ψ_0 uniformly over $[s, T]$. Since $u_n \in X_0$, the function ψ_n is differentiable on the segment $[s, T]$ and

$$\psi_n'(t) = B\,A(t)\,U(t,s)\,u_n = \overline{B\,A(t)}\;U(t,s)\,u_n\,,$$

whence it follows that

$$\psi_n'(t) \longrightarrow \varphi(t) = \overline{B\,A(t)}\;U(t,s)\,u_0$$

as $n \to \infty$. On the basis of the estimate

$$\|\,\psi_n'(t) - \varphi(t)\| \le \sup_{[0,T]} \|\,\overline{B\,A(t)}\,\|\,M\,\exp\big(\,|\,\beta\,|\,T\,\big)\,\|\,u_n - u_0\,\|$$

we deduce that the preceding convergence is uniform over the segment $[s, T]$. Because of this fact, the function ψ_0 is differentiable and $\psi_0' = \varphi$, which completes the proof of the lemma.■

Lemma 6.8.2 *Let* $f \in C\big([0,T]; X\big)$ *and*

$$g(t) = B \int_0^t U(t,s)\,f(s)\;ds\,.$$

Then the function g *is differentiable on the segment* $[0,T]$ *and*

$$g'(t) = \overline{B\,A(t)} \int_0^t U(t,s)\,f(s)\;ds + B\,f(t)\,.$$

Proof From the additivity of integrals it follows that

$$\text{(6.8.16)}\qquad \frac{g(t+h) - g(t)}{h} = \frac{1}{h}\int_t^{t+h} B\,U(t+h,s)\,f(s)\;ds + \int_0^t B\,\frac{U(t+h,s) - U(t,s)}{h}\;f(s)\;ds\,.$$

Since

$$\Big\|\,\frac{1}{h}\int_t^{t+h} B\,U(t+h,s)\,f(s)\;ds - B\,f(t)\Big\|$$

$$= \Big\|\,\frac{1}{h}\int_t^{t+h} B\,U(t+h,s)\,f(s)\;ds - \frac{1}{h}\int_t^{t+h} B\,U(t,t)\,f(t)\;ds\Big\|$$

$$\le \frac{\|\,B\,\|}{h}\int_t^{t+h} \big\|\,U(t+h,s)\,f(s) - U(t,t)\,f(t)\big\|\;ds$$

and the function $H(t,s) = U(t,s)\,f(s)$ is uniformly continuous in the triangle $\Delta(T)$, the first term on the right-hand side of (6.8.16) converges to $B\,f(t)$ as $h \to 0$.

The next step is to show that the limit as $h \to 0$ of the second term on the right-hand side of (6.8.16) is equal to

$$\overline{B\,A(t)} \int_0^t U(t,s)\,f(s)\,ds\,.$$

This fact immediately follows from the relation

$$\lim_{h\to\infty} \int_0^t g_h(s)\,ds = \int_0^t g_0(s)\,ds\,, \tag{6.8.17}$$

where

$$g_h(s) = B\,\frac{U(t+h,s) - U(t,s)}{h}\,f(s)\,,$$

$$g_0(s) = \overline{B\,A(t)}\,U(t,s)\,f(s)\,.$$

First of all observe that by Lemma 6.8.1 the function $g_0(s)$ is a pointwise limit of the functions $g_h(s)$ as $h \to 0$ on $[0,\,T]$. Therefore, relation (6.8.17) will be established if we succeed in showing that the norms of these functions are uniformly bounded. This can be done using the function $\mu_s(t) = B\,U(t,s)\,f(s)$ of the variable t. By Lemma 6.8.1,

$$\mu_s'(t) = \overline{B\,A(t)}\,U(t,s)\,f(s) = g_0(s)\,.$$

By the same token,

$$g_h(s) = \frac{\mu_s(t+h) - \mu_s(t)}{h}$$

and, in agreement with the mean value theorem,

$$\|\,g_h(s)\| = \left\| \frac{\mu_s(t+h) - \mu_s(t)}{h} \right\| \le \sup_{[0,\,T]} \|\,\mu_s'(t)\|\,. \tag{6.8.18}$$

On the other hand, $\mu_s'(t)$ as a function of two independent variables is continuous in $\Delta(T)$ and thus the right-hand side of (6.8.18) is finite. This completes the proof of the lemma.■

Returning to the proof of the theorem we note in passing that the right-hand side of (6.8.17) is differentiable on account of Lemmas 6.8.1–6.8.2. By virtue of the compatibility condition relation (6.8.17) is equivalent to its differential implication taking the form

$$(6.8.19)\quad \psi'(t) = \overline{B\,A(t)}\, U(t,0)\, u_0 + \overline{B\,A(t)} \int_0^t U(t,s)\, f(s)\, ds + B\, f(t)$$

and relying on the formulae from the lemmas we have mentioned above. We claim that if the second equation of the system (6.8.8)–(6.8.9) is replaced by (6.8.19), then an equivalent system can be obtained in a similar way. Indeed, when (6.8.8), (6.8.14) and (6.8.19) are put together, we finally get

$$(6.8.20)\qquad \psi'(t) = \overline{B\,A(t)}\, u(t) + B\, f\big(t, u(t), p(t)\big)\,.$$

It is worth noting here that the preceding relationship up to closing of the operator $B\,A$ can be derived by formally applying the operator B to the initial relation (6.8.1). It is plain to reduce relation (6.8.20) via decomposition (6.8.10) under condition (H6) to the following one:

$$(6.8.21)\qquad f_3\big(t, \psi(t), p(t)\big) = z(t)\,,$$

where

$$(6.8.22)\qquad z(t) = \psi'(t) - \overline{B\,A(t)}\, u(t) - B\, f_1\big(t, u(t)\big)\,.$$

The new variables

$$g_0(t) = \psi'(t) - \overline{B\,A(t)}\, U(t,0)\, u_0 - B\, f_1\big(t, U(t,0)\, u_0\big)\,,$$

$$g_1(t,u) = -B\,\big(f_1(t,u) - f_1\big(t, U(t,0)\, u_0\big)\big)\,,$$

$$K(t,s) = -\overline{B\,A(t)}\, U(t,s)$$

complement our studies and help rearrange the right-hand side of (6.8.22) in simplified form. Putting these together with (6.8.8) and (6.8.22) we arrive at

$$(6.8.23)\qquad z(t) = g_0(t) + g_1\big(t, u(t)\big) + \int_0^t K(t,s)\, f\big(s, u(s), p(s)\big)\, ds\,.$$

Moreover, with the aid of relation (6.8.22) we find that $z(0) = z_0$, where the element z_0 is defined by formula (6.8.11). In light of assumptions (H5) and (H7)–(H9) there exists a small value T for which equation (6.8.21) is equivalent to

$$p(t) = \Phi\big(t,\, z(t)\big)\,. \tag{6.8.24}$$

With the aid of (6.8.23) for the function z the couple of relations (6.8.8) and (6.8.24) can be recast as the system of integral equations

$$u(t) = U(t,0)\,u_0 + \int_0^t U(t,s)\, f\big(s, u(s), p(s)\big)\, ds\,, \tag{6.8.25}$$

$$p(t) = \Phi\Big(t,\, g_0(t) + g_1\big(t, u(t)\big) + \int_0^t K(t,s)\, f\big(s, u(s), p(s)\big)\, ds\Big)\,, \tag{6.8.26}$$

which can be treated as the system (6.3.9), (6.3.12) and so there is no difficulty to prove the unique solvability of (6.8.25)–(6.8.26) in just the same way as we did in proving Theorem 6.3.1, thereby completing the proof of the theorem.■

Of special interest is the **linear case** when

$$f(t,u,p) = L_1(t)\,u + L_2(t)\,p + F(t)\,. \tag{6.8.27}$$

Theorem 6.8.2 *Let conditions* (6.8.7) *and* $(H1)$–$(H4)$ *hold and* $u_0 \in X$, $\psi \in C^1([0, T]; Y)$, $B\,u_0 = \psi(0)$ *and representation* (6.8.23) *take place for* $L_1 \in C([0, T]; \mathcal{L}(X))$, $L_2 \in C([0, T]; \mathcal{L}(Y, X))$ *and* $F \in C([0, T]; X)$. *If for any* $t \in [0, T]$ *the operator* $B\,L_2(t)$ *is invertible in the space* Y *and*

$$\big(B\,L_2\big)^{-1} \in C([0, T];\, \mathcal{L}(Y))\,,$$

then a solution u, p *of the inverse problem* (6.8.1)–(6.8.3) *exists and is unique in the class of functions*

$$u \in C([0, T]; X)\,, \qquad p \in C([0, T]; Y)\,.$$

Proof Under the restrictions imposed above conditions (H5)–(H9) hold true with the appropriate ingredients

$$f_1(t,u) = L_1(t)\,u + F(t)\,,$$

$$f_2(t,u,p) = L_2(t)\,p\,,$$

$$f_3(t,z,p) = B\,L_2(t)\,p\,,$$

$$\Phi(t,z) = \big(B\,L_2(t)\big)^{-1}\,z\,.$$

The proof of Theorem 6.8.1 provides proper guidelines for rearranging the inverse problem (6.8.1)–(6.8.3) as the system of the integral equations (6.8.25)–(6.8.26) taking now the form

$$u(t) = u_0(t) + \int_0^t \big(K_1(t,s)\,u(s) + L_1(t,s)\,p(s)\big)\,ds\,, \tag{6.8.28}$$

$$p(t) = p_0(t) + \int_0^t \big(K_2(t,s)\,u(s) + L_2(t,s)\,p(s)\big)\,ds\,, \tag{6.8.29}$$

where

$$\left\{\begin{aligned} u_0(t) &= U(t,0)\,u_0 + \int_0^t U(t,s)\,F(s)\,ds\,,\\ K_1(t,s) &= U(t,s)\,L_1(s)\,,\\ L_1(t,s) &= U(t,s)\,L_2(s)\,,\\ p_0(t) &= \big(B\,L_2(t)\big)^{-1}\Big[\psi'(t) - \big(\overline{B\,A(t)} + B\,L_1(t)\big)\\ &\qquad \times \Big(U(t,0)\,u_0 + \int_0^t U(t,s)\,F(s)\,ds\Big) - B\,F(t)\Big]\,,\\ K_2(t,s) &= -\big(B\,L_2(t)\big)^{-1}\big(\overline{B\,A(t)} + B\,L_1(t)\big)\,U(t,s)\,L_1(s)\,,\\ L_2(t,s) &= -\big(B\,L_2(t)\big)^{-1}\big(\overline{B\,A(t)} + B\,L_1(t)\big)\,U(t,s)\,L_2(s)\,. \end{aligned}\right. \tag{6.8.30}$$

By assumption, the functions u_0 and p_0 are continuous on the segment $[0,\,T]$ and the operator kernels K_1, L_1, K_2 and L_2 are strongly continuous in $\Delta(T)$. The system (6.8.28)–(6.8.29) falls into the category of second kind linear systems of the Volterra integral equations and, therefore, there exists a solution of (6.8.28)–(6.8.29) and this solution is unique in the class of continuous functions on the whole segment $[0, T]$. This completes the proof of the theorem.■

The next goal of our study is to find out when the continuous solution of the inverse problem (6.8.1)–(6.8.3) satisfies equation (6.8.1) in a pointwise manner. To decide for yourself when the function $u \in C^1([0, T]; X)$, a first step is to check the following statement.

Theorem 6.8.3 *Let conditions* (6.8.7), (6.8.10) *and* $(H1)$–$(H9)$ *be fulfilled and let* $u_0 \in X_0, \psi \in C^1([0, T]; Y)$ *and* $B u_0 = \psi(0)$. *If condition* $(H8)$ *continues to hold with* X_0 *in place of the space* X, *then there is a value* $T_1 > 0$ *such that on the segment* $[0, T_1]$ *a solution* u, p *of the inverse problem* (6.8.1)–(6.8.3) *exists and is unique in the class of functions*

$$u \in C^1([0, T_1]; X), \qquad p \in C([0, T_1]; Y).$$

Proof From conditions (H1)–(H4) immediately follows that the subspace H_0 is invariant with respect to the evolution operator $U(t, s)$, which is strongly continuous in $\Delta(T)$ in the norm of the space X_0. Due to the continuous embedding of X_0 into X condition (6.8.7) remains valid with X_0 in place of the space X if the operator $A(t)$ is replaced by its part acting in the space X_0. Theorem 6.8.1 implies that there is a value $T_1 > 0$ for which the inverse problem (6.8.1)–(2.8.3) becomes uniquely solvable in the class of functions

$$u \in C([0, T_1]; X_0), \qquad p \in C([0, T_1]; Y).$$

Under condition (H8), valid in the norm of the space X_0, decomposition (6.8.10) provides support for the view that the function f defined by (6.8.14) satisfies the condition

$$f \in C([0, T_1]; X_0).$$

Since $u_0 \in X_0$, relation (6.8.8) admits the form

$$u(t) = U(t, 0)\, u_0 + \int_0^t U(t, s)\, f(s)\, ds, \tag{6.8.31}$$

thereby leading to the inclusion $u \in C^1([0, T_1]; X)$ and completing the proof of the theorem.■

In the **linear case** a solution of the inverse problem at hand exists on the whole segment $[0, T]$.

Theorem 6.8.4 *Let conditions* (6.8.7) *and* $(H1)$–$(H4)$ *hold,* $u_0 \in X_0$, $\psi \in \mathcal{C}^1([0, T]; Y)$, $B u_0 = \psi(0)$ *and let representation* (6.8.27) *be valid with*

$$L_1 \in \mathcal{C}([0, T]; \mathcal{L}(X) \bigcap \mathcal{C}([0, T]; \mathcal{L}(X_0)),$$

$$L_2 \in \mathcal{C}([0, T]; \mathcal{L}(Y, X)) \bigcap \mathcal{C}([0, T]; \mathcal{L}(Y, X_0))$$

and

$$F \in \mathcal{C}([0, T]; X) \bigcap \mathcal{C}([0, T]; X_0).$$

If for any $t \in [0, T]$ *the operator* $B L_2(t)$ *is invertible in the space* Y *and*

$$(B L_2)^{-1} \in \mathcal{C}([0, T]; \mathcal{L}(Y)),$$

then a solution u, p *of the inverse problem* (6.8.1)–(6.8.3) *exists and is unique in the class of functions*

$$u \in \mathcal{C}^1([0, T]; X), \qquad p \in \mathcal{C}([0, T]; Y).$$

Proof It follows from the foregoing that Theorem 6.8.2 suits us perfectly in the above framework and, therefore, the inverse problem (6.8.1)–(6.8.3) is uniquely solvable on the whole segment $[0, T]$ and the solution of this problem satisfies the system of the integral equations (6.8.28)–(6.8.29). In light of assumptions (H1)–(H4) the evolution operator $U(t, s)$ becomes strongly continuous in the norm of the space X_0. Under the conditions of the theorem formulae (6.8.30) provide sufficient background for the inclusions $u_0(t) \in \mathcal{C}([0, T]; X_0)$ and $p_0(t) \in \mathcal{C}([0, T]; Y)$ and the conclusions that the kernels $K_1(t, s)$ and $K_2(t, s)$ are strongly continuous in the space X_0 and the kernels $L_1(t, s)$ and $L_2(t, s)$ are strongly continuous from Y into X_0. From such reasoning it seems clear that the system of the integral equations (6.8.28)–(6.8.29) is solvable in the class of functions

$$u \in \mathcal{C}([0, T]; X_0), \qquad p \in \mathcal{C}([0, T]; Y).$$

Due to the restrictions imposed above condition (H8) remains valid for the norm of the space X_0. If this happens, the function f specified by formula (6.8.14) satisfies the condition

$$f \in \mathcal{C}([0, T]; X_0).$$

Since $u_0 \in X_0$, representation (6.8.31) is an alternative form of relation (6.8.8) and no more. All this enables us to conclude that $u \in \mathcal{C}^1([0, T]; X)$, thereby completing the proof of the theorem. ■

6.9 Inverse problems with singular overdetermination: semilinear hyperbolic equations

This section is devoted to more a detailed exploration of the inverse problem that we have posed in Section 6.8, where a pair of the functions $u \in C^1([0, T]; X)$, $p \in C([0, T]; Y)$ is recovered from the set of relations

$$u'(t) = A(t)\,u(t) + f\big(t,\, u(t),\, p(t)\big)\,, \quad 0 \le t \le T\,, \tag{6.9.1}$$

$$u(0) = u_0\,, \tag{6.9.2}$$

$$B\,u(t) = \psi(t)\,, \qquad 0 \le t \le T\,, \tag{6.9.3}$$

under the agreement that equation (6.9.1) is hyperbolic in the sense of assumptions (H1)–(H4). The case when the operator B happens to be unbounded will be of special investigations under the condition

$$B \in \mathcal{L}(X_0, Y)\,. \tag{6.9.4}$$

The solution of the inverse problem concerned necessitates imposing several additional restrictions on the operator function $A(t)$. Let a Banach space X_1 can densely and continuously be embedded into X_0. Assume that condition (H3) of Section 6.8 continues to hold upon replacing X_0 by X_1, that is,

(H3.1) *there exists an operator function $S_1(t)$ defined on the segment $[0, T]$ with values in the space $\mathcal{L}(X_1, X)$; it is strongly continuously differentiable on $[0, T]$; for any $t \in [0, T]$ the operator*

$$\big[\, S_1(t)\big]^{-1} \in \mathcal{L}(X, X_1)$$

and

$$S_1(t)\, A(t)\, \big[\, S_1(t)\big]^{-1} = A(t) + R_1(t)\,,$$

where $R_1(t) \in \mathcal{L}(X)$; the operator function $R_1(t)$ is supposed to be strongly continuous in the space X and to be bounded in the space $\mathcal{L}(X)$.

Also, we take for granted that

(H4.1) *the operator function $A \in C\big([0, T]; \mathcal{L}(X_1, X_0)\big)$.*

In this view, it is reasonable to look for the function f in the form

$$f(t, u, p) = f_1(t, u) + f_2(t, u, p)\,. \tag{6.9.5}$$

Theorem 6.9.1 *Let condition* (6.9.4) *hold and*

$$u_0 \in X_1\,, \qquad \psi \in C^1([0,T];Y)\,, \qquad B\,u_0 = \psi(0)\,.$$

One assumes, in addition, that conditions (6.9.5), $(H3.1)$-$(H4.1)$ *of the present section as well as conditions* $(H1)$-$(H4)$ *and* $(H5)$-$(H9)$ *of Section* 6.8 *are satisfied in the norm of the space* X, *condition* $(H8)$ *continues to hold in the norms of the spaces* X_0 *and* X_1 *both. Then there exists a value* $T_1 > 0$ *such that on the segment* $[0, T_1]$ *a solution* u, p *of the inverse problem* (6.9.1)-(6.9.3) *exists and is unique in the class of functions*

$$u \in C^1([0,T_1];X) \bigcap C([0,T_1];X_1)\,, \qquad p \in C([0,T_1];Y)\,.$$

Proof First of all observe that the inclusions $X_1 \subset X_0$ and $X_0 \subset \mathcal{D}(A(t))$ imply that $X_1 \subset \mathcal{D}(A(t))$. By assumption (H4.1),

$$A \in C([0,T];\mathcal{L}(X_1, X))$$

because embedding of X_0 into X is continuous. Thus, all the assumptions (H1)-(H4) of Section 6.8 remain valid upon substituting X_1 in place of the space X_0. In this case the evolution operator $U(t,s)$ is strongly continuous in $\Delta(T)$ in the norm of the space X_1. Furthermore, by virtue of conditions (H4.1) and (6.9.4) we deduce that

$$BA \in C([0,T];\mathcal{L}(X_1, Y))\,. \tag{6.9.6}$$

We are now in a position to substantiate the solvability of the inverse problem concerned in the class of functions

$$u \in C([0,T];X_1)\,, \qquad p \in C([0,T_1];Y)$$

with some $T_1 > 0$. The function f is defined to be

$$f(t) = f(t, u(t), p(t))\,.$$

Recall that assumption (H8) of Section 6.8 is true in the norm of the space X_1. With relation (6.9.5) in view, we thus have

$$f \in C([0,T_1];X_1)\,. \tag{6.9.7}$$

On the other hand, by the definition of continuous solution the function u should satisfy relation (6.8.8) taking now the form

$$u(t) = U(t,0)\,u_0 + \int_0^t U(t,s)\,f(s)\,ds\,. \tag{6.9.8}$$

Since $u_0 \in X_1$ and the function f is subject to (6.9.7), it follows from representation (6.9.8) that $u \in C^1([0,T_1];X)$.■

In subsequent arguments we shall need yet, among other things, some propositions describing special properties of Banach spaces.

Lemma 6.9.1 *If the Banach space X_0 is continuously embedded into the Banach space X and*

$$\varphi \in \mathcal{C}\big([a,b];\, X_0\,\big) \bigcap \mathcal{C}^1\big([a,b];\, X\,\big)$$

with $\varphi' \in \mathcal{C}\big([a,b];\, X_0\,\big)$, then $\varphi \in \mathcal{C}^1\big([a,b];\, X_0\,\big)$ and the derivatives of φ in the norms of the spaces X and X_0 coincide.

Proof The coincidence of the derivatives we have mentioned above follows from the continuity of embedding of the space X_0 into X. Therefore, it remains to establish the continuous differentiability of the function φ in the norm of the space X_0. This can be done using ψ for the derivative of the function φ in the norm of the space X and then developing the well-established formula

$$\varphi(t) \;=\; \varphi(a) + (X) \int_a^t \psi(s)\; ds\,,$$

where the symbol (X) indicates that the integral is taken in the X-norm. The inclusion $\psi \in \mathcal{C}\big([a,b];\, X_0\big)$ is stipulated by the restrictions imposed above and provides support for the view that the function ψ is integrable on the segment $[a,\, t]$ for any $t \le b$ in the norm of the space X_0. Since X_0 is continuously embedded into X, the equality

$$(X_0) \int_a^t \psi(s)\; ds \;=\; (X) \int_a^t \psi(s)\; ds$$

is certainly true and implies that for any $t \in [a,\, b]$

$$\varphi(t) \;=\; \varphi(a) + (X_0) \int_a^t \psi(s)\; ds\,.$$

Whence it immediately follows that the function φ is continuously differentiable in the norm of the space X_0, thereby completing the proof of Lemma 6.9.1.∎

Lemma 6.9.2 *For any element $u_0 \in \overset{\bullet}{X}_1$ the function*

$$\varphi(t) \;=\; B\,U(t,s)\,u_0$$

is continuously differentiable on the segment $[s,\, T]$ *and*

$$\varphi'(t) \;=\; B\,A(t)\,U(t,s)\,u_0\,.$$

Proof In the proof of this assertion we have occasion to use the function

$$\mu(t) \;=\; U(t,s)\,u_0\,.$$

Since the evolution operator $U(t,s)$ is strongly continuous in $\Delta(T)$ in the norm of the space X_0 and $u_0 \in X_0$, where $X_1 \subset X_0$, the function μ belongs to the class $\mathcal{C}\big([s,T];\, X\,\big)$. Furthermore, the inclusion $u_0 \in X_0$ implies that

$$\mu \,\in\, \mathcal{C}^1\big([s,T];\, X\,\big)$$

and

$$\mu'(t) = A(t)\,U(t,s)\,u_0\,.$$

On the other hand, u_0 belongs to the space X_1 and the operator $U(t,s)$ is strongly continuous on $\Delta(T)$ in the norm of the space X_1. This provides reason enough to conclude that under condition (H4.1)

$$\mu' \,\in\, \mathcal{C}\big([s,T];\, X_0\,\big)\,.$$

In agreement with Lemma 6.9.1 the inclusion $\mu \,\in\, \mathcal{C}^1\big([s,T];\, X_0\big)$ occurs and the derivatives of the function μ coincide with respect to the norms of the spaces X and X_0. The desired assertion follows immediately from relation (6.9.4). ■

Lemma 6.9.3 *If* $f \in \mathcal{C}\big([0,\, T];\, X_1\,\big)$ *and*

$$g(t) \;=\; B \int_0^t U(t,s)\,f(s)\;ds\,,$$

then the function g *is continuously differentiable on* $[0,\, T]$ *and*

$$g'(t) \;=\; B\,A(t) \int_0^t U(t,s)\,f(s)\;ds \,+\, B\,f(t)\,.$$

Proof The proof of this proposition is similar to that of Lemma 6.9.2 with minor changes. The same procedure will work for the function

$$\mu(t) \;=\; \int_0^t U(t,s)\,f(s)\;ds\,.$$

Since the space X_1 is continuously embedded into the space X_0, the inclusion $f \in \mathcal{C}([0, T]; X_0)$ serves to motivate that the function μ is continuously differentiable on the segment $[0, T]$ in the space X and

$$\mu'(t) = A(t) \int_0^t U(t,s)\, f(s)\, ds + f(t).$$

With the relation $f \in \mathcal{C}([0, T]; X_1)$ established, the evolution operator $U(t, s)$ is strongly continuous in $\Delta(T)$ in the norm of the space X_1. Recall that the space X_1 is continuously embedded into the space X_0, so that $\mu' \in \mathcal{C}([0, T]; X_0)$ on the strength of condition (H4.1). By Lemma 6.9.1 the function μ belongs to $\mathcal{C}^1([0, T]; X_0)$ and relation (6.9.4) leads to the final conclusion. Thus, Lemma 6.9.3 is completely proved. ■

Before we undertake the proof of Theorem 6.9.1, let us recall that the main difficulty here lies in the unique solvability of the system of equations (6.8.8)–(6.8.9) in the class of functions

$$u \in \mathcal{C}([0, T_1]; X_1), \qquad p \in \mathcal{C}([0, T_1]; Y)$$

with some value $T_1 > 0$. With regard to the function

$$f(t) = f(t, u(t), p(t))$$

it is not difficult to derive from (6.8.8)–(6.8.9) the equation

$$\psi(t) = B\, U(t,0)\, u_0 + B \int_0^t U(t,s)\, f(s)\, ds,$$

which can be differentiated by the formulae of Lemmas 6.9.2–6.9.3. The outcome of this is

$$\psi'(t) = B\, A(t)\, U(t,0)\, u_0 + B\, A(t) \int_0^t U(t,s)\, f(s)\, ds + B\, f(t). \tag{6.9.9}$$

It is worth noting here that the preceding equation is similar to (6.8.19) and so we are still in the framework of Theorem 6.8.1. Because of (6.8.8), equation (6.9.9) can be rewritten as

$$\psi'(t) = B\, A(t)\, u(t) + B\, f(t, u(t), p(t)), \tag{6.9.10}$$

yielding

$$f_3\big(t,\, \psi(t),\, p(t)\big) \;=\; z(t) \tag{6.9.11}$$

with $z(t) = \psi'(t) - B\,A(t)\,u(t) - B\,f_1\big(t,\, u(t)\big)$ incorporated. Here we have taken into account conditions (6.9.5) and (H6). In the sequel we deal with the new functions

$$g_0(t) \;=\; \psi'(t) - B\,A(t)\,U(t,0)\,u_0 - B\,f_1\,\big(t,\, U(t,0)\,u_0\big)\,,$$

$$g_1(t,u) \;=\; -B\,\big(\,f_1(t,u) - f_1\big(t,\, U(t,0)\,u_0\big)\big)\,,$$

$$K(t,s) \;=\; -B\,A(t)\,U(t,s)\,.$$

Putting these together with (6.8.8) we obtain

$$z(t) \;=\; g_0(t) + g_1(t,\, u(t)) + \int\limits_0^t K(t,s)\,f(s,\, u(s),\, p(s))\; ds\,,$$

showing the new members to be sensible ones. By the definition of the function $z(t)$ the relation $z(0) = z_0$ holds true, where the element z_0 is defined by (6.8.11). We note in passing that in (6.8.11)

$$\overline{B\,A(0)}\;u_0 \;=\; B\,A(0)\;u_0\,.$$

By assumptions (H5) and (H7)–(H9) of Section 6.8 there exists a sufficiently small value t such that equation (6.9.11) becomes equivalent to

$$p(t) \;=\; \Phi\big(t,\, z(t)\big)\,,$$

thereby reducing the system of the equations for recovering the functions u and p to the following one:

$$u(t) \;=\; U(t,0)\,u_0 + \int\limits_0^t U(t,s)\,f\big(\,s,\, u(s),\, p(s)\big)\; ds\,, \tag{6.9.12}$$

$$p(t) \;=\; \Phi\Big(t,\, g_0(t)+g_1(t,\, u(t))+\int\limits_0^t K(t,s)\,f\big(\,s,\, u(s),\, p(s)\big)\; ds\Big)\,. \tag{6.9.13}$$

In accordance with what has been said, the last system is similar to (6.3.9) and (6.3.13). Because of this fact, the rest of the proof is simple to follow and so the reader is invited to complete the remaining part on his/her own (for more detail see the proof of Theorem 6.3.1).■

We now turn to the **linear case** when

$$f(t,u,p) = L_1(t)\,u + L_2(t)\,p + F(t)\,. \tag{6.9.14}$$

In this regard, we will show that the inverse problem of interest becomes solvable on the whole segment $[0,\,T]$.

Theorem 6.9.2 *Let conditions* (6.9.4), $(H3.1)$-$(H4.1)$ *of the present section and conditions* $(H1)$-$(H4)$ *of Section* 6.8 *hold,*

$$u_0 \in X_1\,, \qquad \psi \in C^1([0,\,T];Y)\,, \qquad B\,u_0 = \psi(0)$$

and representation (6.9.14) *take place with*

$$L_1 \in C([0,\,T];\, \mathcal{L}(X) \bigcap \mathcal{L}(X_0) \cap \mathcal{L}(X_1))\,,$$

$$L_2 \in C([0,\,T];\, \mathcal{L}(Y,X) \bigcap \mathcal{L}(Y,X_0) \bigcap \mathcal{L}(Y,X_1))$$

and

$$F \in C([0,\,T];\, X_1)\,.$$

If for any $t \in [0,\,T]$ *the operator* $B\,L_2(t)$ *is invertible in the space* Y *and*

$$(B\,L_2)^{-1} \in C([0,\,T];\, \mathcal{L}(Y))\,,$$

then a solution $u,\,p$ *of the inverse problem* (6.9.1)-(6.9.3) *exists and is unique in the class of functions*

$$u \in C^1([0,\,T];\,X) \bigcap C([0,\,T];\,X_1)\,, \qquad p \in C([0,\,T];\,Y)\,.$$

Proof By the above assumptions the conditions of Theorem 6.9.1 will be valid with the members

$$f_1(t,u) = L_1(t)\,u + F(t)\,,$$

$$f_2(t,u,p) = L_2(t)\,p\,,$$

$$f_3(t,z,p) = B\,L_2(t)\,p\,,$$

$$\Phi(t,z) = \big(B\,L_2(t)\big)^{-1}\,z\,.$$

Therefore, the inverse problem (6.9.1)–(6.9.3) becomes equivalent to the system (6.9.12)–(6.9.13), which in terms of Theorem 6.9.2 takes the form

$$u(t) = u_0(t) + \int_0^t \big(K_1(t,s)\, u(s) + L_1(t,s)\, p(s)\big)\, ds\,, \tag{6.9.15}$$

$$p(t) = p_0(t) + \int_0^t \big(K_2(t,s)\, u(s) + L_2(t,s)\, p(s)\big)\, ds\,, \tag{6.9.16}$$

where the functions $u_0(t)$, $p_0(t)$ and the kernels $K_1(t,s)$, $L_1(t,s)$, $K_2(t,s)$, $L_2(t,s)$ are defined by formulae (6.8.30) with no use of bar over the operator $B\,A(t)$. The initial assumptions ensure the continuity of nonhomogeneous terms in (6.9.15)–(6.9.16) and the strong continuity of the operator kernels in the appropriate spaces. It is sufficient for a solution of the system of the Volterra integral equations (6.9.15)–(6.9.16) of the second kind to exist and to be unique in the class of functions

$$u \in \mathcal{C}([0,T];\, X_1)\,, \qquad p \in \mathcal{C}([0,T];\, Y)\,.$$

Moreover, as we have established at the very beginning of the proof of Theorem 6.9.1, the solvability of the inverse problem in the indicated class of functions implies the continuous differentiability of the function u in the norm of the space X, thereby completing the proof of the theorem.■

6.10 Inverse problems with smoothing overdetermination: semilinear hyperbolic equations and operators with fixed domain

As in the preceding section we focus our attention on the inverse problem which has been under consideration in Section 6.8. For the reader's convenience we quote below its statement: it is required to find a pair of the functions $u \in \mathcal{C}^1([0,T];\, X)$, $p \in \mathcal{C}([0,T];\, Y)$ from the set of relations

$$u'(t) = A(t)\, u(t) + f\big(t,\, u(t),\, p(t)\big)\,, \quad 0 \le t \le T\,, \tag{6.10.1}$$

$$u(0) = u_0\,, \tag{6.10.2}$$

$$B\, u(t) = \psi(t)\,, \qquad 0 \le t \le T\,, \tag{6.10.3}$$

in which for any $t \in [0, T]$ the operator $A(t)$ with a dense domain is supposed to be linear and closed in the space X. Here

$$f\colon [0, T] \times X \times Y \mapsto X$$

and B refers to a linear operator acting from the space X into the space Y. The **smoothing property** of the operator B is well-characterized by the inclusions

$$(6.10.4) \qquad B \in \mathcal{L}(X, Y), \qquad \overline{BA} \in C([0, T]; \mathcal{L}(X, Y)).$$

As a matter of fact, the current questions are similar to those explored in Section 6.8 with only difference relating to another definition for equation (6.10.1) to be of **hyperbolic type**. Conditions (H3) and (H4) imposed in Section 6.8 reflect in abstract form some properties of Friedrichs' t-hyperbolic systems. In turn, the conditions of hyperbolicity to be used here reveal some properties of the first order systems connected with a hyperbolic equation of the second order.

Consider a family of norms $\|\cdot\|_t$ on a Banach space X, the elements of which are equivalent to the original norm of the space X. The same symbol will stand for the induced operator norms on the space $\mathcal{L}(X)$. Also, we take for granted that

(HH1) *the domain of the operator $A(t)$ does not depend on t, that is, $\mathcal{D}(A(t)) = \mathcal{D}$;*

(HH2) *there is a number $a > 0$ such that for any $t \in [0, T]$ all real numbers λ with $|\lambda| > a$ are contained in the resolvent set of the operator $A(t)$ and*

$$\|(A(t) - \lambda I)^{-1}\|_t \le \frac{1}{|\lambda| - a};$$

(HH3) *there are some values $s \in [0, T]$ and λ ($|\lambda| > a$) such that the operator function*

$$C(t) = (A(t) - \lambda I)(A(s) - \lambda I)^{-1}$$

is continuously differentiable on the segment $[0, T]$ in the norm of the space $\mathcal{L}(X)$;

(HH4) *a nondecreasing function $\omega(t)$ is so chosen as to satisfy for any $t \in [0, T]$ and all elements $x \in X$*

$$\|x\|_t \ge \delta \|x\|, \qquad \delta > 0,$$

and for every $t > s$ and all elements $x \in X$

$$\Big| \|x\|_t - \|x\|_s \Big| \le (\omega(t) - \omega(s)) \|x\|.$$

Conditions (HH1)–(HH4) are sufficient for the **Cauchy problem**

$$u'(t) = A(t)\,u(t) + f(t)\,, \quad 0 \le t \le T\,, \tag{6.10.5}$$

$$u(0) = u_0\,, \tag{6.10.6}$$

to be well-posed. It is well-known that under conditions (HH1)–(HH4) there exists an **evolution operator** $U(t, s)$, which is strongly continuous in the triangle

$$\Delta(T) = \big\{(t, s)\colon\ 0 \le s \le T,\ s \le t \le T\big\}$$

and allows to represent a solution of the Cauchy problem (6.10.5)–(6.10.6) by

$$u(t) = U(t,0)\,u_0 + \int_0^t U(t,s)\,f(s)\,ds\,. \tag{6.10.7}$$

In the case when $u_0 \in \mathcal{D}$ and $A\,f \in \mathcal{C}\big([0, T];\, X\,\big)$ formula (6.10.7) gives a unique solution of the Cauchy problem (6.10.5)–(6.10.6) in the class of functions

$$u \in \mathcal{C}^1\big([0, T];\, X\,\big)\,, \qquad A\,u \in \mathcal{C}\big([0, T];\, X\,\big)\,.$$

For each $u_0 \in X$ and any $f \in \mathcal{C}\big([0, T];\, X\,\big)$ formula (6.10.7) gives a unique continuous solution (in a sense of distributions) $u \in \mathcal{C}\big([0, T];\, X\,\big)$ of problem (6.10.5)–(6.10.6). Finally, one thing is worth recalling that for any real number λ with $\|\,\lambda\,| > a$ the operator function

$$W_\lambda(t,s) = \big(A(t) - \lambda\,I\,\big)\,U(t,s)\,\big(A(s) - \lambda\,I\,\big)^{-1} \tag{6.10.8}$$

is strongly continuous in $\Delta(T)$ (see Ikawa (1968)).

Lemma 6.10.1 *For any element $u_0 \in X$ and all values $t \ge s$ the function*

$$\psi_0(t) = B\,U(t,s)\,u_0$$

is differentiable on the segment $[0, T]$ and

$$\psi_0'(t) = \overline{B\,A(t)}\,U(t,s)\,u_0\,.$$

Lemma 6.10.2 *For any function* $f \in C([0, T]; X)$ *the function*

$$g(t) = B \int_0^t U(t,s) f(s)\, ds$$

is differentiable on the segment $[0, T]$ *and*

$$g'(t) = \overline{B A(t)} \int_0^t U(t,s) f(s)\, ds + B f(t).$$

The reader is invited to establish these lemmas on hir/her own, since they are similar to Lemmas 6.8.1–6.8.2, respectively, with $\mathcal{D}$ in place of the space X_0.

The next step is to find out whether the inverse problem concerned has a continuous solution. As in Section 6.8 we say that a pair of the functions $u \in C([0, T]; X)$, $p \in C([0, T]; Y)$ gives a continuous solution of the inverse problem (6.10.1)–(6.10.3) if the following relations occur:

$$u(t) = U(t,0)\, u_0 + \int_0^t U(t,s)\, f(s, u(s), p(s))\, ds, \quad 0 \le t \le T, \tag{6.10.9}$$

$$B\, u(t) = \psi(t), \qquad 0 \le t \le T. \tag{6.10.10}$$

Theorem 6.10.1 *Let conditions* (6.10.4) *and* $B\, u_0 = \psi(0)$ *hold and let*

$$u_0 \in X, \qquad \psi \in C^1([0, T]; Y).$$

If conditions (6.8.10) *and* $(H5)$–$(H9)$ *of Section* 6.8 *as well as conditions* $(HH1)$–$(HH4)$ *of the present section are fulfilled, then there is a value* $T_1 > 0$ *such that on the segment* $[0, T_1]$ *a solution* u, p *of the inverse problem* (6.10.1)–(6.10.3) *exists and is unique in the class of functions*

$$u \in C([0, T_1]; X), \qquad p \in C([0, T_1]; Y).$$

Proof In proving the above assertion one should adopt the arguments and exploit some facts from Theorem 6.8.1. Indeed, with regard to the function

$$f(t) = f(t, u(t), p(t))$$

observe that the initial assumptions ensure that for any continuous functions u and p the function $f(t)$ is continuous on $[0, T]$. Furthermore, having stipulated condition (6.10.10) the equation holds true:

$$\psi(t) = B\,U(t,0)\,u_0 + B \int_0^t U(t,s)\,f(s)\,ds\,.$$

On account of Lemmas 6.10.1–6.10.2 the preceding can be differentiated as follows:

$$\psi'(t) = \overline{B\,A(t)}\;U(t,0)\,u_0 + \overline{B\,A(t)}\int_0^t U(t,s)\,f(s)\,ds + B\,f(t)\,.$$

Because of (6.10.9), it follows from the foregoing that

$$\psi'(t) = \overline{B\,A(t)}\;u(t) + B\,f\big(t,\,u(t),\,p(t)\big)\,,$$

which admits, in view of (6.8.10) and (H6), an alternative form

$$f_3\big(t,\,\psi(t),\,p(t)\big) = z(t)\,, \tag{6.10.11}$$

where

$$z(t) = \psi'(t) - \overline{B\,A(t)}\;u(t) - B\,f_1\big(t,\,u(t)\big)\,.$$

Since $z(0) = z_0$, where z_0 is given by formula (6.8.11), we deduce by successively applying conditions (H5) and (H7)–(H9) of Section 6.8 that for all sufficiently small values t equation (6.10.11) becomes

$$p(t) = \Phi\big(t,\,z(t)\big)\,.$$

Forthcoming substitutions

$$g_0(t) = \psi'(t) - \overline{B\,A(t)}\,U(t,0)\,u_0 - B\,f_1\big(t,\,U(t,0)\,u_0\big)\,,$$

$$g_1(t,u) = -B\,\big(\,f_1\big(t,\,u\big) - f_1\big(t,\,U(t,0)\,u_0\big)\big)\,,$$

$$K(t,s) = -\overline{B\,A(t)}\,U(t,s)$$

make our exposition more transparent and permit us to derive representation (6.8.23) for the function z and thereby reduce the inverse problem (6.10.1)–(6.10.3) to the system of integral equations

$$u(t) = U(t,0)\,u_0 + \int_0^t U(t,s)\,f\big(s,\,u(s),\,p(s)\big)\,ds\,, \tag{6.10.12}$$

(6.10.13)

$$p(t) = \Phi\Big(t,\, g_0(t) + g_1\big(t,\, u(t)\big) + \int\limits_0^t K(t,s)\ f\big(s,\, u(s),\, p(s)\big)\ ds\Big) .$$

It should be noted that the preceding system is almost identical to the system (6.3.9), (6.3.12) and, therefore, we finish the proof of the theorem in just the same way as we did in proving Theorem 6.3.1. ■

In the **linear case** when

(6.10.14)
$$f(t,u,p) = L_1(t)\, u + L_2(t)\, p + F(t) ,$$

the solvability of the inverse problem concerned is revealed on the whole segment $[0, T]$.

Theorem 6.10.2 *Let conditions* (6.10.4) *and* $Bu_0 = \psi(0)$ *hold and let*

$$u_0 \in X , \qquad\qquad \psi \in C^1([0, T]; Y) .$$

One assumes, in addition, that assumptions $(HH1)$–$(HH4)$ *are true and decomposition* (6.10.14) *takes place with*

$$L_1 \in C\big([0, T]; \mathcal{L}(X)\big) ; \quad L_2 \in C\big([0, T]; \mathcal{L}(Y, X)\big) ; \quad F \in C\big([0, T]; X\big) .$$

If for any fixed value $t \in [0, T]$ *the operator* $B\, L_2(t)$ *is invertible in the space* Y *and*

$$\big(B\, L_2\big)^{-1} \in C\big([0, T]; \mathcal{L}(Y)\big) ,$$

then a solution u, p *of the inverse problem* (6.10.1)–(6.10.3) *exists and is unique in the class of functions*

$$u \in C\big([0, T]; X\big) , \qquad\qquad p \in C\big([0, T]; Y\big) .$$

Proof The main idea behind proof is similar to that of Theorem 6.8.2. In this line, we begin by setting the functions

$$f_1(t,u) = L_1(t)\, u + F(t) ,$$

$$f_2(t,u,p) = L_2(t)\, p ,$$

$$f_3(t,z,p) = B\, L_2(t)\, p ,$$

$$\Phi(t,z) = \big(B\, L_2(t)\big)^{-1} z ,$$

being still subject to conditions (H5)–(H9) of Section 6.8. Therefore, Theorem 6.10.1 may be of help in reducing the inverse problem in view to a system of the type (6.10.12)–(6.10.13). In the case which interests us the system in question takes the form of the system of the **Volterra linear equations of the second kind**

$$u(t) = u_0(t) + \int_0^t \big(K_1(t,s)\, u(s) + L_1(t,s)\, p(s) \big)\, ds\,, \tag{6.10.15}$$

$$p(t) = p_0(t) + \int_0^t \big(K_2(t,s)\, u(s) + L_2(t,s)\, p(s) \big)\, ds\,, \tag{6.10.16}$$

where the functions $u_0(t)$, $p_0(t)$ and the kernels $K_1(t,s)$, $L_1(t,s)$, $K_2(t,s)$ and $L_2(t,s)$ are specified by formulae (6.8.30). Under the conditions of Theorem 6.10.2 the implicit representations (6.8.30) imply that the nonhomogeneous terms of equations (6.10.15)–(6.10.16) are continuous and their operator kernels are strongly continuous. Just for this reason the preceding system of integral equations is uniquely solvable and the current proof is completed.■

We now focus our attention on obtaining the conditions under which a continuous solution of (6.10.1)–(6.10.3) will be differentiable. Accepting (6.8.10), that is, involving the approved decomposition

$$f(t,u,p) = f_1(t,u) + f_2(t,u,p)$$

and holding a number λ ($|\lambda| > a$) fixed, we introduce the new functions

$$g_1(t,v) = \big(A(t) - \lambda I\big)\, f_1\big(t, (A(t) - \lambda I)^{-1} v\big)\,,$$

$$g_2(t,v,p) = \big(A(t) - \lambda I\big)\, f_2\big(t, (A(t) - \lambda I)^{-1} v, p\big)\,.$$

Let $u_0 \in \mathcal{D}$. The element v_0 is defined by

$$v_0 = \big(A(0) - \lambda I\big)\, u_0\,. \tag{6.10.17}$$

Also, we require that

(HH8) *for some number $R > 0$ and the element v_0 given by formula (6.10.17) both functions g_1 and g_2 are continuous with respect to the totality of variables on the manifold $S_{X\times Y}\big((v_0, p_0), R, T\big)$ and satisfy thereon the Lipschitz condition with respect to (v,p).*

Theorem 6.10.3 *Let* (6.10.4) *hold,* $u_0 \in \mathcal{D}$, $\psi \in C^1([0, T]; Y)$ *and* $B u_0 = \psi(0)$. *Then the collection of conditions* $(HH1)$*-*$(HH4)$, $(HH8)$, (6.8.10) *and* $(H5)$*-*$(H9)$ *of Section* 6.8 *assure that there is a value* $T_1 > 0$ *such that a solution* u, p *of the inverse problem* (6.10.1)-(6.10.3) *exists and is unique in the class of functions*

$$u \in C^1([0, T_1]; X), \qquad p \in C([0, T_1]; Y).$$

Proof First of all let us stress that the conditions of Theorem 6.10.1 hold true and provide support for decision-making that for all sufficiently small values t a continuous solution of the inverse problem (6.10.1)-(6.10.3) exists and is unique. We are going to show that there is a sufficiently small value T_1 for which this solution satisfies the condition

$$A u \in C([0, T_1]; X). \tag{6.10.18}$$

With this aim, the system of the integral equations (6.10.12)-(6.10.13) is put together with the governing equation for the function

$$v(t) = (A(t) - \lambda I)\, u(t).$$

Relation (6.10.8) yields the identity

$$(A(t) - \lambda I)\, U(t, s) = W_\lambda(t, s)\,(A(s) - \lambda I). \tag{6.10.19}$$

Some progress in deriving a similar equation for the function v will be achieved by subsequent procedures: applying the operator $A(t) - \lambda I$ to (6.10.12), joining the resulting equation with (6.10.17) and (6.10.19) and involving the relevant formulae for the functions g_1 and g_2. The outcome of this is

$$v(t) = v_0(t) + \int_0^t W_\lambda(t, s)\, g(s, v(s))\, ds, \tag{6.10.20}$$

where

$$\begin{cases} v_0(t) = W_\lambda(t, 0)\, v_0, \\ g(t, v) = g_1(t, v) + g_2(t, v, p(t)). \end{cases} \tag{6.10.21}$$

From the conditions of the theorem it follows that the function $v_0(t)$ is continuous on the segment $[0, T]$. Moreover, the function $g(t, v)$ is continuous in a certain neighborhood of the point $(0, v_0)$ with respect to the totality of variables and satisfies in this neighborhood the Lipschitz condition in v. Since the kernel $W_\lambda(t,s)$ involved in (6.10.20) is strongly continuous, equation (6.10.20) is locally solvable in the class of continuous functions. Arguing in inverse order we see that the function

$$u(t) = \big(A(t) - \lambda I\big)^{-1} v(t),$$

where $v(t)$ is a solution to (6.10.20), will satisfy equation (6.10.12). Being a solution of the system (6.10.12)–(6.10.13), the function $u(t)$ is subject to the relation

$$\big(A(t) - \lambda I\big)\, u(t) \in C([0, T_1]; X)$$

if the value $T_1 > 0$ is sufficiently small. We note in passing that the preceding inclusion is equivalent to (6.10.18).

With the aid of (6.10.9) we establish the representation

$$u(t) = U(t,0)\, u_0 + \int\limits_0^t U(t,s)\, f(s)\, ds,$$

where $f(t) = f\big(t,\, u(t),\, p(t)\big)$. Recall that $u_0 \in \mathcal{D}$. The continuous differentiability of the function u will be proved if we succeed in showing that the function $A\,f$ is continuous. The continuity of $A\,f$, in turn, is equivalent to the continuity of $\big(A - \lambda I\big)\, f$ and the last property is ensured by the relation

$$\big(A(t) - \lambda I\big)\, f(t) = g\big(t,\, v(t)\big),$$

where the function g is defined by (6.10.21) and the function

$$v(t) = \big(A(t) - \lambda I\big)\, u(t)$$

is continuous. This completes the proof of the theorem.■

We are interested in the **linear case** where

$$f(t,u,p) = L_1(t)\, u + L_2(t)\, p + F(t), \tag{6.10.22}$$

which is covered by the following proposition.

Theorem 6.10.4 *Let under conditions* (6.10.4) *and* $B\, u_0 = \psi(0)$ *hold and*

$$u_0 \in \mathcal{D}, \qquad \psi \in C^1([0,T]; Y).$$

One assumes, in addition, that under assumptions $(HH1)$*–*$(HH4)$ *decomposition* (6.10.22) *is valid with*

$$L_1, \ (A - \lambda I)\, L_1 \left(A - \lambda I \right)^{-1} \in \mathcal{C}([0, T]; \mathcal{L}(X)) ;$$

$$L_2, \ A\, L_2 \in \mathcal{C}([0, T]; \mathcal{L}(Y, X)) ;$$

$$F, \ A\, F \in \mathcal{C}([0, T]; X) .$$

If for any fixed value $t \in 0 \le t \le T$ *the operator* $B\, L_2(t)$ *is invertible in the space* Y *and*

$$(B\, L_2)^{-1} \in \mathcal{C}([0, T]; \mathcal{L}(Y)) ,$$

then a solution u, p *of the inverse problem* (6.10.1)*–*(6.10.3) *exists and is unique in the class of functions*

$$u \in \mathcal{C}^1([0, T]; X) , \qquad p \in \mathcal{C}([0, T]; Y) .$$

Proof Due to the restrictions imposed above the conditions of Theorem 6.10.3 will be satisfied if we agree to consider

$$f_1(t, u) = L_1(t)\, u + F(t) ,$$

$$f_2(t, u, p) = L_2(t)\, p ,$$

$$f_3(t, z, p) = B\, L_2(t)\, z ,$$

$$\Phi(t, z) = \left(B\, L_2(t) \right)^{-1} z ,$$

$$g_1(t, v) = \left(A(t) - \lambda I \right) L_1(t) \left(A(t) - \lambda I \right)^{-1} v + \left(A(t) - \lambda I \right) F(t)$$

$$g_2(t, v, p) = \left(A(t) - \lambda I \right) L_2(t)\, p .$$

With these ingredients, the system of equations (6.10.12)–(6.10.13), (6.10.20) becomes

$$u(t) = u_0(t) + \int_0^t \big(K_1(t,s)\, u(s) + L_1(t,s)\, p(s)\big)\, ds\,, \tag{6.10.23}$$

$$p(t) = p_0(t) + \int_0^t \big(K_2(t,s)\, u(s) + L_2(t,s)\, p(s)\big)\, ds\,, \tag{6.10.24}$$

$$v(t) = \bar{v}_0(t) + \int_0^t K_3(t,s)\, v(s)\, ds\,, \tag{6.10.25}$$

where

$$\begin{aligned}
u_0(t) &= U(t,0)\, u_0 + \int_0^t U(t,s)\, F(s)\, ds\,, \\
K_1(t,s) &= U(t,s)\, L_1(s)\,, \\
L_1(t,s) &= U(t,s)\, L_2(s)\,, \\
p_0(t) &= \big(B\, L_2(t)\big)^{-1} \Big[\psi'(t) - \big(\overline{B\, A(t)} + B\, L_1(t)\big) \\
&\qquad \times \Big(U(t,0)\, u_0 + \int_0^t U(t,s)\, F(s)\, ds \Big) - B\, F(t) \Big]\,, \\
K_2(t,s) &= -\big(B\, L_2(t)\big)^{-1} \big(\overline{B\, A(t)} + B\, L_1(t)\big)\, U(t,s)\, L_1(s)\,, \\
L_2(t,s) &= -\big(B\, L_2(t)\big)^{-1} \big(\overline{B\, A(t)} + B\, L_1(t)\big)\, U(t,s)\, L_2(s)\,, \\
\bar{v}_0(t) &= W_\lambda(t,0)\, v_0 + \int_0^t W_\lambda(t,s) \big[\big(A(s) - \lambda\, I\big)\, F(s)\big]\, ds
\end{aligned}$$

$$+ \int_0^t W_\lambda(t,s) \left[(A(s) - \lambda I) L_2(s) \right] p(s) \, ds \,,$$

$$K_3(t,s) = W_\lambda(t,s) \left[(A(s) - \lambda I) L_1(s) (A(s) - \lambda I)^{-1} \right].$$

Under the initial premises the functions $u_0(t)$ and $p_0(t)$ are continuous on the segment $[0, T]$ and the operator kernels $K_1(t,s)$, $L_1(t,s)$, $K_2(t,s)$ and $L_2(t,s)$ are strongly continuous in $\Delta(T)$. Relations (6.10.23)–(6.10.24) may be treated as a system of the Volterra linear equations of the second kind for the functions u and p. Under such an approach the existence of a solution to (6.10.23)–(6.10.24) in the class of continuous functions is established on the whole segment $[0, T]$ in light of the results of Section 5.1. On the other hand, for any continuous functions u and p the nonhomogeneous term $\bar{v}_0(t)$ arising from (6.10.25) is continuous and the kernel $K_3(t,s)$ is strongly continuous. Therefore, equation (6.10.25) is also solvable in the class $C([0, T]; X)$. In view of this, another conclusion can be drawn that relation (6.10.18) occurs for $T_1 = T$, thereby justifying the continuous differentiability of the function u on the segment $[0, T]$ in agreement with Theorem 6.10.3 and completing the proof of the theorem.■

Chapter 7

Two-Point Inverse Problems for First Order Equations

7.1 Two-point inverse problems

In this section we deal in a Banach space X with a closed linear operator A with a dense domain. For the purposes of the present section we have occasion to use two mappings Φ: $[0, T] \mapsto \mathcal{L}(X)$ and F: $[0, T] \mapsto X$ and fix two arbitrary elements u_0, $u_1 \in X$. The **two-point inverse problem** consists of finding a function $u \in C^1\big([0, T]; X\big)$ and an element $p \in X$ from the set of relations

$$u'(t) = A\,u(t) + f(t)\,, \qquad 0 \le t \le T\,, \tag{7.1.1}$$

$$u(0) = u_0\,, \tag{7.1.2}$$

$$f(t) = \Phi(t)\,p + F(t)\,, \qquad 0 \le t \le T\,, \tag{7.1.3}$$

$$u(T) = u_1\,. \tag{7.1.4}$$

One assumes, in addition, that the operator A is the generator of a strongly continuous semigroup $V(t)$ or, what amounts to the same things, the

Cauchy (direct) problem (7.1.1)–(7.1.2) is uniformly well-posed. It is known from Fattorini (1983) that for $u_0 \in \mathcal{D}(A)$ and

$$f \in C^1([0, T]; X) + C([0, T]; \mathcal{D}(A))$$

the direct problem (7.1.1)–(7.1.2) has a solution in the class of functions

$$u \in C^1([0, T]; X), \qquad A u \in C([0, T]; X).$$

This solution is unique in the indicated class of functions and is representable by

$$u(t) = V(t)\, u_0 + \int\limits_0^t V(t-s)\, f(s)\, ds. \tag{7.1.5}$$

In contrast to the direct problem the inverse problem (7.1.1)–(7.1.4) involves the function f as the unknown of the prescribed structure (7.1.3), where the element $p \in X$ is unknown and the mappings Φ and F are available. Additional information about the function u in the form of the **final overdetermination** (7.1.4) provides a possibility of determining the element p.

In the further development one more restriction is imposed on the operator-valued function Φ:

$$\Phi \in C^1([0, T]; \mathcal{L}(X)), \tag{7.1.6}$$

which ensures that for any $p \in X$ the function $\Phi(t)\, p$ as a function of the argument t with values in the Banach space X will be continuously differentiable on the segment $[0, T]$. Because of this, the function f of the structure (7.1.3) belongs to the space

$$C^1([0, T]; X) + C([0, T]; \mathcal{D}(A))$$

if and only if the function F belongs to the same space. We note in passing that from relation (5.7.4) and the very definition of the inverse problem solution it follows that the inclusion $u_1 \in \mathcal{D}(A)$ regards as a necessary solvability condition. Before proceeding to careful analysis, it will be sensible to introduce the concept of **admissible input data**.

Definition 7.1.1 *The elements u_0 and u_1 and the function F are called the admissible input data of the inverse problem* (7.1.1)–(7.1.4) *if*

$$u_0,\, u_1 \in \mathcal{D}(A), \qquad F \in C^1([0, T]; X) + C([0, T]; \mathcal{D}(A)). \tag{7.1.7}$$

In the case when the input data are admissible, a solution of the Cauchy problem (7.1.1)–(7.1.2) can be expressed by (7.1.5), thus causing the equivalence between the system joining three relations (7.1.1)–(7.1.3) and the following equation:

$$(7.1.8)\quad u(t) = V(t)\,u_0 + \int_0^t V(t-s)\,\big[\Phi(s)\,p + F(s)\big]\,ds\,, \qquad 0 \le t \le T\,.$$

All this enables us to extract a single equation for the unknown element p. In view of (7.1.8), relation (7.1.4) becomes

$$u_1 = v(T)\,u_0 + \int_0^T V(T-s)\,\big[\Phi(s)\,p + F(s)\big]\,ds$$

and admits an alternative form

$$(7.1.9)\qquad B\,p = g\,,$$

where

$$(7.1.10)\qquad B\,p = \int_0^T V(T-s)\,\Phi(s)\,p\,ds\,,$$

$$(7.1.11)\qquad g = u_1 - V(T)\,u_0 - \int_0^T V(T-s)\,F(s)\,ds\,.$$

This provides sufficient background to deduce that the function u and the element p give a solution of the inverse problem (7.1.1)–(7.1.4) with admissible input data if and only if p solves equation (7.1.9) and u is given by formula (7.1.8). Thus, the inverse problem of interest amounts to equation (7.1.9).

The next step is to reveal some elementary properties of equation (7.1.9) in showing that its right-hand side g defined by formula (7.1.11) is an element of the manifold $\mathcal{D}(A)$. Indeed, recalling that $u_0,\ u_1 \in \mathcal{D}(A)$ we may attempt the sought function F in the form $F = F_1 + F_2$ with $F_1 \in C^1([0,\,T];\,X)$ and $F_2,\ A\,F_2 \in C([0,\,T];\,X)$. From semigroup theory (see Fattorini (1983)) it is known that the function u defined by (7.1.8) is subject to the following relations:

$$u'(t) = V(t)\,\big(A\,u_0 + F_1(0) + \Phi(0)\,p\big) + F_2'(t) + \int_0^t V(t-s)\,\big[\,F_1'(s) + A\,F_2(s) + \Phi'(s)\,p\,\big]\,ds\,, \tag{7.1.12}$$

$$A\,u(t) = V(t)\,\big(A\,u_0 + F_1(0) + \Phi(0)\,p\big) - F_1(t) - \Phi(t)\,p + \int_0^t V(t-s)\,\big[\,F_1'(s) + A\,F_2(s) + \Phi'(s)\,p\,\big]\,ds\,. \tag{7.1.13}$$

At the same time, for the element g defined by (7.1.11), the following relationship takes place:

$$A\,g = A\,u_1 - V(T)\,\big(A\,u_0 + F_1(0)\big) + F_1(T) - \int_0^T V(T-s)\,\big[\,F_1'(s) + A\,F_2(s)\big]\,ds\,, \tag{7.1.14}$$

whence the inclusion $g \in \mathcal{D}(A)$ becomes obvious. On the other hand, for $u_0 = 0$ and $F = 0$ we thus have $g = u_1$. Since u_1 is arbitrarily chosen from the manifold $\mathcal{D}(A)$, any element of $\mathcal{D}(A)$ may appear in place of g. Thus, we arrive at the following assertion.

Corollary 7.1.1 *The unique solvability of the inverse problem* (7.1.1)–(7.1.4) *with any admissible input data is equivalent to the invertibility of the operator B and the equality $\mathcal{D}(B^{-1}) = \mathcal{D}(A)$.*

The operator B^{-1} (if it exists) is closed. This is due to the fact that the operator B is bounded. It is not difficult to establish this property with the aid of (7.1.10). If the operator A is unbounded, thus causing that the manifold $\mathcal{D}(A)$ does not coincide with the space X, then the operator B^{-1} fails to be bounded. However, the Banach theorem on closed operator yields

$$B^{-1} \in \mathcal{L}(\mathcal{D}(A), X)$$

if the space $\mathcal{D}(A)$ is equipped with the **graph norm** of the operator A

$$\| u \|_{\mathcal{D}(A)} = \| u \| + \| A u \| .$$

Corollary 7.1.2 *If a solution of the inverse problem* (7.1.1)-(7.1.4) *exists and is unique for any admissible input data* $u_0, u_1 \in \mathcal{D}(A)$ *and*

$$F = F_1 + F_2$$

with $F_1 \in C^1([0, T]; X)$, $F_2 \in C([0, T]; \mathcal{D}(A))$, *then the stability estimates hold:*

$$(7.1.15) \quad \| u \|_{C^1([0,T];X)} \leq c \Big(\| u_0 \|_{\mathcal{D}(A)} + \| u_1 \|_{\mathcal{D}(A)} + \| F_1 \|_{C^1([0,T];X)} + \| F_2 \|_{C([0,T];\mathcal{D}(A))} \Big) ,$$

$$(7.1.16) \quad \| u \|_{C([0,T];\mathcal{D}(A))} \leq c \Big(\| u_0 \|_{\mathcal{D}(A)} + \| u_1 \|_{\mathcal{D}(A)} + \| F_1 \|_{C^1([0,T];X)} + \| F_2 \|_{C([0,T];\mathcal{D}(A))} \Big) ,$$

$$(7.1.17) \quad \| p \| \leq c \Big(\| u_0 \|_{\mathcal{D}(A)} + \| u_1 \|_{\mathcal{D}(A)} + \| F_1 \|_{C^1([0,T];X)} + \| F_2 \|_{C([0,T];\mathcal{D}(A))} \Big) .$$

Proof Indeed, since the inverse problem of interest is uniquely solvable under any admissible input data, the operator B is invertible and so

$$B^{-1} \in \mathcal{L}(\mathcal{D}(A), X) .$$

In view of this, a solution to equation (7.1.9) satisfies the inequality

$$\| p \| \leq M \| g \|_{\mathcal{D}(A)} ,$$

where $M = \| B^{-1} \|$. With this relation established, estimate (7.1.17) immediately follows from (7.1.11) and (7.1.14). When (7.1.8), (7.1.12)-(7.1.13) and (7.1.17) are put together, we come to (7.1.15) and (7.1.16) as desired. ■

It is worth noting here that in the general case the operator B fails to be invertible, thus causing some difficulties. This obstacle can be illustrated by the example in which the operator A has an eigenvector e with associated eigenvalue λ. One assumes, in addition, that $\Phi(t)$ is a scalar-valued function such that

$$\int_0^T \exp(-\lambda s)\,\Phi(s)\,ds = 0 .$$

Let the value $\Phi(t)$ of the function Φ at the point $t \in [0, T]$ be identified with the operator of multiplication by the number $\Phi(t)$ in the space X. Turning now to the inverse problem (7.1.1)–(7.1.4) observe that the function u and the element p are given by the formulae

$$u(t) = \Big(\int_0^t \exp\big(\lambda(t-s)\big)\,\Phi(s)\,ds \Big)\, e ,$$

$$p = e .$$

This pair constitutes what is called a nontrivial solution of the inverse problem with zero input data u_0, u_1, $F = 0$, thereby clarifying that the operator B transforms the nonzero element e into zero in the space X.

In this connection the conditions, under which the operator B is invertible and equation (7.1.9) is solvable in every particular case, become important and rather urgent. What is more, we need to know at what extent a solution to equation (7.1.9) may be nonunique and find out the conditions under which Fredholm's solvability of the inverse problem (7.1.1)–(7.1.4) arises. To overcome difficulties involved in a study of equation (7.1.9) and stipulated by the fact that this equation is of the first kind, it is reasonable to reduce it to an equation of the second kind by imposing extra restrictions on the input data.

Lemma 7.1.1 *Let $\Phi \in C^1([0, T]; \mathcal{L}(X))$, the operator $\Phi(T)$ be invertible and $\Phi(T)^{-1} \in \mathcal{L}(X)$. If the element g defined by (7.1.11) belongs to the manifold $\mathcal{D}(A)$ and $\lambda \in \rho(A)$, then equation (7.1.9) is equivalent to the following one:*

$$p - B_1 p = h , \tag{7.1.18}$$

where

$$B_1 = \Phi(T)^{-1} \Big[\int_0^T V(T-s) \times \big[\Phi'(s) - \lambda \, \Phi(s) \big] \, ds + V(T) \, \Phi(0) \Big] , \tag{7.1.19}$$

$$h = -\Phi(T)^{-1} (A - \lambda I) \, g . \tag{7.1.20}$$

Proof One well-known fact from **semigroup theory** may be useful in the further development: for any function $f \in C^1 \big([0, T]; X \big)$

$$A \int_0^t V(t-s) \, f(s) \, ds = \int_0^t V(t-s) \, f'(s) \, ds + V(t) \, f(0) - f(t) . \tag{7.1.21}$$

Since $\lambda \in \rho(A)$ and $g \in \mathcal{D}(A)$, equation (7.1.9) is equivalent to the following one:

$$\big(A - \lambda I \big) \, B p = \big(A - \lambda I \big) \, g , \tag{7.1.22}$$

which, in turn, can be rewritten as

$$\int_0^T V(T-s) \, \Phi'(s) \, p \, ds + V(T) \, \Phi(0) \, p - \Phi(T) \, p - \lambda \int_0^T V(T-s) \, \Phi(s) \, p \, ds = \big(A - \lambda I \big) \, g . \tag{7.1.23}$$

Here we have taken into account (7.1.10) and formula (7.1.21) with $f(t) = \Phi(t) \, p$ incorporated. Collecting two integral terms in (7.1.23) and applying then the operator $\Phi(T)^{-1}$ to both sides of the resulting equality, we derive (7.1.18), thereby completing the proof of Lemma 7.1.1.■

In light of the well-known results from semigroup theory there exist constants $M \geq 1$ and β such that the estimate

$$\| V(t) \| \leq M \, \exp (\beta \, t) \tag{7.1.24}$$

is valid for any $t \geq 0$. This provides support for the view that $V(t)$ is a **contraction semigroup** if $M = 1$ and $\beta \leq 0$. For a wide range of applications the corresponding semigroups turn out to be contractive. Moreover, in some practical problems the constant β arising from (7.1.24) is strictly negative and in every such case the semigroup is said to be **exponentially decreasing**.

Theorem 7.1.1 *If under the conditions of Lemma* 7.1.1 *estimate* (7.1.24) *holds and*

$$\int_0^T \Big\| \big[\Phi'(s) - \lambda \Phi(s) \big] \Phi(T)^{-1} \Big\| \exp \big(\beta \, (T - s) \big) \, ds + \Big\| \Phi(0) \, \Phi(T)^{-1} \Big\| \exp \big(\beta \, T \big) < \frac{1}{M} \tag{7.1.25}$$

then the operator B is invertible,

$$\mathcal{D}(B^{-1}) = \mathcal{D}(A)$$

and, in particular, the inverse problem (7.1.1)-(7.1.4) *is uniquely solvable for any admissible input data.*

Proof By virtue of Lemma 7.1.1 equation (7.1.9) can be replaced by (7.1.18) with

$$p' = \Phi(T) \, p \, .$$

Multiplying the resulting equality by $\Phi(T)$ from the left yields the governing equation for the element p':

$$p' - B_2 \, p' = h' , \tag{7.1.26}$$

where the expressions

$$B_2 = \int_0^T V(T-s) \big[\Phi'(s) - \lambda \, \Phi(s) \big] \, \Phi(T)^{-1} \, ds + V(T) \, \Phi(0) \, \Phi(T)^{-1} ,$$

$$h' = - \big(A - \lambda \, I \big) \, g$$

are derived from representations (7.1.19)-(7.1.20). Putting these together with (7.1.24)-(7.1.25) we conclude that the operator B_2 has the bound

$$\| \, B_2 \| < 1 ,$$

whence the unique solvability of (7.1.26) follows for any right-hand side $h' \in X$. Therefore, equation (7.1.18) is also uniquely solvable for any right-hand side, since the elements h and h' are related by

$$h' = \Phi(T) \; h \, ,$$

where $\Phi(T)$, $\Phi(T)^{-1} \in \mathcal{L}(X)$. For the same reason as before, Lemma 7.1.1 asserts the unique solvability of (7.1.9) for any admissible input data. Furthermore, on account of Corollary 7.1.1 the operator B is invertible, so that

$$\mathcal{D}(B^{-1}) = \mathcal{D}(A) ,$$

thereby completing the proof of the theorem. ∎

Theorem 7.1.1 applies equally well to one particular problem in which a scalar-valued function Φ is defined and continuously differentiable on the segment $[0, T]$. We identify the number $\Phi(t)$ and the operator of multiplication by this number $\Phi(t)$ in the space X bearing in mind the inverse problem (7.1.1)–(7.1.4). Then

$$\| \Phi(T) \| = | \Phi(T) |$$

and Φ as an operator function belongs to the space $C^1([0, T]; \mathcal{L}(X))$. The operator $\Phi(T)$ is invertible if and only if $\Phi(T) \neq 0$ and its inverse coincides with the operator of multiplication by the number

$$\Phi(T)^{-1} = 1/\Phi(T)\,.$$

Also, for all values $t \in [0, T]$ we accept

$$\Phi(t) > 0\,, \qquad \Phi'(t) \geq 0\,. \tag{7.1.27}$$

Theorem 7.1.2 *One assumes that the semigroup $V(t)$ generated by the operator A obeys estimate* (7.1.24) *with $M = 1$ and $\beta < 0$. If the function $\Phi \in C^1[0, T]$ is in line with* (7.1.27), *then a solution of the inverse problem* (7.1.1)–(7.1.4) *exists and is unique for any admissible input data.*

Proof From estimate (7.1.24) it follows that $\| V(T) \| < 1$, giving $1 \in \rho(V(T))$. The theorem on mapping of the semigroup spectrum yields the inclusion $0 \in \rho(A)$, making it possible to apply Theorem 7.1.1 to $\lambda = 0$. Denote by Q the left-hand side of inequality (7.1.25). Then

$$Q = \big| \Phi(T) \big|^{-1} \int\limits_0^T \big| \Phi'(s) \big| \exp\big(\beta\,(T - s)\big)\, ds + \big| \Phi(0)\,/\,\Phi(T) \big| \exp(\beta\, T)\,.$$

Since $\beta < 0$, we obtain for any $s \in [0, T]$ the quantity

$$\exp\big(\beta\,(T - s)\big) \leq 1$$

and the following chain of relations as an immediate implication of conditions (7.1.27):

$$\begin{aligned} Q &\leq \Phi(T)^{-1} \int\limits_0^T \Phi'(s)\, ds + \frac{\Phi(0)}{\Phi(T)} \exp(\beta\, T) \\ &= \Phi(T)^{-1} \big(\Phi(T) - \Phi(0) \big) + \frac{\Phi(0)}{\Phi(T)} \exp(\beta\, T) \\ &= 1 - \frac{\Phi(0)}{\Phi(T)} \big(1 - \exp(\beta\, T) \big)\,. \end{aligned}$$

Recall that, by assumption, $\Phi(0) > 0$, $\Phi(T) > 0$ and $\beta < 0$ and, therefore, estimate (7.1.25) remains valid. This provides reason enough to refer to Theorem 7.1.1 which asserts the unique solvability of the inverse problem (7.1.1)–(7.1.4) with any admissible input data. ■

Of special interest is one particular case $\Phi(t) \equiv 1$ for which the inverse problem (7.1.1)–(7.1.4) can be written as

$$u'(t) = A\,u(t) + p + F(t)\,, \qquad 0 \le t \le T\,, \tag{7.1.28}$$

$$u(0) = u_0\,, \qquad u(T) = u_1\,. \tag{7.1.29}$$

The function $\Phi \equiv 1$ satisfies both conditions (7.1.27) and so Theorem 7.1.2 is followed by

Corollary 7.1.3 *If estimate* (7.1.24) *with* $M = 1$ *and* $\beta < 0$ *is obtained for the semigroup* $V(t)$ *generated by the operator* A, *then a solution of the inverse problem* (7.1.28)–(7.1.29) *exists and is unique for any admissible input data.*

In this line, we claim that Corollary 7.1.3 will be valid under the weaker condition

$$1 \in \rho(\,V(T))\,. \tag{7.1.30}$$

Indeed, a first look at equation (7.1.9) may be of help in this matter. Because of (7.1.30), the theorem on mapping of the semigroup spectrum yields the inclusion $0 \in \rho(A)$, due to which the integral on the right-hand side of (7.1.10) can be found for $\Phi \equiv 1$. This can be done using (7.1.21) with $f(t) \equiv p$. The outcome of this is

$$A \int\limits_0^T V(T-s)\,p\,ds = \big(\,V(T) - I\,\big)\,p\,. \tag{7.1.31}$$

Comparison of (7.1.31) and (7.1.10) gives

$$B\,p = A^{-1}\,(\,V(T) - I\,)\,p\,,$$

implying that

$$B^{-1} = \big(\,V(T) - I\,\big)^{-1} A\,. \tag{7.1.32}$$

With this relation in view, the element p can explicitly be expressed by

$$p = \big(V(T) - I\big)^{-1} A\, g\,. \tag{7.1.33}$$

Being concerned with the element f, we find the function u by formula (7.1.8):

$$u(t) = V(t)\, u_0 + \int_0^t V(t-s)\, \big[\, p + F(s)\big]\cdot ds\,, \qquad 0 \le t \le T\,, \tag{7.1.34}$$

since $\Phi \equiv 1$. Thus, we arrive at the following assertion.

Corollary 7.1.4 *If the semigroup $V(t)$ generated by the operator A satisfies condition* (7.1.30), *then a solution of the inverse problem* (7.1.28)–(7.1.29) *exists and is unique for any admissible input data. Moreover, this solution is explicitly found by formulae* (7.1.11), (7.1.33) *and* (7.1.34).

In trying to verify condition (7.1.30) some difficulties do arise. However, for a wide class of problems the theorem on the spectrum mapping is much applicable and, by virtue of its strong version, we thus have

$$\exp\big(\, t\, \sigma(A)\big) = \sigma\big(\, V(t)\big) \setminus \{\, 0\,\}\,.$$

For example, the problem with a self-adjoint operator A falls into the category of such problems and so condition (7.1.30) can be replaced by $A^{-1} \in \mathcal{L}(X)$. Furthermore, the class we have mentioned above contains also problems with parabolic equations, since the operator A generates an analytic semigroup. In this case the meaning of condition (7.1.30) is that the spectrum of the operator A contains no points of the type $2\pi k i / T$, where k is an integer and i is the imaginary unit.

As stated above, the inverse problem under consideration may have, in general, a nontrivial solution even if all of the input data functions become nonzero. If this inverse problem possesses a compact semigroup, then **Fredholm-type solvability** of this problem can be achieved. Recall that the semigroup V is said to be **compact** if the operator $V(t)$ is compact for all $t > 0$.

Theorem 7.1.3 *If the operator A generates a compact semigroup,*

$$\Phi \in C^1([0, T];\, \mathcal{L}(X))\,, \qquad \Phi(T)^{-1} \in \mathcal{L}(X)\,, \qquad \lambda \in \rho(A)$$

and the element h is defined by (7.1.20), *then the following assertions are valid:*

(1) *for the inverse problem* (7.1.1)–(7.1.4) *to be solvable for any admissible input data it is necessary and sufficient that this problem has only a trivial solution under the zero input data;*

(2) *the set of all solutions of the inverse problem* (7.1.1)–(7.1.4) *with zero input data forms in the space* $C^1([0, T]; X) \times X$ *a finite-dimensional subspace;*

(3) *there exist elements* $l_1, l_2, \dots, l_n \in X^*$ *such that the inverse problem* (7.1.1)–(7.1.4) *is solvable if and only if* $l_i(h) = 0$, $1 \le i \le n$.

Proof It is clear that all the conditions of Lemma 7.1.1 hold true, by means of which the inverse problem concerned reduces to equation (7.1.18). Recall that the semigroup V is compact. Due to this property V will be continuous for $t > 0$ in the operator topology of the space $\mathcal{L}(X)$. Since the set of all compact operators constitutes a closed two-sided ideal in this topology, the operator

$$B' = \int_0^{T-\varepsilon} V(T-s)\left[\Phi'(s) - \lambda\,\Phi(s)\right]\,ds \tag{7.1.35}$$

is compact for any $\varepsilon > 0$ as a limit of the corresponding Riemann sums in the space $\mathcal{L}(X)$. On the other hand, the norm of the function

$$f(s) = V(T-s)\left[\Phi'(s) - \lambda\,\Phi(s)\right]$$

is bounded on the segment $[0, T]$. Consequently, the integral in (7.1.35) converges as $\varepsilon \to 0$ in the operator topology of the space $\mathcal{L}(X)$, implying that the operator

$$B' = \int_0^{T} V(T-s)\left[\Phi'(s) - \lambda\,\Phi(s)\right]\,ds$$

is compact. It should be noted that in Lemma 7.1.1 this integral is understood in the sense of the strong topology of the space $\mathcal{L}(X)$.

If the operator $V(T)$ is supposed to be compact, then so is the operator

$$B'' = V(T)\,\Phi(0)\,.$$

For the same reason as before the compactness of the operators B' and B'' both implies this property for the operator

$$B_1 = \Phi(T)^{-1}\left(B' + B''\right).$$

This serves as a basis for special investigations of equation (7.1.18) from the viewpoint of Fredholm's theory.

It is easily seen that the element g defined by (7.1.11) runs over the entire manifold $\mathcal{D}(A)$ if the input data run over the set of all admissible elements. Formula (7.1.20) implies that the element h runs over the entire space X, while the element g lies within the manifold $\mathcal{D}(A)$. From such reasoning it seems clear that the solvability of the inverse problem (7.1.1)–(7.1.4) under any admissible data is equivalent to that of equation (7.1.18) for any $h \in X$. Therefore, the first assertion of the theorem is an immediate implication of Fredholm's alternative.

If all of the input data become zero, then $h = 0$, so that the set comprising all of the solutions to equation (7.1.18) will coincide with the characteristic subspace of the operator B_1 if this subspace is associated with the unit eigenvalue. Since the operator B_1 is compact, the characteristic subspace so constructed is finite-dimensional. In order to arrive at the second assertion of the theorem it remains to take into account that (7.1.8) admits now the form

$$u(t) = \int_0^t V(t-s)\,\Phi(s)\,p\;ds\,.$$

Let $\{l_i\}$ form a basis of the finite-dimensional space comprising all of the solutions to the homogeneous equation $l - B_1^* l = 0$. From Fredholm's theory it follows that equation (7.1.18) will be solvable if and only if $l_i(h) = 0$, $1 \le i \le n$. This proves the third assertion, thereby completing the proof of the theorem. ■

7.2 Inverse problems with self-adjoint operator and scalar function Φ

This section is connected with one possible statement of the **two-point inverse problem**, special investigations of which lead to a final decision-making about the existence and uniqueness of its solution. One assumes that X is a Hilbert space, an operator A is self-adjoint and the inclusion $\Phi \in C^1[0,T]$ holds. The main goal of our study is to find a function

$u \in C^1([0, T]; X)$ and an element $p \in X$ from the set of relations

$$u'(t) = A\,u(t) + f(t)\,, \qquad 0 \le t \le T\,, \tag{7.2.1}$$

$$u(0) = u_0\,, \tag{7.2.2}$$

$$f(t) = \Phi(t)\,p + F(t)\,, \qquad 0 \le t \le T\,, \tag{7.2.3}$$

$$u(T) = u_1\,, \tag{7.2.4}$$

when the function $F \in C^1([0, T]; X) + C([0, T]; \mathcal{D}(A))$ and the elements $u_0,\ u_1 \in \mathcal{D}(A)$ are available, that is, the input data comprising F, u_0 and u_2 are supposed to be admissible in the sense of Definition 7.1.1.

The operator A is the generator of a strongly continuous semigroup if and only if it is semibounded from above. In that case there is a real number b such that the operator A is representable by

$$A = \int_{-\infty}^{b} \lambda\; dE_\lambda\,,$$

where E_λ refers to the **spectral resolution** of unity of the operator A (see Akhiezer and Glasman (1966)). Every element $h \in X$ can be put in correspondence with the **measure** on the real line

$$d\mu_h(\lambda) = d(E_\lambda h,\, h)\,,$$

where $(f,\, g)$ denotes the inner product of elements f and g in the space X.

A key role in "milestones" with regard to problem (7.2.1)–(7.2.4) is played by the following lemma.

Lemma 7.2.1 *Let the operator A be self-adjoint in the Hilbert space X and E_λ be its spectral resolution of unity. One assumes, in addition, that the function φ belongs to the space $C(\mathbf{R})$ and the set of all its zeroes either is empty or contains isolated points only. Then the equation*

$$\left(\int_{-\infty}^{\infty} \varphi(\lambda)\; dE_\lambda \right) p = h \tag{7.2.5}$$

is solvable with respect to p if and only if

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^{-2}\; d(E_\lambda h,\, h) < \infty\,. \tag{7.2.6}$$

Moreover, equation (7.2.5) *with $h = 0$ has a unique trivial solution if and only if the point spectrum of the operator A contains no zeroes of the function φ.*

Proof The uniqueness of a solution to equation (7.2.5) is of our initial concern. We proceed as usual. This amounts to considering one of the zeroes of the function φ, say λ_0, setting $h = 0$ and adopting λ_0 and f as an eigenvalue of the operator A and the associated eigenvector, respectively. In accordance with what has been said,

$$dE_\lambda f = \delta(\lambda - \lambda_0)\, f\,,$$

where $\delta(\lambda)$ stands for **Dirac's measure**. Therefore,

$$\Big(\int\limits_{-\infty}^{\infty} \varphi(\lambda)\; dE_\lambda \Big) f \;=\; \varphi(\lambda_0) f \;=\; 0$$

and equation (7.2.5) has a nontrivial solution $p = f$.

Arguing in reverse direction we assume that the point spectrum of the operator A contains no zeroes of the function φ and

$$h = \Big(\int\limits_{-\infty}^{\infty} \varphi(\lambda)\; dE_\lambda \Big) p \;=\; 0\,.$$

Then

$$\| h \|^2 = \int\limits_{-\infty}^{\infty} |\varphi(\lambda)|^2 \; d\big(E_\lambda p,\, p \big) \;=\; 0\,. \tag{7.2.7}$$

Since each zero of the function φ is isolated, relation (7.2.7) implies that the function

$$\mu_p(\lambda) = \big(E_\lambda p,\, p \big)$$

may exercise jumps only at zeroes of the function φ. Let now $p \neq 0$. It follows from the equality

$$\| p \|^2 = \int\limits_{-\infty}^{\infty} d\big(E_\lambda p,\, p \big)$$

that one of the zeroes of the function φ, say λ_0, should coincide with a point at which the measure $\mu_p(\lambda)$ has a jump. Consequently, λ_0 of such a kind would be a jump point of the resolution E_λ, implying that λ_0 falls into the collection of the eigenvalues of the operator A. The contradiction obtained shows that $\rho = 0$.■

In this context, the question of solvability of equation (7.2.5) arises naturally. Granted (7.2.6), denote by X_1 a **cyclic subspace** generated by the element h in the space X. This subspace will be a closure of the linear span of all elements of the type $\left(E_\beta - E_\alpha\right) h$, where α, $\beta \in \mathbf{R}$ (see Akhiezer and Glasman (1966)).

Moreover, the operator A is reduced by the subspace X_1 and the component A_1 of the operator A acting in the space X_1 is a self-adjoint operator with a **simple spectrum** (see Plesner (1965)). The resolution of unity of the operator A_1 coincides with the restriction of E_λ onto X_1. Since h is a generating element, the **spectral type** of the measure $\mu_h(\lambda)$ will be maximal with respect to the operator A_1. Therefore, condition (7.2.6) implies that the **spectral measure** with respect to the operator A_1 of the zero set of the function φ is equal to zero. The operator $\varphi(A_1)$ is invertible due to this fact. Furthermore, the inclusion $h \in \mathcal{D}\left(\varphi(A_1)^{-1}\right)$ is stipulated by condition (7.2.6), whence another conclusion can be drawn that equation (7.2.5) is solvable in the space X_1 and thus in the space X (see Plesner (1965)).

Arguing in inverse direction we assume that (7.2.5) has a solution, say p, in the space X. Relation (7.2.5) yields

$$\left(E_\lambda h,\, h \right) \;=\; \int\limits_{-\infty}^{\lambda} |\varphi(\mu)|^2 \; d\left(E_\mu p,\, p \right) .$$

This serves to motivate that the measure μ_h is subordinate to the measure μ_p and permits us to establish

$$d\left(E_\lambda h,\, h \right) \;=\; |\varphi(\lambda)|^2 \, d\left(E_\lambda p,\, p \right) .$$

With this relationship in view, it is not difficult to derive that

$$\int\limits_{-\infty}^{\infty} |\varphi(\lambda)|^{-2} \, d\left(E_\lambda h,\, h \right) \;=\; \int\limits_{-\infty}^{\infty} d\left(E_\lambda p,\, p \right) \;=\; \|\, p \,\|^2 < \infty .$$

Now condition (7.2.6) follows and this completes the proof of the lemma. ■

A study of the inverse problem (7.2.1)–(7.2.4) involves the function

$$\varphi(\lambda) \;=\; \int\limits_0^T \Phi(s) \, \exp\left(\lambda \,(T - s)\right) \, ds . \tag{7.2.8}$$

Theorem 7.2.1 *If the operator A is self-adjoint and semibounded from above in the Hilbert space X, $\Phi \in C^1[0,T]$ and $\Phi \not\equiv 0$, then the following assertions are true:*

(1) *the inverse problem (7.2.1)–(7.2.4) under the fixed input data is solvable if and only if*

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g) < \infty, \tag{7.2.9}$$

where E_λ refers to the resolution of unity of the operator A, the function φ is specified by (7.2.8), the element g is given by relation (7.1.11), that is,

$$g = u_1 - V(T)\, u_0 - \int_0^T V(T-s)\, F(s)\, ds$$

and V is a semigroup generated by the operator A;

(2) *if the inverse problem (7.2.1)–(7.2.4) is solvable, then its solution will be unique if and only if the point spectrum of the operator A contains no zeroes of the function φ specified by (7.2.8).*

Proof The inverse problem (7.2.1)–(7.2.4) is a particular case of the two-point problem (7.1.1)–(7.1.4) and, as stated in Section 7.1, can be reduced to equation (7.1.9) of the form $B\,p = g$, where

$$B\,p = \int_0^T V(T-s)\,\Phi(s)\,p\; ds\,. \tag{7.2.10}$$

Here V refers, as usual, to the semigroup generated by the operator A.

Since the function u is explicitly expressed via the element p by formula (7.1.8), the existence or uniqueness of a solution of the inverse problem concerned is equivalent to the same properties for the equation $B\,p = g$, respectively.

As we have mentioned above, there is a real number b such that

$$A = \int_{-\infty}^{b} \lambda\; dE_\lambda\,.$$

On the other hand, from calculus of self-adjoint operators it is known that

$$V(t) = \int_{-\infty}^{b} \exp(\lambda t)\, dE_\lambda ,$$

which is substituted into (7.2.10) and leads to the formula

$$B = \int_0^T \Big(\int_{-\infty}^{b} \Phi(s) \exp\big(\lambda(T-s)\big)\, dE_\lambda \Big)\, ds . \tag{7.2.11}$$

It is worth noting here that the measure $dE_\lambda \otimes ds$ is bounded on $(-\infty, b] \times [0,\, T]$ and the function

$$K(\lambda, s) = \Phi(s) \exp\big(\lambda\, (T-s)\big)$$

is continuous and bounded on $(-\infty, b] \times [0,\, T]$. Due to the indicated properties the Fubini theorem suits us perfectly for the integrals in (7.2.11) and allows one to change the order of integration. By minor manipulations with these integrals we arrive at

$$B = \int_{-\infty}^{b} \varphi(\lambda)\, dE_\lambda ,$$

where the function φ is specified by (7.2.8). What is more, it follows from (7.2.8) that φ is an entire function which does not identically equal zero. In this case the zero set of the function φ contains isolated points only. To complete the proof of the theorem, it remains to take into account that the equation $Bp = g$ is of the form (7.2.5) and apply then Lemma 7.2.1, thereby justifying both assertions.■

Suppose that the zero set of the function φ does not intersect the spectrum of the operator A. Under this premise item (2) of Theorem 7.2.1 is certainly true and condition (7.2.9) is still valid for $\Phi(T) \neq 0$. A simple observation may be of help in verifying the fact that the integral in (7.2.9) can be taken from $-\infty$ to a finite upper bound. Also, the integral

$$\int_a^b |\varphi(\lambda)|^{-2}\, d(E_\lambda g,\, g) \tag{7.2.12}$$

is always finite for any finite numbers a and b, since the function φ is continuous, its zero set consists of the isolated points only and the spectrum of the operator A is closed. Indeed, if $\varphi(\lambda_0) = 0$, then the resolvent set $\rho(A)$ contains not only this point λ_0, but also its certain neighborhood in which $E_\lambda = \text{const}$. In this view, it is reasonable to take the integral in (7.2.12) over the segment $[a, b]$ except for certain neighborhoods of all points from the zero set of the function φ. Therefore, the integral in (7.2.12) is finite and it remains to establish the convergence of the integral in (7.2.9) at $-\infty$. To that end, we integrate by parts in (7.2.8) and establish the relation

$$\varphi(\lambda) = \frac{\Phi(0) \exp \lambda T - \Phi(T)}{\lambda} + \frac{1}{\lambda} \times \int_0^T \Phi'(s) \exp\big(\lambda (T - s)\big)\, ds\,, \tag{7.2.13}$$

implying that

$$\varphi(\lambda) \sim \Phi(T) / |\lambda|$$

as $\lambda \to -\infty$. Therefore, the integral in (7.2.9) converges at $-\infty$ if and only if there exists a real number a, ensuring the convergence of the integral

$$\int_{-\infty}^{a} \lambda^2 \, d(E_\lambda g, g)\,,$$

which is just finite due to the inclusion $g \in \mathcal{D}(A)$ (see Riesz and Sz.-Nagy (1972)). Recall that the element g was defined by (7.1.11) and its belonging to the manifold $\mathcal{D}(A)$ was justified at the very beginning of Section 7.1.

From the theory of operators it is known that the spectrum of any self-adjoint operator is located on the real axis. Hence the function φ becomes non-zero on the operator spectrum, provided that none of the real numbers is included in the zero set of this function. Furthermore, it is supposed that the function Φ is nonnegative on the segment $[0, T]$ and does not identically equal zero. On the basis of definition (7.2.8) the condition imposed above is sufficient for the function φ to become non-zero on the operator spectrum. Summarizing, we deduce the following corollary.

Corollary 7.2.1 *Let the operator A be self-adjoint and semibounded from above in the Hilbert space X. If the function $\Phi \in C^1[0, T]$ is nonnegative and $\Phi(T) > 0$, then a solution of the inverse problem (7.2.1)–(7.2.4) exists and is unique for any admissible input data.*

We now turn to the inverse problem (7.2.1)–(7.2.4) for any self-adjoint operator with a **discrete spectrum**. In this case the conditions of Theorem 7.2.1 become more illustrative. The foregoing example clarifies what is done.

Let $\{ e_k \}_{k=1}^{\infty}$ form an **orthonormal basis** of the eigenvectors of the operator A and

$$A e_k = \lambda_k e_k .$$

With the aid of evident expansions with respect to that basis such as

$$g = \sum_{k=1}^{\infty} g_k e_k , \qquad p = \sum_{k=1}^{\infty} p_k e_k$$

it is plain to realize the search of the members p_k for the given sequence $\{ g_k = (g, e_k) \}$. Since

$$V(T-s) e_k = \exp \big(\lambda_k (T-s) \big) e_k ,$$

equation (7.1.9) being written in the form

$$\sum_{k=1}^{\infty} \Big(\int_0^T \Phi(s) \exp \big(\lambda_k (T-s) \big) \, ds \Big) p_k e_k = g$$

becomes equivalent to the infinite sequence of equalities

$$\varphi(\lambda_k) p_k = g_k , \qquad k = 1, 2, \dots , \tag{7.2.14}$$

where φ is specified by (7.2.8). With the aid of (7.2.14) we may clarify a little bit the sense of item (2) of Theorem 7.2.1. Indeed, in the case when $\varphi(\lambda_k) = 0$ for some subscript k, the coefficient p_k cannot be uniquely recovered and can take an arbitrarily chosen value. A necessary condition of solvability in this case amounts to the equality $g_k = 0$. If we succeed in showing that the assertion of item (2) of Theorem 7.2.1 is valid, then all of the Fourier coefficients of the element p will be uniquely determined from the system of equations (7.2.14).

By appeal to the spectral theory of self-adjoint operators from Riesz and Sz.-Nagy (1972) we write down the relation

$$\int_{-\infty}^{\infty} | \varphi(\lambda) |^{-2} \, d(E_\lambda g, g) = \sum_{k=1}^{\infty} |g_k|^2 \, | \varphi(\lambda_k) |^{-2} . \tag{7.2.15}$$

Due to item (1) of Theorem 7.2.1 the condition for the series on the right-hand side of (7.2.15) to be convergent is equivalent to the solvability of the equation $B p = g$ and thereby of the inverse problem (7.2.1)–(7.2.4).

The trace of the same condition can clearly be seen in the system of equations (7.2.14). First, the solvability of (7.2.14) implies the necessity of another conclusion that if g_k is nonzero for some subscript k, then so is the value $\varphi(\lambda_k)$ for the same subscript k. All this enables us to find that

$$p_k = g_k / \varphi(\lambda_k) .$$

Second, for the set of numbers $\{ p_k \}$ to be the sequence of the Fourier coefficients of some element p with respect to the orthonormal basis $\{ e_k \}$ it is necessary and sufficient that the condition

$$\sum_{k=1}^{\infty} \| p_k \|^2 < \infty$$

holds true. Note that this condition coincides with the requirement for the right-hand side of (7.2.15) to be finite and so is equivalent to the statement of item (1) of Theorem 7.2.1.

Having involved the available eigenvalues and eigenvectors of the operator A, the reader can derive on his/her own by the Fourier method the explicit formula for the element p as a solution of the inverse problem concerned. The outcome of this is

$$p = \sum_{k=1}^{\infty} (g, e_k) \, \varphi(\lambda_k)^{-1} e_k . \tag{7.2.16}$$

As such, formula (7.2.16) also will be useful in developing the relevant successive approximations.

The results obtained can be generalized for **anti-Hermitian operators**. Let X be a complex Hilbert space. We agree to consider $A = i A_1$, where A_1 is a self-adjoint operator in the space X. In this case A is the generator of a strongly continuous semigroup of unitary operators

$$V(t) = \int_{-\infty}^{\infty} \exp (i \lambda t) \, dE_\lambda ,$$

where E_λ stands for the resolution of unity of the operator A_1.

In just the same way as we did for self-adjoint operators it is possible to justify that in the current situation the inverse problem (7.2.1)–(7.2.4) amounts to equation (7.1.9) of the form $B p = g$, where

$$B = \int_{-\infty}^{\infty} \varphi_1(\lambda) \, dE_\lambda , \tag{7.2.17}$$

$$\varphi_1(\lambda) = \int_0^T \Phi(s) \exp (i \lambda (T - s)) \, ds . \tag{7.2.18}$$

Lemma 7.2.1 suits us perfectly in investigating the equation $B p = g$ via representation (7.2.17) and leads to the following proposition.

Corollary 7.2.2 *Let X be a complex Hilbert space and $A = i A_1$, where A_1 is a self-adjoint operator in the space X. If, in addition, $\Phi \in C^1[0, T]$ and $\Phi \not\equiv 0$, then the following assertions are true:*

(1) *the inverse problem (7.2.1)–(7.2.4) is solvable for the fixed admissible input data u_0, u_1 and F if and only if*

$$\int\limits_{-\infty}^{\infty} |\varphi_1(\lambda)|^{-2} \, d(E_\lambda g, g) < \infty, \tag{7.2.19}$$

where E_λ is the resolution of unity of the operator A_1, the function φ_1 is defined by (7.2.18), the element g is representable by (7.1.11) as follows:

$$g = u_1 - V(T) u_0 - \int\limits_0^T V(T-s) F(s) \, ds$$

(here V is the group generated by the operator A);

(2) *if the inverse problem (7.2.1)–(7.2.4) is solvable, then for its solution to be unique it is necessary and sufficient that the point spectrum of the operator A contains no zeroes of the function φ defined by (7.2.8).*

The second item of the above proposition needs certain clarification. It is worth bearing in mind some circumstances involved. Being multiplied by i, the point spectrum of the operator A_1 turns into the point spectrum of the operator A. At the same time, the zeroes of the function φ_1 "move" to the zeroes of the function φ.

Several conditions quoted below assure us of the validity of items (1)–(2) of Corollary 7.2.2. Recall that the spectrum of the operator A is located on the imaginary axis. Now the statement of item (3) will be proved if we succeed in showing that the function φ has no zeroes on the imaginary axis or, what amounts to the same, the function φ_1 has no zeroes on the real axis. If, in addition, the inequality $|\Phi(0)| < |\Phi(T)|$ holds true, then condition (7.2.19) is also true. Indeed, if the function φ_1 does not vanish on the real axis, then it is not equal to zero on the spectrum of the operator A_1 and, therefore, for any $D > 0$ the integral

$$\int_{-D}^{D} |\varphi_1(\lambda)|^{-2}\, d(E_\lambda g, g)$$

is finite.

On the other hand, by integrating (7.2.18) by parts we obtain

$$\varphi_1(\lambda) = \frac{\Phi(0)\exp(i\lambda T) - \Phi(T)}{i\lambda} + \frac{1}{i\lambda}\int_0^T \Phi'(s)\exp\big(i\lambda(T-s)\big)\, ds\,. \tag{7.2.20}$$

After replacing s by $T - s$ the integral term in (7.2.20) becomes

$$-\frac{1}{i\lambda}\int_0^T \Phi_1'(s)\exp(i\lambda s)\, ds\,,$$

where $\Phi_1(s) = \Phi(T-s)$. Whence, by the Riemann lemma it follows that

$$\frac{1}{i\lambda}\int_0^T \Phi'(s)\exp\big(i\lambda(T-s)\big)\, ds = o\Big(\frac{1}{\lambda}\Big)\,, \qquad \lambda \to \infty\,. \tag{7.2.21}$$

In conformity with (7.2.21) and the inequality $|\Phi(0)| < |\Phi(T)|$, relation (7.2.20) implies as $\lambda \to \infty$ that

$$|\varphi_1(\lambda)| \geq c/|\lambda|\,,$$

so that for all sufficiently large numbers D

$$\int_{|\lambda|>D} |\varphi_1(\lambda)|^{-2}\, d(E_\lambda g, g) \leq c^{-2}\int_{|\lambda|>D} \lambda^2\, d(E_\lambda g, g)$$

$$\leq c^{-2}\int_{-\infty}^{\infty} \lambda^2\, d(E_\lambda g, g)\,.$$

Since $g \in \mathcal{D}(A)$, the latter integral is finite. The same continues to hold for the integral in (7.2.19).

At the next stage we state that if the function Φ is nonnegative and strictly increasing on the segment $[0, T]$ (the inequality $|\Phi(0)| < |\Phi(T)|$ is due to this fact), then the function φ_1 defined by (7.2.18) has no zeroes on the real axis. Being nonnegative and strictly decreasing on the segment $[0, T]$, the function $\Phi_1(t) = \Phi(T - t)$ is subject to the following relationships:

$$\begin{aligned}
\operatorname{Re} \varphi_1(\lambda) &= \int\limits_0^T \Phi(s) \cos\big(\lambda\,(T-s)\big)\, ds \\
&= \int\limits_0^T \Phi(T-s) \cos \lambda s\, ds \\
&= \int\limits_0^T \Phi_1(s) \cos \lambda s\, ds\,, \\
\operatorname{Im} \varphi_1(\lambda) &= \int\limits_0^T \Phi(s) \sin\big(\lambda\,(T-s)\big)\, ds \\
&= \int\limits_0^T \Phi_1(s) \sin \lambda s\, ds\,.
\end{aligned}$$

Here we used also (7.2.18). Since

$$\operatorname{Re} \varphi_1(0) = \int\limits_0^T \Phi_1(s)\, ds > 0$$

and $\operatorname{Im} \varphi_1(\lambda)$ as a function of λ is odd, it remains to show that for any $\lambda > 0$

$$\int\limits_0^T \Phi_1(s) \sin \lambda s\, ds > 0\,. \tag{7.2.22}$$

If $\lambda > \pi/T$, then $\sin \lambda s \geq 0$ for all $s \in [0, T]$ and inequality (7.2.22) immediately follows. Let $\lambda < \pi/T$ and $T \neq 2\pi n/\lambda$. Allowing n to be integer we extend the function $\Phi_1(s)$ by zero up to the point of the type $2\pi n/\lambda$, which is nearest to T from the right. After that, the integral in (7.2.22) equals

$$\sum_{k=1}^{n} \int\limits_{2\pi(k-1)/\lambda}^{2\pi k/\lambda} \Phi_1(s) \sin \lambda s \, ds . \tag{7.2.23}$$

We claim that every term of the sum in (7.2.23) is positive. Indeed,

$$\int\limits_{2\pi(k-1)/\lambda}^{2\pi k/\lambda} \Phi_1(s) \sin \lambda s \, ds = \int\limits_{2\pi(k-1)/\lambda}^{\pi(2k-1)/\lambda} \Phi_1(s) \sin \lambda s \, ds + \int\limits_{\pi(2k-1)/\lambda}^{2\pi k/\lambda} \Phi_1(s) \sin \lambda s \, ds .$$

Upon substituting $s + \pi/\lambda$ for λ the second integral transforms into the following ones:

$$\int\limits_{2\pi(k-1)/\lambda}^{2\pi k/\lambda} \Phi_1(s) \sin \lambda s \, ds = \int\limits_{2\pi(k-1)/\lambda}^{\pi(2k-1)/\lambda} \big(\Phi_1(s) - \Phi_1(s+\pi/\lambda) \big) \sin \lambda s \, ds .$$

Due to the strict decrease of the function Φ_1 on the segment $[0, T]$ it remains to note that

$$\Phi_1(s) - \Phi_1(s+\pi/\lambda) > 0$$

and $\sin \lambda s \geq 0$ on the segments $[2\pi(k-1)/\lambda, \pi(2k-1)/\lambda]$. In accordance with what has been said, the following result is obtained.

Corollary 7.2.3 *Let X be a complex Hilbert space and $A = i A_1$, where A_1 is a self-adjoint operator in the space X. If, in addition, the function $\Phi \in C^1[0, T]$ is nonnegative and strictly increasing on the segment $[0, T]$, then a solution of the inverse problem (7.2.1)–(7.2.4) exists and is unique for any admissible input data.*

7.3 Two-point inverse problems in Banach lattices

Let X be a **Banach lattice**, that is, a Banach space equipped with the partial ordering relation $\leq$; meaning that any pair of its elements f, $g \in X$ has the **least upper bound** $\sup(f, g)$ and the **greatest lower bound** $\inf(f, g)$. In this case the linear operations and norm are consistent with the **partial ordering relation** as follows:

(1) *$f \leq h$ implies $f + h \leq g + h$ for any $h \in X$;*

(2) *if $f \geq 0$, then $\lambda f \geq 0$ for any number $\lambda \geq 0$;*

(3) *$|f| \leq |g|$ implies $\|f\| \leq \|g\|$.*

We note in passing that item (3) involves the **absolute value** of the element f in the sense of the relation

$$|f| = \sup(f, -f).$$

Before proceeding, it will be sensible to outline some relevant information which will be needed in the sequel. More a detailed exposition of Banach lattices is available in Arendt et al. (1986); Batty and Robinson (1984), Birkhoff (1967), Clement et al. (1987), Kantorovich and Akilov (1977), Krasnoselskii (1962), Schaefer (1974). Any element $f \in X$ admits **Jordan's decomposition**

$$f = f^+ - f^-,$$

where the elements $f^+ \geq 0$ and $f^- \geq 0$ are defined to be

$$f^+ = \sup(f, 0), \qquad f^- = \sup(-f, 0).$$

These members are involved in the useful relations

$$|f| = f^+ + f^-, \qquad \inf(f^+, f^-) = 0. \tag{7.3.1}$$

The set X_+ consisting of all nonnegative elements of the space X forms a closed **cone** of the space X. Any functional from X^* is said to be **positive** if it takes on nonnegative values on each element of the set X_+. For any element $f > 0$ there exists a positive functional $\varphi \in X^*$ such that $\varphi(f) = 1$, thereby justifying that a nonnegative element of the space X equals zero if and only if all of the positive functionals vanish on this element.

Lemma 7.3.1 *If $f \in C([a,b]; X_+)$, then*

$$\int_a^b f(t)\, dt \geq 0. \tag{7.3.2}$$

Moreover, if the integral in (7.3.2) becomes zero, then the function f is identically equal to zero on the segment $[a, b]$.

Proof By assumption, the function f takes nonnegative values only. Hence the Riemann sums for the integral in (7.3.2) are also nonnegative. Then so is the integral itself as a limit of such sums, since the cone X_+ is closed. This proves the first assertion of the lemma.

Equating now integral (7.3.2) to zero we assume that there is at least one point $t_0 \in [a, b]$ at which $f(t_0) > 0$ and have occasion to use a positive functional $\varphi \in X^*$ such that

$$\varphi\big(f(t_0)\big) = 1 .$$

Since $\varphi\big(f(t)\big) \geq 0$ for any $t \in [a, b]$, the inequality

$$\int_a^b \varphi\big(f(t)\big)\, dt > 0$$

holds. On the other hand,

$$\int_a^b \varphi\big(f(t)\big)\, dt = \varphi\Big(\int_a^b f(t)\, dt \Big) = 0 .$$

The contradiction obtained shows that the above assumption fails to be true and in this case the function f is identically equal to zero on the segment $[a, b]$, thereby completing the proof of the lemma.■

In later discussions we shall need the concepts of positive operator and positive semigroup. An operator $U \in \mathcal{L}(X)$ is said to be **positive** if $U(X_+) \subset X_+$. The positiveness of the operator U will be always indicated by the relation $U \geq 0$. A semigroup V is said to be **positive** if the operator $V(t) \geq 0$ for any $t \geq 0$. Advanced theory of positive operators and positive semigroups is available in Arendt et al. (1986), Batty and Robinson (1984), Clement et al. (1987).

We now consider in the Banach lattice X the two-point inverse problem of finding a function $u \in C^1\big([0, T]; X \big)$ and an element $p \in X$ from the set of relations

(7.3.3) $$u'(t) = A\, u(t) + \Phi(t)\, p + F(t) , \qquad 0 \leq t \leq T ,$$

(7.3.4) $$u(0) = u_0 ,$$

(7.3.5) $$u(T) = u_1 ,$$

for all admissible input data

$$u_0,\ u_1 \in \mathcal{D}(A), \qquad F \in C^1([0,T];X) + C([0,T];\mathcal{D}(A)).$$

Theorem 7.3.1 *One assumes that the spectrum of the operator A is located in the half-plane $\{\lambda\colon \operatorname{Re}\lambda < 0\}$ and this operator generates a positive semigroup. If*

$$\Phi \in C^1([0,T],\mathcal{L}(X))$$

and for any $t \in [0,T]$

$$\Phi(t) \geq 0, \qquad \Phi'(t) \geq 0,$$

then a solution u, p of the inverse problem (7.3.3)–(7.3.5) *is unique under the constraints*

$$\Phi(T)^{-1} \in \mathcal{L}(X), \qquad \Phi(T)^{-1} \geq 0.$$

Proof The theorem will be proved if we succeed in showing that the inverse problem (7.3.3)–(7.3.5) with zero input data u_0, u_1, $F = 0$ can have the trivial solution $u = 0$, $p = 0$ only. To that end, the following system

$$u'(t) = A\,u(t) + \Phi(t)\,p, \qquad 0 \leq t \leq T, \tag{7.3.6}$$

$$u(0) = 0, \tag{7.3.7}$$

$$u(T) = 0, \tag{7.3.8}$$

complements later discussions. In conformity with the results of Section 7.1, relations (7.3.6)–(7.3.8) take place if and only if

$$u(t) = \int_0^t V(t-s)\,\Phi(s)\,p\,ds, \tag{7.3.9}$$

where $V(t)$ denotes, as usual, the semigroup generated by the operator A and the element p satisfies the equation

$$B\,p = 0, \tag{7.3.10}$$

where the operator B acts in accordance with the rule

$$B\,p = \int_0^T V(T-s)\,\Phi(s)\,p\,ds. \tag{7.3.11}$$

Via **Jordan's decomposition** $p = p^+ - p^-$ of the element p in the Banach lattice X we are led to the two associated Cauchy problems

$$\left\{ \begin{aligned} u_+'(t) &= A\, u_+(t) + \Phi(t)\, p^+, \quad 0 \le t \le T, \\ u_+(0) &= 0, \end{aligned} \right. \tag{7.3.12}$$

and

$$\left\{ \begin{aligned} u_-'(t) &= A\, u_-(t) + \Phi(t)\, p^-, \quad 0 \le t \le T, \\ u_-(0) &= 0, \end{aligned} \right. \tag{7.3.13}$$

whose solutions are given by the formulae

$$u_+(t) = \int_0^t V(t-s)\, \Phi(s)\, p^+ \, ds$$

and

$$u_-(t) = \int_0^t V(t-s)\, \Phi(s)\, p^- \, ds,$$

respectively. From the theory of semigroups we know that for any $f \in C^1\big([0, T]; X\big)$ the function

$$u(t) = \int_0^t V(t-s)\, f(s)\, ds$$

is continuously differentiable on the segment $[0, T]$ and

$$u'(t) = \int_0^t V(t-s)\, f'(s)\, ds + V(t)\, f(0).$$

From such reasoning it seems clear that

$$u_+'(T) = \int_0^T V(T-s)\, \Phi'(s)\, p^+ \, ds + V(T)\, \Phi(0)\, p^+,$$

$$u_-'(T) = \int_0^T V(T-s)\, \Phi'(s)\, p^- \, ds + V(T)\, \Phi(0)\, p^-.$$

Recall that the elements p^+ and p^- are nonnegative, while the operators $\Phi(t)$, $\Phi'(t)$ and the semigroup $V(t)$ are positive. Hence, from the preceding formulae and Lemma 7.3.1 it follows that

$$u'_+(T) \geq 0\,, \qquad u'_-(T) \geq 0\,. \tag{7.3.14}$$

Observe that on account of (7.3.12)–(7.3.13) the function $w = u_+ - u_-$ satisfies the Cauchy problem

$$\begin{cases} w'(t) = A\,w(t) + \Phi(t)\,p\,, & 0 \leq t \leq T\,, \\ w(0) = 0\,, \end{cases} \tag{7.3.15}$$

and, in so doing, can be written in the form

$$w(t) = \int_0^t V(t-s)\,\Phi(s)\,p\;ds\,.$$

With the aid of equalities (7.3.10)–(7.3.11) we find that

$$w(T) = 0\,,$$

whence the coincidence of u_+ and u_- at the moment $t = T$ is obvious. In what follows we will use a common symbol for the final values of u_+ and u_-:

$$\varphi = u_+(T) = u_-(T)\,.$$

Upon substituting $t = T$ into (7.3.12) and (7.3.13) we get

$$u'_+(T) = A\,\varphi + \Phi(T)\;p^+\,,$$

$$u'_-(T) = A\,\varphi + \Phi(T)\;p^-\,.$$

On the strength of (7.3.14) the preceding relations imply that

$$\Phi(T)\,p^+ \geq -A\,\varphi$$

and

$$\Phi(T)\,p^- \geq -A\,\varphi\,,$$

so that

$$-A\,\varphi \leq \inf\big(\,\Phi(T)\,p^+\,,\;\Phi(T)\,p^-\big)\,. \tag{7.3.16}$$

The value $z = \inf\big(\Phi(T)\,p^+,\ \Phi(T)\,p^-\big)$ is aimed to show that the right-hand side of (7.3.16) equals zero. Indeed, $z \geq 0$, since the elements p^+ and p^- are nonnegative and the operator $\Phi(T)$ is positive. On the other hand, by the definition of greatest lower bound we derive the inequalities

$$z \leq \Phi(T)\, p^+$$

and

$$z \leq \Phi(T)\, p^-,$$

yielding, due to the positiveness of the operator $\Phi(T)^{-1}$, the relations

$$\Phi(T)^{-1}\, z \leq p^+$$

and

$$\Phi(T)^{-1}\, z \leq p^- .$$

Combination of the last estimates gives

$$\Phi(T)^{-1} z \leq \inf\big(p^+, p^-\big) .$$

We are led by (7.3.1) to the inequality $\Phi(T)^{-1}\, z \leq 0$, giving $z \leq 0$, because the operator $\Phi(T)$ is positive. This serves as a basis for the equality $z = 0$. Returning to (7.3.16) we see that

$$A\,\varphi \geq 0 . \tag{7.3.17}$$

From the initial assumptions on the spectrum of the operator A and spectral properties of generators of strongly continuous positive semigroups it follows that the spectrum of the operator A lies, in fact, in a half-plane $\operatorname{Re}\lambda \leq -\delta$, where $\delta > 0$. Furthermore, in that case the resolvent $\big(\lambda I - A\big)^{-1}$ is a positive operator in the domain $\operatorname{Re}\lambda > -\delta$. In particular, the operator $-A^{-1}$ is also positive. Hence, $\varphi \leq 0$ by virtue of (7.3.17). On the other hand,

$$\varphi = u_+(T) = \int_0^T V(T-s)\,\Phi(s)\, p^+\, ds\,, \tag{7.3.18}$$

since the function $u_+(t)$ solves the Cauchy problem (7.3.12). Recall that the element p^+ is nonnegative, while the operator $\Phi(t)$ and the semigroup V are positive. By Lemma 7.3.1 formula (7.3.18) implies the inequality $\varphi \geq 0$, giving $\varphi = 0$ in the case which interests us. Observe that the integral in (7.3.18) equals zero and the integrand

$$f(t) = V(T-t)\,\Phi(t)\,p^+$$

is a continuous function with values in X_+. On account of Lemma 7.3.1 this function is identically equal to zero on the segment $[0, T]$ and, in particular,

$$0 = f(T) = \Phi(T)\, p^+,$$

implying that $p^+ = 0$, because the operator $\Phi(T)$ is invertible. Along similar lines, the equality

$$u_-(T) = \int_0^T V(T-s)\, \Phi(s)\, p^- \, ds = 0$$

in combination with Lemma 7.3.1 assures us of the validity of the equality $p^- = 0$. Consequently, having stipulated condition (7.3.9), the element $p = p^+ - p^-$ and the function u are equal to zero, thereby completing the proof of the theorem. ■

Corollary 7.3.1 *If under the conditions of Theorem* 7.3.1 *the semigroup generated by the operator A is compact, then a solution u, p of the inverse problem* (7.3.3)–(7.3.5) *exists and is unique for any admissible input data.*

Proof To prove this assertion, we first refer to Theorem 7.3.1 and thus ensure the uniqueness here. On the other hand, by Theorem 7.1.3 of Section 7.1 the solvability of the inverse problem (7.3.3)–(7.3.5) is of Fredholm's character, due to which the desired result immediately follows from the uniqueness property. ■

In concluding this section we note that if the operator A generates a strongly continuous semigroup, then its spectrum is located in a half-plane $\{ \lambda \colon \operatorname{Re} \lambda < \omega \}$ and by virtue of the conditions of Theorem 7.3.1 some restrictions on this operator do arise naturally. In trying to overcome the difficulties involved the substitution $u(t) = v(t)\, e^{\omega t}$ helps set up the inverse problem (7.3.3)–(7.3.5) as follows:

$$\text{(7.3.19)} \qquad \begin{cases} v'(t) = A_1\, v(t) + \Phi_1(t)\, p + F_1(t), & 0 \le t \le T, \\ v(0) = v_0, \\ v(T) = v_1, \end{cases}$$

where

$$A_1 = A - \omega I ,$$

$$\Phi_1(t) = \Phi(t)\, e^{-\omega t},$$

$$F_1(t) = F(t)\, e^{-\omega t},$$

$$v_0 = u_0 ,$$

$$v_1 = u_1\, e^{-\omega T}.$$

By applying the results obtained in Theorem 7.3.1 and Corollary 7.3.1 to problem (7.3.19) we arrive at the following assertions.

Corollary 7.3.2 *Let the spectrum of the operator A lie in a half-plane $\{\operatorname{Re}\lambda < \omega\}$ and A be the generator of a positive semigroup. One assumes, in addition, that for any $t \in [0, T]$*

$$\Phi \in C^1\big([0, T];\, \mathcal{L}(X)\big) ,$$

$\Phi(t) \geq 0$ and $\Phi'(t) - \omega\, \Phi(t) \geq 0$. If $\Phi(T)^{-1} \in \mathcal{L}(X)$ and $\Phi(T)^{-1} \geq 0$, then a solution u, p of the inverse problem (7.3.3)–(7.3.5) is unique.

Corollary 7.3.3 *If under the conditions of Corollary 7.3.2 the semigroup generated by the operator A is compact, then a solution of the inverse problem (7.3.3)–(7.3.5) exists and is unique for any admissible input data.*

Chapter 8

Inverse Problems for Equations of Second Order

8.1 Cauchy problem for semilinear hyperbolic equations

In Banach spaces X and Y we deal with a closed linear operator A with a dense domain, the elements $u_0,\ u_1 \in X$ and the mappings

$$F\colon\ [0, T] \times X \times X \times Y \mapsto X$$

and

$$B\colon\ X \mapsto Y\,, \qquad\qquad \psi\colon\ [0, T] \mapsto Y\,.$$

It is required to recover a pair of the functions

$$u \in C^2([0, T];\, X)\,, \qquad\qquad p \in C([0, T];\, Y)\,,$$

which comply with the following relations:

(8.1.1) $$u''(t) = A\,u(t) + F\big(t,\, u(t),\, u'(t),\, p(t)\big)\,, \qquad 0 \le t \le T\,,$$

(8.1.2) $$u(0) = u_0\,, \qquad u'(0) = u_1\,;$$

(8.1.3) $$B\,u(t) = \psi(t)\,, \qquad 0 \le t \le T\,.$$

The basic restriction imposed on the operator A is connected with the requirement for the linear Cauchy (direct) problem

(8.1.4) $$u''(t) = A\,u(t) + F(t)\,, \quad 0 \le t \le T\,,$$

(8.1.5) $$u(0) = u_0\,, \qquad u'(0) = u_1\,,$$

to be well-posed. Therefore, the operator A is supposed to generate a **strongly continuous cosine function** $C(t)$, that is, an operator function which is defined for all $t \in \mathbf{R}$ with values in the space $\mathcal{L}(X)$ is continuous on the real line $\mathbf{R}$ in the strong topology of the space $\mathcal{L}(X)$ and is subject to the following two conditions:

(1) $C(0) = I$;

(2) $C(t+s) + C(t-s) = 2\,C(t)\,C(s)$ *for all* $t,\, s \in \mathbf{R}$.

It should be noted that the operator A can be recovered from its cosine function as a strong second derivative at zero

$$A\,x = C''(0)\,x$$

with the domain $\mathcal{D}(A) = \big\{x\colon\ C(t)\,x \in \mathcal{C}^2(\mathbf{R})\big\}$. Recall that the Cauchy problem (8.1.4)–(8.1.5) is uniformly well-posed if and only if the operator A generates a strongly continuous cosine function. In the sequel we will exploit some facts concerning the solvability of the Cauchy problem and relevant elements of the theory of cosine functions. For more detail we recommend to see Fattorini (1969a,b), Ivanov et al. (1995), Kisynski (1972), Kurepa (1982), Lutz (1982), Travis and Webb (1978), Vasiliev (1990).

The cosine function $C(t)$ is associated with the **sine function**

$$S(t)\,x = \int\limits_0^t C(s)\,x\,ds$$

and the subspace

$$E = \{ x \colon C(t)\, x \in \mathcal{C}^1(\mathbf{R}) \}, \tag{8.1.6}$$

which becomes a Banach space with the appropriate norm

$$\| x \|_E = \| x \| + \sup_{0 \le t \le 1} \| C'(t)\, x \| . \tag{8.1.7}$$

There exist two constants $M \geq 1$ and $\omega \geq 0$ such that the cosine and sine operator functions satisfy on the real line the appropriate estimates

$$\| C(t) \| \le M \exp(\omega t), \qquad \| S(t) \| \le M \, | t | \exp(\omega t) . \tag{8.1.8}$$

For the operator A to generate a strongly continuous cosine function satisfying estimate (8.1.8) it is necessary and sufficient that any number λ with $\lambda > \omega$ complies with the inclusion $\lambda^2 \in \rho(A)$ and the collection of inequalities

$$\left\| \frac{d^n}{d\lambda^n} \left[\lambda R(\lambda^2, A) \right] \right\| \le \frac{M\, n!}{(\lambda - \omega)^{n+1}}, \qquad n = 0, 1, 2, \dots ,$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ refers to the resolvent of the operator A. Any self-adjoint operator in a Hilbert space generates a strongly continuous cosine f unction if and only if this operator is semibounded from above.

For any $u_0 \in \mathcal{D}(A)$, $u_1 \in E$, $F \in \mathcal{C}^1([0, T]; X) \cup \mathcal{C}([0, T]; \mathcal{D}(A))$ a solution of the Cauchy problem (8.1.4)–(8.1.5) exists and is unique in the class of functions

$$u \in \mathcal{C}^2([0, T]; X) \bigcap \mathcal{C}^1([0, T]; E) \bigcap \mathcal{C}([0, T]; \mathcal{D}(A)) .$$

Moreover, this solutions is given by the formula

$$u(t) = C(t)\, u_0 + S(t)\, u_1 + \int_0^t S(t-s)\, F(s)\, ds . \tag{8.1.9}$$

In real-life situations there is a need for certain reduction of the governing equation of the second order to a first order equation. One way of proceeding is due to Kisynski (1972) and involves the Banach space $\mathcal{X} = E \times X$ and the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \tag{8.1.10}$$

with the domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times E$. If the operator A is the generator of the strongly continuous cosine function $C(t)$ associated with the sine function $S(t)$, then the operator $\mathcal{A}$ will generate in the space $\mathcal{X}$ the strongly continuous group

$$V(t) = \begin{pmatrix} C(t) & S(t) \\ A\,S(t) & C(t) \end{pmatrix}. \tag{8.1.11}$$

Upon substituting

$$w(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \qquad \mathcal{F}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}, \qquad w_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{8.1.12}$$

the Cauchy problem (8.1.4)–(8.1.5) amounts to the Cauchy problem associated with the equation of the first order

$$\begin{cases} w'(t) = \mathcal{A}\,w(t) + \mathcal{F}(t), & 0 \le t \le T, \\ w(0) = w_0 . \end{cases} \tag{8.1.13}$$

Without loss of generality the operator $\mathcal{A}$ is supposed to be invertible. Indeed, making in equation (8.1.1) the replacement

$$u(t) = v(t) \exp(\lambda t)$$

with any λ being still subject to the inclusion $\lambda^2 \in \rho(A)$ we set up another inverse problem for the functions v and p

$$v''(t) = A_\lambda\, v(t) + F_\lambda\big(t,\, v(t),\, v'(t),\, p(t)\big), \qquad 0 \le t \le T,$$

$$v(0) = v_0, \qquad v'(0) = v_1,$$

$$B\, v(t) = \psi_\lambda(t), \qquad 0 \le t \le T,$$

where

$$F_\lambda(t, v, v_1, p) = \exp(-\lambda t) \times F\big(t,\, \exp(\lambda t)\, v,\, \exp(\lambda t)\,(v_1 + \lambda v),\, p\big) - 2\,\lambda\, v_1,$$

$$v_0 = u_0, \qquad v_1 = u_1 - \lambda\, u_0,$$

$$\psi_\lambda(t) = \exp(-\lambda t)\, \psi(t),$$

$$A_\lambda = A - \lambda^2 I.$$

The emerging inverse problem is of the same type as the initial one, but here the operator A is invertible and generates a strongly continuous cosine function.

For the given function $\psi \in C^2([0, T]; Y)$ and element $u_0 \in \mathcal{D}(BA)$ the element z_0 is defined to be

$$z_0 = \psi''(0) - B A u_0 - B F_1(0, u_0, u_1), \tag{8.1.14}$$

where F_1 is involved in the approved decomposition

$$F(t, u, v, p) = F_1(t, u, v) + F_2(t, u, v, p). \tag{8.1.15}$$

With these ingredients, we impose the following conditions:

(A) *the equation $B F_2(0, u_0, u_1, p) = z_0$ with respect to p has a unique solution $p_0 \in Y$;*

(B) *there exists a mapping*

$$F_3\colon [0, T] \times Y \times Y \times Y \mapsto Y$$

such that

$$B F_2(t, u, v, p) = F_3(t, B u, B v, p); \tag{8.1.16}$$

(C) *there is a number $R > 0$ such that for any $t \in [0, T]$ the mapping $z = F_3(t, \psi(t), \psi'(t), p)$ as a function of p has in the ball $S_Y(p_0, R)$ the inverse*

$$p = \Phi(t, z). \tag{8.1.17}$$

Let the operator B comply with

$$B \in \mathcal{L}(\mathcal{D}(A^m), Y), \tag{8.1.18}$$

where m is a nonnegative integer and the manifold $\mathcal{D}(A^m)$ is equipped with the graph norm of the operator A^m. Extra smoothness conditions are required to be valid by relating the operator A to be invertible:

(D) *there is a number $R > 0$ such that on the manifold*

$$S_0 = \{(t, u, v, p)\colon 0 \leq t \leq T,\ \|u - u_0\|_E \leq R,$$
$$\|v - u_1\|_X \leq R,\ \|p - p_0\|_Y \leq R\}$$

the functions F_1 *and* F_2 *are continuous in* t *and satisfy the Lipschitz condition with respect to* (u, v, p) *in the norm of the space* X. *On the manifold*

$$S_1 = \Big\{(t, u, v, p)\colon 0 \le t \le T\,, \ \|u - A\,u_0\|_X \le R\,,$$
$$\|v - u_1\|_E \le R\,, \ \|p - p_0\|_Y \le R\Big\}$$

the functions

$$F_1(t, A^{-1}u, v)$$

and

$$F_2(t, A^{-1}u, v, p)$$

are continuous in t *and satisfy the Lipschitz condition with respect to* (u, v, p) *in the norm of the space* E. *On the manifold*

$$S_m = \Big\{(t, u, v, p)\colon 0 \le t \le T\,, \ \|u - A^{m+1}\,u_0\|_X \le R\,,$$
$$\|v - A^m\,u_1\|_E \le R\,, \ \|p - p_0\|_Y \le R\Big\}$$

both functions

$$A^m\,F_1\big(t,\, A^{-m-1}\,u,\, A^{-m}\,v\big)$$

and

$$A^m\,F_2\big(t,\, A^{-m-1}\,u,\, A^{-m}\,v\big)$$

are continuous in t *and satisfy the Lipschitz condition with respect to* (u, v, p) *in the norm of the space* E;

(E) *there is a value* $R > 0$ *such that on the manifold*

$$S_Y(z_0, R, T) = \Big\{(t, z)\colon \ 0 \le t \le T\,, \ \|z - z_0\|_Y \le R\Big\}$$

the mapping (8.1.17) *is continuous in* t *and satisfies the Lipschitz condition in* z.

Theorem 8.1.1 *Let the operator* A *generate a strongly continuous cosine function in the space* X, $A^{-1} \in \mathcal{L}(X)$ *and condition* (8.1.18) *hold. If the inclusions* $u_0 \in \mathcal{D}(A^{m+1})$, $u_1 \in \mathcal{D}(A^m)$ *and* $A^m\,u_1 \in E$ *occur, where the space* E *is the same as in* (8.1.6), *and all the conditions* (A)–(E) *are fulfilled, then there exists a number* $T_1 > 0$ *such that on the segment* $[0, T_1]$ *a solution* u, p *of the inverse problem* (8.1.1)–(8.1.3) *exists and is unique in the class of functions*

$$u \in C^2([0, T_1];\, X)\,, \qquad p \in C([0, T_1];\, Y)\,.$$

Proof We are led by (8.1.12) to

$$w(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \qquad \mathcal{F}(t, w, p) = \begin{pmatrix} 0 \\ F(t, u, u', p) \end{pmatrix}, \qquad w_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

thereby reducing (8.1.1)–(8.1.2) to the Cauchy problem for the first order equation

$$(8.1.19) \qquad w'(t) = \mathcal{A}\, w(t) + \mathcal{F}\big(t,\, w(t),\, p(t)\big), \qquad 0 \le t \le T,$$

$$(8.1.20) \qquad w(0) = w_0,$$

where the operator $\mathcal{A}$ is specified by (8.1.10).

Plain calculations show that the operator $\mathcal{A}$ is invertible and

$$\mathcal{A}^{-1} = \begin{pmatrix} 0 & A^{-1} \\ I & 0 \end{pmatrix}.$$

Because of its structure, for each nonnegative integer n we thus have

$$\mathcal{A}^{2n+1} = \begin{pmatrix} 0 & A^n \\ A^{n+1} & 0 \end{pmatrix}, \qquad \mathcal{A}^{2n} = \begin{pmatrix} A^n & 0 \\ 0 & A^n \end{pmatrix},$$

$$\mathcal{A}^{-2n-1} = \begin{pmatrix} 0 & A^{-n-1} \\ A^{-n} & 0 \end{pmatrix}, \qquad \mathcal{A}^{-2n} = \begin{pmatrix} A^{-n} & 0 \\ 0 & A^{-n} \end{pmatrix}.$$

and

$$(8.1.21) \quad \mathcal{A}^{2n+1}\, \mathcal{F}\big(t,\, \mathcal{A}^{-2n-1}(z_1,\, z_2),\, p\big) = \begin{pmatrix} A^n\, F\big(t, A^{-n-1} z_2, A^{-n} z_1, p\big) \\ 0 \end{pmatrix}.$$

By the initial assumptions relation (8.1.21) yields the inclusion

$$w_0 \in \mathcal{D}(\mathcal{A}^{2m+1}),$$

due to which the function $\mathcal{F}$ must satisfy the conditions of Theorem 6.5.2 with $2\,m$ in place of m. For all sufficiently small values T a solution of problem (8.1.19)–(8.1.20) is subject to the following relations:

$$(8.1.22) \qquad \begin{cases} w \in C\big([0,\, T];\, \mathcal{D}(\,\mathcal{A}^{2m+1})\big), \\ w' \in C\big([0,\, T];\, \mathcal{D}(\,\mathcal{A}^{2m})\big), \\ \big(\mathcal{A}^{2m}\, w(t)\big)' = \mathcal{A}^{2m}\, w'(t), \end{cases}$$

which serve to motivate that

$$u \in C^1([0, T]; \mathcal{D}(A^m))$$

and

$$(A^m u(t))' = A^m u'(t),$$

since

$$\mathcal{D}(\mathcal{A}^{2m}) = \mathcal{D}(A^m) \times \mathcal{D}(A^m).$$

From (8.1.18) it follows that for each solution of problem (8.1.1)–(8.1.3)

(8.1.23) $$B u'(t) = \psi'(t), \qquad 0 \le t \le T.$$

Common practice involves the operator

$$\mathcal{B}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = B w_2$$

and the function $\Psi(t) = \psi'(t)$, by means of which it is easily verified that condition (8.1.3) is equivalent on the strength of (8.1.23) to the following one:

(8.1.24) $$\mathcal{B} w(t) = \Psi(t), \qquad 0 \le t \le T.$$

Having stipulated condition (8.1.18), the inclusion

$$\mathcal{B} \in \mathcal{L}(\mathcal{D}(\mathcal{A}^{2m}), Y)$$

occurs and provides reason enough to reduce the inverse problem (8.1.1)–(8.1.3) to the inverse problem (6.5.1)–(6.5.3) we have posed in Section 6.5 for a first order equation. Under the conditions of Theorem 8.1.1 problem (8.1.19)–(8.1.20), (8.1.24) satisfies the premises of Theorem 8.5.3 with $2m$ in place of m. Thus, the desired assertion is an immediate implication of Theorem 6.5.3, thereby completing the proof of the theorem.■

Of special interest is the case when the function F involved in (8.1.1) is linear with respect to the variables u, u' and p, that is,

(8.1.25) $$F(t, u, u', p) = L_1(t)\, u + L_2(t)\, u' + L_3(t)\, p + F(t),$$

where for each fixed numbers $t \in [0, T]$ the operators $L_1(t) \in \mathcal{L}(E, X)$, $L_2(t) \in \mathcal{L}(X)$, $L_3(t) \in \mathcal{L}(Y, X)$ and the value $F(t) \in X$. All this enables us to prove the unique solvability of the inverse problem concerned on the whole segment $[0, T]$.

Theorem 8.1.2 *Let the operator A generate a strongly continuous cosine function in the space X, $A^{-1} \in \mathcal{L}(X)$, condition* (8.1.18) *hold, $u_0 \in \mathcal{D}(A^{m+1})$, $u_1 \in \mathcal{D}(A^m)$ and $A^m u_1 \in E$, where the space E is the same as in* (8.1.6). *One assumes, in addition, that $\psi \in C^2([0,T];Y)$, $B u_0 = \psi(0)$ and $B u_1 = \psi'(0)$. Let representation* (8.1.25) *take place with the operator functions $L_1 \in C([0,T];\mathcal{L}(E,X))$, $L_1 A^{-1} \in C([0,T];\mathcal{L}(X,E))$, $A^m L_1 A^{-m-1} \in C([0,T];\mathcal{L}(X,E))$,*

$$L_2 \in C([0,T];\mathcal{L}(X) \cap \mathcal{L}(E)), \qquad A^m L_2 A^{-m} \in C([0,T];\mathcal{L}(E))$$

and

$$A^m L_3 \in C([0,T];\mathcal{L}(Y,X)).$$

If the function $A^m F$ belongs to the space $C([0,T];E)$, the operator $B L_3(t)$ is invertible for each $t \in [0,T]$ and

$$(B L_3)^{-1} \in C([0,T];\mathcal{L}(Y)),$$

then a solution u, p of the inverse problem (8.1.1)–(8.1.3) *exists and is unique in the class of functions*

$$u \in C^2([0,T];X), \qquad p \in C([0,T];Y).$$

Proof Exploiting the fact that the inverse problem (8.1.1)–(8.1.3) can be reduced to problem (8.1.19)–(8.1.20), (8.1.24) we may attempt the function $\mathcal{F}(t,w,p)$ arising from (8.1.19) in the form

$$\mathcal{F}(t,w,p) = \mathcal{L}_1(t)\, w + \mathcal{L}_2(t)\, p + \mathcal{F}(t), \tag{8.1.26}$$

where

$$\mathcal{L}_1(t) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ L_1(t)\, w_1 + L_2(t)\, w_2 \end{pmatrix},$$

$$\mathcal{L}_2(t)\, p = \begin{pmatrix} 0 \\ L_3(t)\, p \end{pmatrix}, \qquad \mathcal{F}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.$$

It is straightforward to verify that

$$\mathcal{A}\mathcal{L}_1(t)\mathcal{A}^{-1} = \begin{pmatrix} L_2(t) & L_1(t)\, A^{-1} \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{A}^{2m+1}\mathcal{L}_1(t)\mathcal{A}^{-2m-1} = \begin{pmatrix} A^m L_2(t)\, A^{-m} & A^m L_1(t)\, A^{-m-1} \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{A}^{2m+1}\mathcal{L}_2(t)\, p = \begin{pmatrix} A^m L_3(t)\, p \\ 0 \end{pmatrix},$$

$$\mathcal{A}^{2m+1}\mathcal{F}(t) = \begin{pmatrix} A^m F(t) \\ 0 \end{pmatrix}, \qquad \mathcal{B} L_2(t) = B L_3(t).$$

Now, by assumption, problem (8.1.19)–(8.1.20), (8.1.24), (8.1.26) satisfies all the conditions of Theorem 6.5.4 with replacing m by $2m$ and so the desired assertion is an immediate implication of Theorem 6.5.4. Thus, we complete the proof of the theorem. ■

Assume now that the smoothing effect of the operator B is well-characterized by the inclusions

$$B \in \mathcal{L}(X,Y), \qquad \overline{BA} \in \mathcal{L}(E,Y). \tag{8.1.27}$$

In such a setting it is sensible to introduce the notion of weak solution and there is a need for some reduction. With the aid of relations (8.1.12) problem (8.1.4)–(8.1.5) amounts to problem (8.1.13). In view of this, the function u is called a **weak solution** of problem (8.1.4)–(8.1.5) if the function w gives a continuous solution of the reduced problem (8.1.13). As stated in Section 5.2, a continuous solution of problem (8.1.13) exists and is unique for any $w_0 \in \mathcal{X}$ and any $\mathcal{F} \in C([0,T];\mathcal{X})$. Furthermore, this solution is given by the formula

$$w(t) = V(t)\,w_0 + \int_0^t V(t-s)\,\mathcal{F}(s)\,ds,$$

where $V(t)$ refers to the semigroup generated by the operator $\mathcal{A}$. As a matter of fact, $V(t)$ is a group and is representable by (8.1.11), establishing the appropriate link with the cosine function of the operator A. Coming back to problem (8.1.4)–(8.1.5) we observe that its weak solution is nothing more than a function u being continuously differentiable on the segment $[0,T]$ and solving equation (8.1.4) in a sense of distributions. This solution exists and is unique for any $u_0 \in E$, $u_1 \in X$ and $F \in C([0,T];X)$, and is given by formula (8.1.9) as follows:

$$u(t) = C(t)\,u_0 + S(t)\,u_1 + \int_0^t S(t-s)\,F(s)\,ds,$$

so that

$$u'(t) = A\,S(t)\,u_0 + C(t)\,u_1 + \int_0^t C(t-s)\,F(s)\,ds. \tag{8.1.28}$$

The concept of weak solution of the inverse problem at hand needs certain clarification. A pair of the functions

$$u \in C^1([0,T];X), \qquad p \in C([0,T];Y)$$

is said to be a **weak solution** of the inverse problem (8.1.1)–(8.1.3) if for the function

$$F(t) = F\big(t, u(t), u'(t), p(t)\big)$$

the function u gives a weak solution of the direct problem (8.1.4)–(8.1.5) and relation (8.1.3) is satisfied in a pointwise manner. It is easy to verify that a pair of the functions u, p gives a weak solution of the inverse problem (8.1.1)–(8.1.3) if and only if the pair w, p as the outcome of the reduction procedure (8.1.12) is a continuous solution of the inverse problem (8.1.19)–(8.1.20), (8.1.24) in the sense of the definition of Section 5.3.

The next step is to redefine the element z_0 involved in condition (A) by means of the relation

$$z_0 = \psi''(0) - \overline{B\,A}\; u_0 - B\,F_1(0,\, u_0,\, u_1)\,, \tag{8.1.29}$$

thereby excluding (8.1.15) from further consideration, and replace condition (D) by the following one:

(D1) *there exists a number $R > 0$ such that on the manifold S_0 the functions F_1 and F_2 are continuous in t and satisfy the Lipschitz condition with respect to (u, v, p).*

Recall that the manifold S_0 was defined earlier in condition (D) of the present section.

Theorem 8.1.3 *Let the operator A generate a strongly continuous cosine function in the space X, conditions* (8.1.27) *hold, $u_0 \in E$, $u_1 \in X$, $\psi \in C^2\big([0, T]; Y\big)$, $B\,u_0 = \psi(0)$ and $B\,u_1 = \psi'(0)$. Under conditions (A)–(C), (D1) and (E) there exists a value $T_1 > 0$ such that on the segment $[0, T_1]$ a weak solution u, p of the inverse problem* (8.1.1)–(8.1.3) *exists and is unique.*

Proof As indicated above, replacement (8.1.12) may be of help in reducing the question of existence and uniqueness of a weak solution of the inverse problem (8.1.1)–(8.1.3) to the question of existence and uniqueness of a continuous solution of the inverse problem (8.1.19)–(8.1.20), (8.1.24). From condition (8.1.27) it seems clear that the inclusions

$$\mathcal{B}\,,\, \overline{\mathcal{B}\mathcal{A}} \in \mathcal{L}(\mathcal{X}, Y)$$

hold true, since

$$\mathcal{B}\mathcal{A}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = B\,A\,w_1\,.$$

As can readily be observed, the conditions of Theorem 6.3.1 are fulfilled and assure us of the validity of the desired assertion.■

Following the same procedure one can establish the conditions under which the function u solving the inverse problem (8.1.1)–(8.1.3) is twice continuously differentiable. Before giving further motivations, let us replace conditions (D1) and (E) by the following ones:

(D2) *there exists a number $R > 0$ such that the functions F_1 and F_2 as mappings from S_0 into X are Frechet differentiable and their partial derivatives in every direction are continuous in t and satisfy the Lipschitz condition with respect to (u, v, p) in the operator norm;*

(E2) *there exists a number $R > 0$ such that the function defined by (8.1.17) is Frechet differentiable as a mapping from $S_Y(z_0, R, T)$ into Y and its partial derivatives Φ_t and Φ_z are continuous in t and satisfy the Lipschitz condition with respect to z in the operator norm.*

By simply applying Theorem 6.3.2 to the inverse problem (8.1.19)–(8.1.20), (8.1.24) we arrive at the following assertion.

Corollary 8.1.1 *Let the operator A generate a strongly continuous cosine function in the space X, conditions* (8.1.27) *hold, $u_0 \in \mathcal{D}(A)$, $u_1 \in E$, $\psi \in C^3([0, T]; Y)$, $B u_0 = \psi(0)$ and $B u_1 = \psi'(0)$. Under conditions (A)–(C), (D2) and (E2) there exists a value $T_1 > 0$ such that on the segment $[0, T_1]$ a solution u, p of the inverse problem* (8.1.1)–(8.1.3) *exists and is unique in the class of functions*

$$u \in C^2([0, T_1]; X) \bigcap C([0, T_1]; \mathcal{D}(A)), \qquad p \in C^1([0, T_1]; Y).$$

We begin by investigating the linear case (8.1.25) and approving for further development the same framework as we dealt before. To obtain here the solvability of the inverse problem concerned on the whole segment $[0, T]$, we apply Theorem 6.4.1 of Section 6.4 to the inverse problem (8.1.19)–(8.1.20), (8.1.24), (8.1.26), whose use permits us to establish the following proposition.

Corollary 8.1.2 *Let the operator A generate a strongly continuous cosine function in the space X, conditions* (8.1.27) *hold, the inclusions $u_0 \in E$, $u_1 \in X$ and $\psi \in C^2([0, T]; Y)$ occur and the equalities*

$$B u_0 = \psi(0), \qquad B u_1 = \psi'(0)$$

hold. Let representation (6.1.25) *hold with the operator functions*

$$L_1 \in C([0, T]; \mathcal{L}(E, X)), \qquad L_2 \in C([0, T]; \mathcal{L}(X))$$

and

$$L_3 \in C\big([0,\, T];\, \mathcal{L}(Y, X)\big)\,.$$

If the function F belongs to the space $C\big([0,\, T];\, X\,\big)$, the operator $B\,L_3(t)$ in the space Y is invertible for each $t \in [0,\, T]$ and

$$\big(\,B\,L_3\,\big)^{-1} \in C\big([0,\, T];\, \mathcal{L}(Y)\big)\,,$$

then a weak solution u, p of the inverse problem (8.1.1)–(8.1.3) *exists and is unique on the whole segment* $[0,\, T]$.

One may wonder when a weak solution becomes differentiable. The answer to this is obtained by applying Theorem 6.4.2 to the inverse problem (8.1.19)–(8.1.20), (8.1.24), (8.1.26).

Corollary 8.1.3 *Let the operator A generate a strongly continuous cosine function in the space X, condition* (8.1.27) *hold, the inclusions $u_0 \in \mathcal{D}(A)$, $u_1 \in E$ and $\psi \in C^3\big([0,\, T];\, Y\,\big)$ occur and*

$$B\,u_0 = \psi(0)\,, \qquad\qquad B\,u_1 = \psi'(0)\,.$$

Let representation (8.1.25) *take place with the operator functions $L_1 \in C^1\big([0,\, T];\, \mathcal{L}(E, X)\big)$, $L_2 \in C^1\big([0,\, T];\, \mathcal{L}(X)\big)$ and $L_3 \in C^1\big([0,\, T];\, \mathcal{L}(Y, X)\big)$. If the function F belongs to the space $C^1\big([0,\, T];\, X\,\big)$, the operator $B\,L_3(t)$ in the space Y is invertible for each $t \in [0,\, T]$ and*

$$\big(\,B\,L_3\,\big)^{-1} \in C\big([0,\, T];\, \mathcal{L}(Y)\big)\,,$$

then on the segment $[0,\, T]$ a solution u, p of the inverse problem (8.1.1)–(8.1.3) *exists and is unique in the class of functions*

$$u \in C^2\big([0,\, T];\, X\,\big) \bigcap C^1\big([0,\, T];\, E\,\big) \bigcap C\big([0,\, T];\, \mathcal{D}(A)\big)\,,$$

$$p \in C^1\big([0,\, T];\, Y\,\big)\,.$$

In just the same way as we did for the first order equations it is plain to derive an **equation "in variations"** for the inverse problem (8.1.1)–(8.1.3). Rigorous assertions follow from Corollary 6.4.2 and are formulated below.

Corollary 8.1.4 *Let, in addition to the conditions of Corollary* 8.1.1, *the functions* F_1 *and* F_2 *be twice Frechet differentiable on the manifold* S_0 *and the function defined by* (8.1.17) *be twice Frechet differentiable on the manifold* $S_Y(z_0, R, T)$. *If* $\psi \in \mathcal{C}^4([0, T]; Y)$, $u_1 \in \mathcal{D}(A)$ *and*

$$A u_0 + F(0, u_0, u_1, p_0) \in E ,$$

then there exists a value $0 < T_1 \le T$ *such that on the segment* $[0, T_1]$ *the functions*

$$u \in \mathcal{C}^3([0, T_1]; X)$$

and

$$p \in \mathcal{C}^2([0, T_1]; Y) .$$

Moreover, the functions $v = u'$ *and* $q = p'$ *give a solution of the inverse problem*

$$(8.1.30) \quad \begin{cases} v''(t) = A v(t) + K_1(t) v(t) + K_2(t) v'(t) + K_3(t) q(t) + h(t) , \\ v(0) = v_0 , \qquad v'(0) = v_1 , \\ B v(t) = g(t) , \end{cases}$$

where

$$\begin{aligned} K_1(t) &= F_u\big(t, u(t), u'(t), p(t)\big) , \\ K_2(t) &= F_{u'}\big(t, u(t), u'(t), p(t)\big) , \\ K_3(t) &= F_p\big(t, u(t), u'(t), p(t)\big) , \\ h(t) &= F_t\big(t, u(t), u'(t), p(t)\big) , \\ v_0 &= u_1 , \qquad v_1 = A u_0 + F(0, u_0, u_1, p_0) , \\ g(t) &= \psi'(t) . \end{aligned}$$

Corollary 8.1.5 *Let, in addition to the conditions of Corollary* 8.1.3, $L_1 \in \mathcal{C}^2([0, T]; \mathcal{L}(E, X))$, $L_2 \in \mathcal{C}^2([0, T]; \mathcal{L}(X))$, $L_3 \in \mathcal{C}^2([0, T]; \mathcal{L}(Y, X))$ *and* $F \in \mathcal{C}^2([0, T]; X)$, $\psi \in \mathcal{C}^4([0, T]; Y)$, $u_1 \in \mathcal{D}(A)$,

$$A u_0 + L_1(0) u_0 + L_2(0) u_1 + L_3(0) p_0 + F(0) \in E .$$

Then $u \in C^3([0,T]; X)$, $p \in C^2([0,T]; Y)$ *and the functions* $v = u'$ *and* $q = p'$ *give a solution of the inverse problem* (8.1.30) *with*

$$K_1(t) = L_1(t), \qquad K_2(t) = L_2(t), \qquad K_3(t) = L_3(t),$$

$$h(t) = L_1'(t)\, u(t) + L_2'(t)\, u'(t) + L_3'(t)\, p(t) + F'(t),$$

$$v_0 = u_1, \quad v_1 = A\, u_0 + L_1(0)\, u_0 + L_2(0)\, u_1 + L_3(0)\, p_0 + F(0),$$

$$g(t) = \psi'(t).$$

By successively applying Corollaries 6.1.4–6.1.5 it is not difficult to derive the conditions under which solutions of the corresponding inverse problems as smooth as we like.

8.2 Two-point inverse problems for equations of hyperbolic type

We consider in a Banach space X the **inverse problem** for the hyperbolic equation of the second order

(8.2.1) $$u''(t) = A\, u(t) + \Phi(t)\, p + F(t), \qquad 0 \le t \le T,$$

(8.2.2) $$u(0) = u_0, \qquad u'(0) = u_1,$$

(8.2.3) $$u(T) = u_2.$$

Being concerned with the operator A, the operator function $\Phi(t)$ with values in the space $\mathcal{L}(X)$, the function F with values in the space X and the elements u_0, u_1 and u_2, we are looking for a function $u \in C^2([0,T]; X) \cap C([0,T]; \mathcal{D}(A))$ and an element $p \in X$. Equation (8.2.1) is said to be **hyperbolic** if the operator A generates a **strongly continuous cosine function**. As in Section 8.1 the main attention in the study of the inverse problem (8.2.1)–(8.2.3) is paid to the question of well-posedness of the direct problem

(8.2.4) $$u''(t) = A\, u(t) + f(t), \quad 0 \le t \le T,$$

(8.2.5) $$u(0) = u_0, \qquad u'(0) = u_1.$$

In order to retain the notations of Section 8.1 we will use in this section the symbol $C(t)$ for the cosine function generated by the operator A and the symbol $S(t)$ for the associated sine function. The subspace comprising all continuously differentiable vectors of the cosine function is denoted by E. This subspace with the accompanying norm (8.1.7) becomes a Banach space. If $u_0 \in \mathcal{D}(A)$, $u_1 \in E$ and $f \in \mathcal{C}^1([0, T]; X) + \mathcal{C}([0, T]; \mathcal{D}(A))$, then a solution of the Cauchy problem (8.2.4)–(8.2.5) exists and is unique in the class of functions

$$u \in \mathcal{C}^2([0, T]; X) \bigcap \mathcal{C}([0, T]; \mathcal{D}(A))$$

(for more detail see Fattorini (1969a,b), Kurepa (1982), Travis and Webb (1978), Vasiliev et al. (1990)). Furthermore, this solution is expressed by (5.3.12) of Section 5.3 as follows:

$$u(t) = C(t)\, u_0 + S(t)\, u_1 + \int_0^t S(t-s)\, f(s)\, ds\,. \tag{8.2.6}$$

Note that (8.2.1) and (8.2.2) coincide with (8.2.4) and (8.2.5), respectively, in one particular case when

$$f(t) = \Phi(t)\, p + F(t)\,. \tag{8.2.7}$$

The inverse problem (8.2.1)–(8.2.3) is solvable only under the agreement $u_2 \in \mathcal{D}(A)$. In what follows we keep $\Phi \in \mathcal{C}^1([0, T]; X)$. In that case, for any $p \in X$, the function f specified by (8.2.7) belongs to the space $\mathcal{C}^1([0, T]; X) + \mathcal{C}([0, T]; \mathcal{D}(A))$ if and only if so does the function F. In this regard, it will be sensible to introduce the notion of **admissible input data**. By the input data of the inverse problem concerned we mean the elements u_0, u_1, u_2 and the function F.

Definition 8.2.1 *The input data of the inverse problem* (8.2.1)–(8.2.3) *is said to be admissible if* u_0, $u_2 \in \mathcal{D}(A)$, $u_1 \in E$ *and*

$$F \in \mathcal{C}^1([0, T]; X) + \mathcal{C}([0, T]; \mathcal{D}(A))\,.$$

We note in passing that for any admissible input data the inverse problem (8.2.1)–(8.2.3) reduces to a single equation for the unknown p. Indeed, if this element is sought by substituting into (8.2.1), then the function u is given by formulae (8.2.7)–(8.2.8) as a solution of the direct problem (8.2.1)–(8.2.2). On the other hand, condition (8.2.3) is equivalent to relation (8.2.6) for $t = T$ with u_2 in place of $u(T)$, that is,

$$u_2 = C(T)\, u_0 + S(T)\, u_1 + \int_0^T S(t-s)\left[\Phi(s)\, p + F(s)\right] ds\,, \tag{8.2.8}$$

where all the terms are known except for the element p. Equation (8.2.8) can be rewritten as

$$B\,p = g\,, \tag{8.2.9}$$

where

$$B = \int_0^T S(T-s)\,\Phi(s)\,ds\,, \tag{8.2.10}$$

$$g = u_2 - C(T)\,u_0 - S(T)\,u_1 - \int_0^T S(T-s)\,F(s)\,ds\,. \tag{8.2.11}$$

The integral on the right-hand side of (8.2.10) is understood in the sense of the strong topology of the space $\mathcal{L}(X)$. True, it is to be shown that the element g defined by (8.2.11) belongs to the manifold $\mathcal{D}(A)$ for any admissible input data. Indeed, from the theory of cosine functions (see Fattorini (1969a,b), Kurepa (1982), Travis and Webb (1978)) it is clear that

$$\begin{aligned} A\,g &= A\,u_2 - C(T)\,A\,u_0 - A\,S(T)\,u_1 \\ &\quad - \int_0^T C(T-s)\,F_1'(s)\,ds \\ &\quad - C(T)\,F_1(0) + F_1(T) \\ &\quad - \int_0^T S(T-s)\,A\,F_2(s)\,ds \end{aligned} \tag{8.2.12}$$

for the decomposition $F = F_1 + F_2$ with the members

$$F_1 \in C^1([0,T];\,X)$$

and

$$F_2 \in C([0,T];\,\mathcal{D}(A))\,.$$

Since $A\,S(T) \in \mathcal{L}(E, X)$ for any value T, relation (8.2.12) yields the estimate

$$\| A g \| \le c \Big(\| A u_2 \| + \| A u_0 \| + \| u_1 \|_E + \| F_1 \|_{C^1([0,T];\, X)} + \| F_2 \|_{C([0,T];\, \mathcal{D}(A))} \Big) , \tag{8.2.13}$$

where the constant c depends only on T and the operator A. If the admissible input data of the inverse problem concerned include the elements $u_0 = 0$ and $u_1 = 0$ and the function $F = 0$, then the element g coincides with u_2 and, therefore, may be arbitrarily chosen from the manifold $\mathcal{D}(A)$. That is why the question of solvability of the inverse problem (8.2.1)–(8.2.3) for any admissible input data and the question of the solution uniqueness are equivalent to the possibility of the occurrence of the inclusion

$$B^{-1} \in \mathcal{L}(\mathcal{D}(A), X) .$$

Here $\mathcal{D}(A)$ is endowed with the graph norm of the operator A. The final conclusion immediately follows from the Banach theorem on closed operator.

Corollary 8.2.1 *If the inverse problem* (8.2.1)–(8.2.3) *is uniquely solvable for any admissible input data $u_0, u_2 \in \mathcal{D}(A)$, $u_1 \in E$ and the decomposition $F = F_1 + F_2$ with*

$$F_1 \in C^1([0, T];\, X) , \qquad F_2 \in C([0, T];\, \mathcal{D}(A)) ,$$

then the stability estimates are valid:

$$\| u \|_{C^2([0,T];\, X)} \le c \Big(\| u_0 \|_{\mathcal{D}(A)} + \| u_1 \|_E + \| u_2 \|_{\mathcal{D}(A)} + \| F_1 \|_{C^1([0,T];\, X)} + \| F_2 \|_{C([0,T];\, \mathcal{D}(A))} \Big) , \tag{8.2.14}$$

$$\| u \|_{C([0,T];\, \mathcal{D}(A))} \le c \Big(\| u_0 \|_{\mathcal{D}(A)} + \| u_1 \|_E + \| u_2 \|_{\mathcal{D}(A)} + \| F_1 \|_{C^1([0,T];\, X)} + \| F_2 \|_{C([0,T];\, \mathcal{D}(A))} \Big) , \tag{8.2.15}$$

$$\| p \| \le c \Big(\| u_0 \|_{\mathcal{D}(A)} + \| u_1 \|_E + \| u_2 \|_{\mathcal{D}(A)} + \| F_1 \|_{C^1([0,T];\, X)} + \| F_2 \|_{C([0,T];\, \mathcal{D}(A))} \Big) . \tag{8.2.16}$$

Proof A simple observation may be of help in verifying this assertion. The conditions imposed above are equivalent to the inclusion

$$B^{-1} \in \mathcal{L}(\mathcal{D}(A), X),$$

implying that

$$\| p \| \leq M \| g \|_{\mathcal{D}(A)}$$

with constant $M = \| B^{-1} \|$. Combination of the preceding inequality and (8.2.13) gives (8.2.16). Exploiting some facts from the theory of cosine functions and using the decomposition $f = f_1 + f_2$, valid for $f_1 \in C^1([0, T]; X)$ and $f_2 \in C([0, T]; \mathcal{D}(A))$, it is plain to show by minor manipulations that the function u defined by (8.2.6) satisfies the relations

$$u''(t) = C(t)\, A\, u_0 + A\, S(t)\, u_1 + \int\limits_0^t C(t-s)\, f_1'(s)\, ds \tag{8.2.17}$$

$$+ C(t)\, f_1(0) + \int\limits_0^t S(t-s)\, A\, f_2(s)\, ds + f_2(t),$$

$$A\, u(t) = C(t)\, A\, u_0 + A\, S(t)\, u_1 + \int\limits_0^t C(t-s)\, f_1'(s)\, ds \tag{8.2.18}$$

$$+ C(t)\, f_1(0) + \int\limits_0^t S(t-s)\, A\, f_2(s)\, ds - f_1(t)$$

(see Fattorini (1969a,b), Kurepa (1982), Travis and Webb (1978)). Putting these together with the representations $f_1 = \Phi(t)\, p + F_1(t)$ and $f_2(t) = F_2(t)$ and involving estimate (8.2.16), we derive estimates (8.2.14)–(8.2.15) from (8.2.17)–(8.2.18), respectively.

There seem to be at least two possible ways of reducing (8.2.9) to a second kind equation. Each of them necessitates imposing different restrictions on the operator function Φ.

Lemma 8.2.1 *If $\Phi \in C^1([0, T]; \mathcal{L}(X))$, the operator $\Phi(t)$ is invertible and $\Phi(T)^{-1} \in \mathcal{L}(X)$, $\lambda^2 \in \rho(A)$, then equation (8.2.9) is equivalent to the following one:*

$$p - B_1\, p = g_1, \tag{8.2.19}$$

where

$$B_1 = \Phi(T)^{-1} \Bigg(\int_0^T \Big[C(T-s)\,\Phi'(s) - \lambda^2\, S(T-s)\,\Phi(s) \Big]\, ds$$

$$+ C(T)\,\Phi(0) \Bigg) ,$$

$$g_1 = -\Phi(T)^{-1} \left(A - \lambda^2 I \right) g .$$

Proof Since $\lambda^2 \in \rho(A)$ and $g \in \mathcal{D}(A)$, equation (8.2.9) is equivalent to

$$(8.2.20) \qquad \left(A - \lambda^2 I \right) B\, p = \left(A - \lambda^2 I \right) g .$$

From the theory of cosine functions it is known that relations (8.2.6) and (8.2.18) with $u_0 = 0$, $u_1 = 0$ and $f_2 = 0$ are followed by

$$(8.2.21) \qquad A \int_0^t S(t-s)\, f_1(s)\, ds = \int_0^t C(t-s)\, f_1'(s)\, ds$$

$$+ C(t)\, f_1(0) - f_1(t)$$

if the function f_1 belongs to the space $C^1\left([0, T]; X\right)$ (Fattorini (1969a,b), Kurepa (1982), Travis and Webb (1978)). By inserting $t = T$ and $f_1(t) = \Phi(t)\, p$ both in (8.2.21) we establish the relationship

$$(8.2.22) \qquad A \int_0^T S(T-s)\, \Phi(s) p\, ds = \int_0^T C(T-s)\, \Phi'(s) p\, ds$$

$$+ C(T)\, \Phi(0)\, p - \Phi(T)\, p .$$

Recall that the operator B was specified by (8.2.10). Then, in view of (8.2.22), equation (8.2.20) becomes

$$(8.2.23) \qquad \int_0^T C(T-s)\, \Phi'(s)\, p\, ds + C(T)\, \Phi(0)\, p$$

$$- \Phi(T)\, p - \lambda^2 \int_0^T S(T-s)\, \Phi(s)\, p\, ds$$

$$= \left(A - \lambda^2 I \right) g .$$

After that, we are able to collect the integral terms in (8.2.23) and apply then the operator $\Phi(T)^{-1}$ to the resulting relation, yielding equation (8.2.19) and thereby completing the proof of the lemma.■

Lemma 8.2.2 *If $\Phi \in C^2([0, T]; \mathcal{L}(X))$, the operator*

$$D = \Phi(T) - C(T)\,\Phi(0) \tag{8.2.24}$$

is invertible and $D^{-1} \in \mathcal{L}(X)$, $\lambda^2 \in \rho(A)$, then equation (8.2.9) is equivalent to the following one:

$$p - B_2\, p = g_2\,, \tag{8.2.25}$$

where

$$B_2 = D^{-1}\Big(\int_0^T S(T-s)\,\Big[\Phi''(s) - \lambda^2\,\Phi(s)\Big]\,ds + S(T)\,\Phi'(0)\Big)\,,$$

$$g_2 = -D^{-1}\,(A - \lambda^2 I)\,g\,.$$

Proof With the inclusions $\lambda^2 \in \rho(A)$ and $g \in \mathcal{D}(A)$ in view, we deduce that equation (8.2.9) is equivalent to (8.2.20). We are going to show that if the function $f_1 \in C^2([0, T]; X)$, then

$$A \int_0^t S(t-s)\,f_1(s)\,ds \tag{8.2.26}$$

$$= \int_0^t S(t-s)\,f_1''(s)\,ds + C(t)\,f_1(0) + S(t)\,f_1'(0) - f_1(t)\,.$$

With this aim, let us transform the integral term on right-hand side of (8.2.21) in a natural way. By definition, the sine function is strongly continuously differentiable and $S'(t) = C(t)$. By the well-established rules from calculus we find that

$$\begin{aligned} C(t-s)\,f_1'(s) &= -\Big(\frac{d}{ds}\,S(t-s)\Big)\,f_1'(s) \\ &= -\frac{d}{ds}\,\big(S(t-s)\,f_1'(s)\big) \\ &\quad + S(t-s)\,f_1''(s)\,, \end{aligned}$$

which can be integrated over s from 0 to t. The outcome of this is

$$\int_0^t C(t-s)\, f_1'(s)\, ds = -\, S(t-s)\, f_1'(s)\Big|_0^t + \int_0^t S(t-s)\, f_1''(s)\, ds$$

$$= S(t)\, f_1'(0) + \int_0^t S(t-s)\, f_1''(s)\, ds\,.$$

Upon substituting the final result into the right-hand side of (8.2.21) we are led by minor manipulations to (8.2.26).

By merely setting $t = T$ and $f_1(t) = \Phi(t)\, p$ both in (8.2.26) we arrive at the chain of relations

$$\begin{aligned} A\,B\; p &= A \int_0^T S(T-s)\, \Phi(s)\, p\; ds \\ &= \int_0^T S(T-s)\, \Phi''(s)\; ds + C(T)\, \Phi(0)\, p \\ &\quad + S(T)\, \Phi'(0)\, p - \Phi(T)\, p\,. \end{aligned}$$

Let us substitute the preceding representation for the operator $A\,B$ into equation (8.2.20). This procedure permits us to write a final result as follows:

$$\begin{aligned} &\int_0^T S(T-s)\, \Big[\, \Phi''(s) - \lambda^2\, \Phi(s) \Big]\; p\; ds + S(T)\, \Phi'(0)\, p \\ &\qquad - \big[\, \Phi(T) - C(T)\, \Phi(0) \big]\, p = \big(\, A - \lambda^2\, I\, \big)\, g\,. \end{aligned}$$

Applying the operator $D^{-1} = \big[\, \Phi(T) - C(T)\, \Phi(0) \big]^{-1}$ to both sides of the governing equation yields (8.2.25), thereby completing the proof of the lemma. ■

As indicated in Section 8.1, there are constants $M \geq 1$ and $\omega \geq 0$ such that the operator functions $C(t)$ and $S(t)$ obey the estimates

(8.2.27) $\quad \| C(t) \| \leq M \exp\big(\omega\, t\big)\,, \qquad \| S(t) \| \leq M\, |\, t\, |\, \exp\big(\omega\, t\big)\,,$

which are followed by

$$\| B_1 \| \leq \| \Phi(T)^{-1} \| \left(\int_0^T \Big(\| \Phi'(s) \| + |\lambda|^2 (T-s) \| \Phi(s) \| \Big) \right.$$

$$\left. \times \exp\big(\omega (T-s)\big)\, ds + \| \Phi(0) \| \exp\big(\omega T\big) \right) M ,$$

$$\| B_2 \| \leq \| D^{-1} \| \left(\int_0^T (T-s) \| \Phi''(s) - \lambda^2 \Phi(s) \| \exp\big(\omega (T-s)\big)\, ds \right.$$

$$\left. + T \| \Phi'(0) \| \exp\big(\omega T\big) \right) M ,$$

where the operators B_1 and B_2 arose from Lemmas 8.2.1–8.2.2, respectively. This provides support for decision-making that the bounds $\| B_1 \| < 1$ and $\| B_2 \| < 1$ are sufficient for equations (8.2.19) and (8.2.20) to be uniquely solvable for any right-hand sides. Thus, we arrive at the following assertions.

Corollary 8.2.2 *If under the conditions of Lemma* 8.2.1 *both estimates* (8.2.27) *take place and*

$$\| \Phi(T)^{-1} \| \left(\int_0^T \Big(\| \Phi'(s) \| + |\lambda|^2 (T-s) \| \Phi(s) \| \Big) \exp\big(\omega (T-s)\big)\, ds \right.$$

$$\left. + \| \Phi(0) \| \exp\big(\omega T\big) \right) < \frac{1}{M} ,$$

then a solution u, p of the inverse problem (8.2.1)–(8.2.3) *exists and is unique for any admissible input data.*

Corollary 8.2.3 *If under the conditions of Lemma* 8.2.2 *both estimates* (8.2.27) *hold and*

$$\| D^{-1} \| \left(\int_0^T (T-s) \| \Phi''(s) - \lambda^2 \Phi(s) \| \exp\big(\omega (T-s)\big)\, ds \right.$$

$$\left. + T \| \Phi'(0) \| \exp\big(\omega T\big) \right) < \frac{1}{M} ,$$

then a solution u, p of the inverse problem (8.2.1)–(8.2.3) *exists and is unique.*

In light of some peculiarities of the cosine function equation (8.2.25) will be much more convenient in later discussions than equation (8.2.19). For the same reason as before it is interesting to establish the conditions under which the operator D specified by (8.2.24) becomes invertible.

Lemma 8.2.3 *Let the operator $\Phi(T)$ be invertible and*

$$\Phi(T)^{-1} \in \mathcal{L}(X).$$

One assumes, in addition, that

$$\| C(T)\, \Phi(0)\, \Phi(T)^{-1} \| < 1. \tag{8.2.28}$$

Then the operator D specified by (8.2.24) *is invertible. Moreover, the inclusion $D^{-1} \in \mathcal{L}(X)$ occurs and the estimate is valid:*

$$\| D^{-1} \| \le \frac{\| \Phi(T)^{-1} \|}{1 - \| C(T)\, \Phi(0)\, \Phi(T)^{-1} \|}. \tag{8.2.29}$$

Proof Owing to the inclusion $\Phi(T)^{-1} \in \mathcal{L}(X)$ relation (8.2.24) implies that

$$D = \left(I - C(T)\, \Phi(0)\, \Phi(T)^{-1} \right) \Phi(T).$$

Because of (8.2.28), the operator $D_1 = I - C(T)\, \Phi(0)\, \Phi(T)^{-1}$ is invertible in the space $\mathcal{L}(X)$ and

$$D_1^{-1} = \sum_{n=0}^{\infty} \left[C(T)\, \Phi(0)\, \Phi(T)^{-1} \right]^n,$$

yielding

$$\| D_1^{-1} \| \le \sum_{n=0}^{\infty} \| C(T)\, \Phi(0)\, \Phi(T)^{-1} \|^n = \frac{1}{1 - \| C(T)\, \Phi(0)\, \Phi(T)^{-1} \|}.$$

Since $D = D_1\, \Phi(T)$, the relationship $D^{-1} = \Phi(T)^{-1}\, D_1^{-1}$ is simple to follow. In accordance with what has been said, we thus have

$$\| D^{-1} \| \le \| \Phi(T)^{-1} \| \cdot \| D_1^{-1} \|,$$

which assures us of the validity of estimate (8.2.29) and thereby completes the proof of the lemma.■

It is worth noting that in the general case the set of all solutions of the inverse problem (8.2.1)–(8.2.3) with zero input data may not coincide with zero subspace in $C^2([0,T];X) \times X$. To understand the nature of the obstacle involved more deeply, we give one possible example. Let the operator A have an eigenvector e with associated eigenvalue $\lambda > 0$ and a numerical function Φ satisfy the condition

$$\int_0^T \Phi(s) \sin\left(\sqrt{-\lambda}\,(T-s)\right) ds = 0 .$$

We identify the value $\Phi(t)$ with the operator of multiplication by the number $\Phi(t)$ in the space X and pass to the inverse problem (8.2.1)–(8.2.3). It is straightforward to verify that the function u and the element p such that

$$u(t) = \left(\int_0^t \frac{\sin\left(\sqrt{-\lambda}\,(t-s)\right)}{\sqrt{-\lambda}}\, \Phi(s)\, ds \right) e , \qquad p = e$$

give a nontrivial solution of problem (8.2.1)–(8.2.3) with zero input data $u_0 = 0$, $u_1 = 0$, $u_2 = 0$ and $F = 0$. Equation (8.6.25) may be of help in obtaining the conditions of **Fredholm's solvability** of the inverse problem at hand.

Theorem 8.2.1 *If the operator A generates a strongly continuous cosine function, whose associated sine function is compact, that is, the operator $S(t)$ is compact for each t,*

$$\Phi \in C^2([0,T];\mathcal{L}(X)) ,$$

the operator D specified by (8.2.24) *is invertible, $D^{-1} \in \mathcal{L}(X)$, $\lambda^2 \in \rho(A)$ and $h = -D^{-1}\left(A - \lambda^2 I\right) g$, where the element g is given by formula* (8.2.11), *then the following assertions are valid:*

(1) *for the inverse problem* (8.2.1)–(8.2.3) *to be solvable for any admissible input data it is necessary and sufficient that it has only a trivial solution under the zero input data;*

(2) *the set of all solutions of the inverse problem* (8.2.1)–(8.2.3) *with zero input data forms in the space $C^2([0,T];X) \times X$ a finite-dimensional subspace;*

(3) *there exist elements $l_1, l_2, \ldots, l_n \in X^*$ such that the inverse problem* (8.2.1)–(8.2.3) *is solvable if and only if $l_i(h) = 0$, $1 \le i \le n$.*

Proof By Lemma 8.2.2 the inverse problem at hand reduces to equation (8.2.25). It is well-known that the sine function $S(t)$ is continuous on the real line $\mathbf{R}$ in the operator topology of the space $\mathcal{L}(X)$, in which the set of compact operators constitutes a closed two-sided ideal. In view of this, the operator

$$B' = \int_0^T S(T-s)\left[\Phi''(s) - \lambda^2\,\Phi(s)\right]\,ds$$

is compact as a limit in the space $\mathcal{L}(X)$ of the corresponding Riemann sums. These are compact, since $S(t)$ is compact.

On the same grounds as before, the operators

$$B'' = S(T)\,\Phi'(0)$$

and

$$B_2 = D^{-1}\left(B' + B''\right)$$

are compact and, therefore, equation (8.2.25) can be analysed from the standpoint of Fredholm's theory.

The element h arising from the conditions of the theorem coincides with the right-hand side g_2 of equation (8.2.25) and can run over the entire space along with the element g over the manifold $\mathcal{D}(A)$. Thus, the solvability of the inverse problem (8.2.1)–(8.2.3) for any admissible input data is equivalent to the question whether equation (8.2.25) is solvable for each $g_2 \in X$ and the first desired assertion follows from Fredholm's alternative.

In the case when the input data are zero, $g_2 = 0$ and the set of all solutions to equation (8.2.25) coincides with the characteristic subspace of the operator B_2 if one associates this subspace with the unit eigenvalue. Since the operator B_2 is compact, the characteristic subspace just mentioned appears to be finite-dimensional. It remains to note that the function u and the element p are related by the linear formula

$$u(t) = \int_0^t S(t-s)\,\Phi(s)\,p\;ds\,,$$

leading to the second assertion. In conformity with Fredholm's theory, equation (8.2.25) is solvable if and only if $l_i(g_2) = 0,\ 1 \leq i \leq n$, where $\{\,l_i\,\}_{i=1}^{n}$ is a basis of the finite-dimensional space comprising all solutions to the homogeneous adjoint equation $l - B_2^*\,l = 0$. This proves the third assertion and thereby completes the proof of the theorem.■

Quite often, one can encounter situations in which the cosine function generated by the operator A obeys estimate (8.2.27) with $M = 1$ and $\omega = 0$. In that case Lemma 8.2.3 implies that the operator specified by (8.2.24) is invertible if

$$\| \Phi(0) \, \Phi(T)^{-1} \| < 1 . \tag{8.2.30}$$

Thus, we come to the following proposition.

Corollary 8.2.4 *Let the operator A generate a strongly continuous cosine function, whose associated sine function is compact, that is, for which the operator $S(t)$ is compact for each t, and estimate* (8.2.27) *with $M = 1$ and $\omega = 0$ be valid together with estimate* (8.2.30). *If $\Phi \in C^2([0, T]; \mathcal{L}(X))$, $\lambda^2 \in \rho(A)$ and $h = -D^{-1} \left(A - \lambda^2 I \right) g$, where the element g is defined by* (8.2.11), *then the inverse problem* (8.2.1)–(8.2.3) *is of Fredholm's type, that is, the assertions of items* (1)–(3) *of Theorem* 8.2.1 *are true.*

Equation (8.2.1) can find a wide range of applications when the operator A in the Hilbert space X is unbounded, self-adjoint and nonpositive. Such an operator always generates a strongly continuous cosine function satisfying estimate (8.2.27) with $M = 1$ and $\omega = 0$. For the associated sine function to be compact it suffices that the spectrum of the operator is discrete. Corollary 8.2.4 can serve as a basis for this type of situation.

Corollary 8.2.5 *Let the operator A be unbounded, self-adjoint and nonpositive and its spectrum be purely discrete in the Hilbert space X. If*

$$\Phi \in C^2([0, T]; \mathcal{L}(X)) , \qquad \Phi(T) \in \mathcal{L}(X)$$

and the estimate

$$\| \Phi(0) \; \Phi(T)^{-1} \| < 1$$

is valid, then the inverse problem (8.2.1)–(8.2.3) *is of Fredholm's type.*

It is very interesting to compare the results just established with those of Theorem 7.1.3 from Section 7.1. Suppose that in the Hilbert space X the operator A is unbounded, self-adjoint and negative and its spectrum is discrete. Every such operator generates both an exponentially decreasing contraction semigroup and a cosine function, whose associated sine function is compact and satisfies (8.2.27) with $M = 1$ and $\omega = 0$. If Φ is smooth and $\Phi(T)^{-1} \in \mathcal{L}(X)$, then the inverse problem (7.1.1)–(7.1.4) for the first order equation is of Fredholm's character. At the same time for the inverse problem (8.2.1)–(8.2.3) to be of Fredholm's character it is required, in addition, that condition (8.2.30) is also satisfied. When condition (8.2.30) fails to hold, the inverse problem (8.2.1)–(8.2.3) is not of Fredholm's character. The following example confirms our statement. We introduce in the space $X = L_2(0, l)$ the operator $A u = u''$ with the domain

$$\mathcal{D}(A) = W_2^2(0, l) \bigcap \overset{\circ}{W}{}_2^1(0, l),$$

whose aim is to demonstrate that problem (8.2.1)–(8.2.3) is not of Fredholm's character for $\Phi(t) \equiv I$.

We note in passing that for $\Phi(t) \equiv I$ and $A^{-1} \in \mathcal{L}(X)$ the operator $D = I - C(T)$ is invertible if and only if problem (8.2.1)–(8.2.3) is well-posed. Indeed, in that case all the conditions of Lemma 8.2.1 are satisfied. We are led by merely setting $\lambda = 0$ to

$$B_1 = C(T),$$

which makes it possible to reduce equation (8.2.25) to the following one:

$$p - C(T)\, p = g_1 .$$

Corollary 8.2.6 *If the operator A generates a strongly continuous cosine function $C(t)$ in the Banach space X, $\Phi(t) = I$ and $0 \in \rho(A)$, then the inverse problem* (8.2.1)–(8.2.3) *is uniquely solvable for any admissible input data if and only if* $1 \in \rho\big(C(T)\big)$.

We now assume that X is the Hilbert space and the operator A is self-adjoint and semibounded from above. For any numerical function Φ, the value $\Phi(t)$ will be identified with the operator of multiplication by the number $\Phi(t)$ in the space X. The function $\varphi(\lambda)$ on the negative semi-axis is defined by

$$\varphi(\lambda) = \frac{1}{\sqrt{-\lambda}} \int\limits_0^T \Phi(s) \sin\big(\sqrt{-\lambda}\,(T - s)\big)\, ds . \tag{8.2.31}$$

Note that we might extend the function φ from the negative semi-axis to compose an **entire function** of the complex variable λ. If, in particular, $\Phi(t) \not\equiv 0$, then the zeroes of the function φ are isolated.

In what follows we denote by E_λ the **spectral resolution of unity** of the operator A. Within this notation, we write down

$$A = \int\limits_{-\infty}^{b} \lambda\, dE_\lambda ,$$

where b is a real number. Therefore, the main functions in question become

$$C(t) = \cos\left(\sqrt{-A}\, t\right), \qquad S(t) = \frac{1}{\sqrt{-A}} \sin\left(\sqrt{-A}\, t\right)$$

and equation (8.2.9) involves

$$B = \int_0^T \Phi(s)\, \frac{\sin\left(\sqrt{-A}\,(T-s)\right)}{\sqrt{-A}}\, ds\,.$$

In that case $B = \varphi(A)$, where φ stands for the analytical continuation of the function defined by (8.2.31). In this line, the solvability of equation (8.2.9) can be derived from Lemma 7.2.1 of Section 7.2. Thus, we obtain the following result.

Theorem 8.2.2 *If the operator A is self-adjoint and semibounded from above in the Hilbert space X, $\Phi \in C^1[0, T]$ and $\Phi(t) \not\equiv 0$, then the following assertions are valid:*

(1) *the inverse problem* (8.2.1)–(8.2.3) *with the fixed admissible input data u_0, u_1, u_2, F is solvable if and only if*

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g) < \infty\,, \tag{8.2.32}$$

where E_λ is the spectral resolution of unity of the operator A, φ is defined by (8.2.31) *and the element g is given by* (8.2.11);

(2) *if the inverse problem* (8.2.1)–(8.2.3) *is solvable, then its solution is unique if and only if the point spectrum of the operator A contains no zeroes of the entire function φ defined by* (8.2.31).

We cite below sufficient conditions under which items (1)–(2) of the preceding theorem will be true. Since the spectrum of any self-adjoint operator is located on the real axis, the assertion of item (2) will be proved if we succeed in showing that the function φ has no real zeroes. This is certainly true for the case when the function Φ is nonnegative and strictly increasing on the segment $[0, T]$. Indeed, for any positive λ

$$\varphi(\lambda) = \frac{1}{\sqrt{\lambda}} \int_0^T \Phi(s)\, \mathrm{sh}\left(\sqrt{\lambda}\,(T-s)\right)\, ds > 0$$

and so the function φ has no positive zeroes. Moreover,

$$\varphi(0) = \int_0^T (T-s)\,\Phi(s)\,ds > 0\,.$$

For $\lambda < 0$ set

$$\Phi_1(s) = \Phi(T-s)$$

and substitute $T-s$ for s in the integral involving the function φ, whose use permits us to write down

$$\varphi(\lambda) = \frac{1}{\sqrt{-\lambda}} \int_0^T \Phi_1(s)\,\sin\big(\sqrt{-\lambda}\,s\big)\,ds\,.$$

Recall that the positivity of the right-hand side of the above expression has been already justified in deriving inequality (7.2.22) obtained in Section 7.2.

Integrating by parts in (8.2.31) and making the substitution $T-s$ for s, we establish one useful representation

$$\varphi(\lambda) = \frac{1}{\lambda}\Big(\Phi(T) - \Phi(0)\,\cos\big(\sqrt{-\lambda}\,T\big)\Big) - \frac{1}{\lambda}\int_0^T \Phi'(T-s)\,\cos\big(\sqrt{-\lambda}\,s\big)\,ds\,. \tag{8.2.33}$$

By the Riemann lemma relation (8.2.33) implies, as $\lambda \to -\infty$, that

$$\varphi(\lambda) = \frac{1}{\lambda}\Big(\Phi(T) - \Phi(0)\,\cos\big(\sqrt{-\lambda}\,T\big)\Big) + o\Big(\frac{1}{\lambda}\Big), \tag{8.2.34}$$

whence it follows that the inequality

$$|\,\Phi(0)| < |\,\Phi(T)|$$

ensures, as $\lambda \to -\infty$, the validity of the estimate

$$|\,\varphi(\lambda)| \geq \frac{c}{|\,\lambda\,|}\,.$$

Therefore, there is a sufficiently large number $D > 0$ such that

$$\begin{aligned}\int_{-\infty}^{-D} |\,\varphi(\lambda)|^{-2}\,d(E_\lambda\,g,\,g) &\leq \frac{1}{c^2}\int_{-\infty}^{-D} |\,\lambda\,|^2\,d(E_\lambda\,g,\,g)\\ &\leq \frac{1}{c^2}\int_{-\infty}^{\infty} \lambda^2\,d(E_\lambda\,g,\,g)\end{aligned} \tag{8.2.35}$$

and all the integrals in (8.2.35) are finite. This is due to the fact that $g \in \mathcal{D}(A)$. If, in addition, the spectrum of the operator A contains no zeroes of the function φ, then the integral

$$\int_{-D}^{\infty} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g) = \int_{-D}^{b} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g)$$

is also finite. This implies the validity of the assertion of item (1) of Theorem 8.2.2 and leads us to the following statement.

Corollary 8.2.7 *If the operator A is self-adjoint and semibounded from above in the Hilbert space X and the function $\Phi \in C^1[0, T]$ is nonnegative and strictly increasing on the segment $[0, T]$, then a solution u, p of the inverse problem* (8.2.1)–(8.2.3) *exists and is unique for any admissible input data.*

Of special interest is one particular case where $\Phi(t) \equiv 1$. By calculating the integral in (8.2.31) we deduce that

$$\varphi(\lambda) = \begin{cases} \dfrac{1 - \cos\left(\sqrt{-\lambda}\, T\right)}{\lambda}, & \lambda \neq 0, \\[2ex] \dfrac{T^2}{2}, & \lambda = 0. \end{cases}$$

Let us introduce the set of points

$$Z = \left\{ \cos\left(\sqrt{-\lambda}\, T\right) : \ \lambda \in \sigma(A),\ \lambda \neq 0 \right\}. \tag{8.2.36}$$

If $1 \notin Z$, then the conditions of item (2) of Theorem 8.2.2 are satisfied. Under the stronger condition $1 \notin \bar{Z}$, where $\bar{Z}$ is the closure of the set Z, the estimate $|\varphi(\lambda)| \geq c/|\lambda|$ is valid on the spectrum of the operator A as $\lambda \to -\infty$. Thus, the integral in (8.2.32) is finite by virtue of the inclusion $g \in \mathcal{D}(A)$, leading to another conclusion.

Corollary 8.2.8 *If the operator A is self-adjoint and semibounded from above in the Hilbert space X, $\Phi(t) \equiv 1$ and $1 \notin \bar{Z}$, where the set Z is prescribed by* (8.2.36), *then a solution u, p of the inverse problem* (8.2.1)–(8.2.3) *exists and is unique for any admissible input data.*

Consider now the inverse problem (8.2.1)–(8.2.3) in the case when the spectrum of the operator A is purely discrete. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis of the space X consisting of the eigenvectors of the operator A and

$$A e_k = \lambda_k e_k .$$

Then the elements g and p can be represented as follows:

$$g = \sum_{k=1}^{\infty} g_k e_k , \qquad p = \sum_{k=1}^{\infty} p_k e_k ,$$

thus causing an alternative form of equation (8.2.9) $\varphi(A)\, p = g$ in connection with the infinite system of equations

$$\varphi(\lambda_k)\, p_k = g_k , \qquad k = 1, 2, \ldots . \tag{8.2.37}$$

Because of (8.2.37), a solution to equation (8.2.9) is unique. Then a solution of the inverse problem at hand will be unique if and only if $\varphi(\lambda_k) \neq 0$ for each k. This coincides with the assertion of item (2) of Theorem 8.2.2.

In conformity with the spectral theory of self-adjoint operators,

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g) = \sum_{k=1}^{\infty} |\varphi(\lambda_k)|^{-2}\, |g_k|^2 \tag{8.2.38}$$

if the series on the right-hand side contains only those terms for which $g_k \neq 0$. When all the terms should be taken into account, we write down $\infty \cdot 0 = 0$. In light of the results from item (1) of Theorem 8.2.2 the convergence of the series on the right-hand side of (8.2.38) is equivalent to the solvability of equation (8.2.9) and consequently the solvability of the inverse problem. The same condition can be derived from the system (8.2.37). Indeed, if equation (8.2.9) is solvable, then so is every equation from the collection (8.2.37). Therefore, if $g_k \neq 0$ for some k, then a similar relation will be true for the value $\varphi(\lambda_k)$ with the same subscript k, so that

$$p_k = \frac{g_k}{\varphi(\lambda_k)} \tag{8.2.39}$$

and the remaining components of the vector p may be arbitrarily chosen. The series on the right-hand side of (8.2.38) is a sum of the squares of only those components p_k for which $g_k \neq 0$. On the other hand, the sum containing the squares of all components of the vector p is finite, since it is equal to its norm squared. Thus, the series in (8.2.39) is convergent.

Conversely, if the series in (8.2.38) is convergent, then one can define, in complete agreement with (8.2.39), all of those values p_k for which $\varphi(\lambda_k) \neq 0$ and take the remaining values with $\varphi(\lambda_k) = 0$ to be zero. When this is the case, the sequence $\{ p_k \}_{k=1}^{\infty}$ satisfies the system (8.2.37) and the convergence of the series in (8.2.38) provides the existence of such a vector p, whose components are equal to the values p_k.

By means of the Fourier method one can derive for the solution to equation (8.2.9) the explicit formula

$$p = \sum_{k=1}^{\infty} \varphi(\lambda_k)^{-1} (g, e_k) e_k , \tag{8.2.40}$$

which may be of help in approximating the element p.

The preceding example with the space $X = L_2(0, l)$, the governing equation $A u = u''$, the domain

$$\mathcal{D}(A) = W_2^2(0, l) \cap \overset{\circ}{W}{}_2^1(0, l)$$

and the function $\Phi(t) \equiv 1$ involved helps clarify what is done. Observe that in such a setting the spectrum of the operator A is purely discrete, that is,

$$\lambda_k = -(\pi k / l)^2, \qquad k = 1, 2, \ldots . \tag{8.2.41}$$

The values

$$\mu_n = -(2 \pi n / T)^2, \qquad n = 1, 2, \ldots , \tag{8.2.42}$$

give zeroes of the function φ defined by (8.2.31). If the ratio T/l is rational, both sequences (8.2.41) and (8.2.42) have an infinite number of the coinciding terms. Thus, the uniqueness of the inverse problem (8.2.1)–(8.2.3) is violated. What is more, it follows from relations (8.2.37) that the space of all solutions to equation (8.2.9) for the case $g = 0$, corresponding, in particular, to the zero input data, is infinite-dimensional and so the inverse problem (8.2.1)–(8.2.3) fails to be of Fredholm's type. If the ratio T/l is irrational, then sequences (8.2.41) and (8.2.42) are not intersecting. This serves to motivate the uniqueness of a solution of the inverse problem (8.2.1)–(8.2.3). Also, the inverse problem concerned will be solvable on a dense set once we use any linear combination of the eigenvectors of the operator A instead of g. On the other hand, when the subscript k is large enough, the distance between λ_k and μ_k can be made as small as we like. This provides support for the view that the series in (8.2.38) does not converge for some $g \in \mathcal{D}(A)$, thereby justifying that the inverse problem (8.2.1)–(8.2.3) fails to be of Fredholm's character.

Let us consider one more example with $X = L_2(0, l)$ and $A u = u''$. Here the domain $\mathcal{D}(A)$ consists of all functions $u \in W_2^2(0, l)$ with zero boundary values $u(0) = u'(l) = 0$. For the case $\Phi(t) \equiv 1$ the spectrum of the operator A is of the form

$$\lambda_k = -((2k+1)\pi / 2l)^2, \qquad k = 1, 2, \ldots , \tag{8.2.43}$$

and the zeroes of the function φ are given by formula (8.2.42). If the ratio T/l is irrational, then, as before, a solution of the inverse problem exists and is unique only for a certain dense set of admissible input data (not for all of these data) and there is no reason here for Fredholm's character.

In this regard, one thing is worth noting. In the present and preceding examples we have $1 \in Z$, where the set Z is prescribed in (8.2.36), and, consequently, the number $\lambda = 1$ enters the continuous spectrum of the operator $C(T)$.

Allowing the ratio T/l to be rational of the form

$$\frac{T}{l} = \frac{2k+1}{4n} \tag{8.2.44}$$

we find that $\lambda_k = \mu_k$. Having multiplied the numerator and the denominator by one and the same odd number the fraction on the right-hand side of (8.2.44) is once again of this type. Thus, there exists an infinite number of pairs (k, n) such that (8.2.44) takes place. All this reflects the situation in which the number $\lambda = 1$ is an eigenvalue of the operator $C(T)$ of infinite multiplicity. If this happens, a solution of the inverse problem is nonunique and the space of its solutions with zero input data turns out to be infinite-dimensional.

When the ratio T/l is rational, but representation (8.2.44) fails to hold, none of the numbers of the type (8.2.43) falls into the set of zeroes of the function φ. If so, $1 \notin \bar{Z}$, since the set Z is discrete and finite. Moreover, as $\lambda \to -\infty$ the function

$$\varphi(\lambda) = \left(1 - \cos(\sqrt{-\lambda}\, T)\right)/\lambda$$

obeys on the spectrum of the operator A the estimate $|\varphi(\lambda)| \geq c/|\lambda|$, implying that the integral in (8.2.32) is finite for each $g \in \mathcal{D}(A)$. For this reason the inverse problem (8.2.1)–(8.2.3) is well-posed. If, for example, $T = 2l$, condition (8.2.44) is violated, so that the inverse problem concerned is well-posed. It is straightforward to verify that $C(T) = -I$, $S(T) = 0$ and all the conditions of Corollary 8.2.6 hold true. Equation (8.2.19) becomes $p - C(T)p = g_1$, from which the element p can be found in the explicit form as follows:

$$p = \tfrac{1}{2}\, g_1 = -A g\,.$$

8.3 Two-point inverse problems for equations of the elliptic type

We consider in a Banach space X the inverse problem

$$(8.3.1) \qquad u''(t) = A\,u(t) + \Phi(t)\,p + F(t)\,, \qquad 0 \le t \le T\,,$$

$$(8.3.2) \qquad u(0) = u_0\,, \qquad u'(0) = u_1\,,$$

$$\alpha\,u(T) + \beta\,u'(T) = u_2\,,$$

where A is a closed linear operator with a dense domain,

$$\Phi\colon\ [0,\,T] \mapsto \mathcal{L}(X)\,, \qquad F\colon [0,\,T] \mapsto X\,;$$

the values u_0, u_1, $u_2 \in X$ and real numbers α, β are such that $\alpha^2+\beta^2 \neq 0$.

Throughout the entire section, equation (8.3.1) is supposed to be **elliptic**, meaning the positivity of the operator A. By definition, its resolvent set contains all positive numbers and for any $\lambda \ge 0$ the estimate is valid:

$$\left\| \left(A + \lambda\,I \right)^{-1} \right\| \le \frac{c}{1+\lambda}\,.$$

When the operator A happens to be positive, it is possible to define its **fractional powers**. Moreover, the operator $-A^{1/2}$ generates a strongly continuous **analytic semigroup** $V(t)$. In each such case the **spectral radius** of the operator $V(t)$ is less than 1 for any $t > 0$ (for more detail see Krein (1967), Krein and Laptev (1962, 1966a, b)) and so the operator $I - V(t)$ is invertible for any $t > 0$. We are now interested in finding a function $u \in C^2([0,\,T];\,X\,) \cap C^1([0,\,T];\,\mathcal{D}(A^{1/2}))$ and an element $p \in X$ from relations (8.3.1)–(8.3.2).

Before proceeding to careful analysis, we give some necessary conditions for the preceding inverse problem to be solvable:

$$u_0 \in \mathcal{D}(A)\,, \qquad u_1 \in \mathcal{D}(A^{1/2})\,.$$

The element $u_2 \in \mathcal{D}(A)$ for $\beta = 0$ and $u_2 \in \mathcal{D}(A^{1/2})$ for other cases.

As a direct problem associated with equation (8.3.1) the boundary value problem is given first:

$$(8.3.3) \qquad u''(t) = A\,u(t) + f(t)\,, \qquad 0 \le t \le T\,,$$

$$(8.3.4) \qquad u(0) = u_0\,, \qquad \alpha\,u(T) + \beta\,u'(T) = u_2\,.$$

In the sequel we shall need as yet the operator

$$(8.3.5) \qquad \Delta(T) = \alpha\left(I - V(2\,T)\right) + \beta\,A^{1/2}\left(\ I + V(2\,T)\right).$$

As stated in Krein and Laptev (1962, 1966a), the direct problem (8.3.3)–(8.3.4) will be well-posed once we require that

$$\Delta(T)^{-1} \in \mathcal{L}(X)\,. \tag{8.3.6}$$

Under this agreement one can find the corresponding **Green function**

$$\begin{aligned} G(t,s) = -\frac{1}{2}\,\Delta(T)^{-1}\Big\{&\big(\beta I - \alpha A^{-1/2}\big) \\ &\times \Big[V(2T-t-s) - V\big(2T - |t-s|\big)\Big] \\ &+ \big(\beta I + \alpha A^{-1/2}\big)\,\big[V\big(|t-s|\big) - V(t+s)\big]\Big\} \end{aligned} \tag{8.3.7}$$

Observe that condition (8.3.6) is fulfilled in the particular cases $\alpha = 0$ or $\beta = 0$ as an immediate implication of the estimate $r\big(V(2T)\big) < 1$, where r denotes, as usual, the spectral radius.

In the case when $\beta \neq 0$ the domains of the operators $\Delta(T)$ and $A^{1/2}$ coincide. Furthermore, by the Banach theorem on closed graph condition (8.3.6) ensures for $\beta \neq 0$ that

$$A^{1/2}\,\Delta(T)^{-1} \in \mathcal{L}(X)\,. \tag{8.3.8}$$

To derive the formula for the inverse problem solution, it will be sensible to introduce the new elements

$$\begin{cases} a = \Delta(T)^{-1}\left[\big(\alpha I + \beta A^{1/2}\big)\,u_0 - V(T)\,u_2\right], \\ b = \Delta(T)^{-1}\left[u_2 - \big(\alpha I - \beta A^{1/2}\big)\,V(T)\,u_0\right]. \end{cases} \tag{8.3.9}$$

From the theory of semigroups it is known that the operator A commutates with the semigroup $V(t)$ and thereby with the operator $\Delta(T)$. From such reasoning it seems clear that the elements

$$a,\, b \in \mathcal{D}(A)\,. \tag{8.3.10}$$

For $\beta = 0$ inclusion (8.3.10) follows immediately from the belonging of the elements u_0 and u_2 to the manifold $\mathcal{D}(A)$. For $\beta \neq 0$ the validity of (8.3.10) is ensured by the inclusions $u_0 \in \mathcal{D}(A)$, $u_2 \in \mathcal{D}(A^{1/2})$ and (8.3.8).

Assume now that the function f satisfies on the segment $[0, T]$ either Hölder's condition in the norm of the space X or

$$f,\, A^{1/2}f \in C\big([0, T];\, X\big)\,.$$

It was shown by Krein (1967), Krein and Laptev (1962, 1966a) that under these constraints a solution u of the direct problem (8.3.3)–(8.3.4) exists and is unique in the class of functions

$$u \in C^2([0,\, T];\, X) \bigcap C^1([0,\, T];\, \mathcal{D}(A^{1/2}))\,.$$

Moreover, this solution is given by the formula

$$(8.3.11) \qquad u(t) \;=\; V(t)\, a + V(T-t)\, b + \int\limits_0^T G(t,s)\, f(s)\; ds\,,$$

so that

$$(8.3.12) \qquad u'(t) \;=\; -V(t)\, A^{1/2}\, a + V(T-t)\, A^{1/2}\, b + \int\limits_0^T G_t(t,s)\, f(s)\; ds\,.$$

Let us introduce the notion of **admissible input data** which will be needed in the sequel. In what follows we assume that the function $\Phi(t)$ with values in the space $\mathcal{L}(X)$ satisfies Hölder's condition on the segment $[0,\, T]$.

Definition 8.3.1 *The elements u_0, u_1, u_2 and the function F are referred to as the admissible input data of the inverse problem (8.3.1)–(8.3.2) if $u_0 \in \mathcal{D}(A)$, $u_1 \in \mathcal{D}(A^{1/2})$, the element u_2 belongs either $\mathcal{D}(A)$ for $\beta = 0$ or $\mathcal{D}(A^{1/2})$ for $\beta \neq 0$, the function F admits the decomposition $F = F_1 + F_2$, where F_1 satisfies on the segment $[0,\, T]$ Hölder's condition in the norm of the space X and F_2, $A^{1/2}\, F_2 \in C([0,\, T];\, X)$.*

We note in passing that the admissible input data may be involved in formulae (8.3.11)–(8.3.12) by merely setting

$$(8.3.13) \qquad f(t) \;=\; \Phi(t)\, p + F(t)\,.$$

Likewise, the inverse problem (8.3.1)–(8.3.2) can be reduced to a single equation for the element p as before. Substitution of (8.3.13) into (8.3.12) could be useful in deriving this equation by successively applying $t = 0$ and u_1 in place of $u'(0)$. The outcome of this is

$$(8.3.14) \qquad B\, p \;=\; g\,,$$

where

$$B = \int_0^T G_t(0, s)\, \Phi(s)\; ds\,, \tag{8.3.15}$$

$$\begin{aligned} g = & \; u_1 + A^{1/2}\, a - V(T)\, A^{1/2}\, b \\ & - \int_0^T G_t(0, s)\, F(s)\; ds\,. \end{aligned} \tag{8.3.16}$$

In conformity with semigroup theory the inclusion $g \in \mathcal{D}(A^{1/2})$ shall enter into force for any admissible input data. Moreover, we claim that the element g runs over the entire manifold $\mathcal{D}(A^{1/2})$ for the varying admissible input data. Indeed, the values $u_0 = 0$, $u_2 = 0$ and $F = 0$ lead to the equality $g = u_1$, which confirms our statement.

It is easy to verify that a pair $\{\, u,\, p \,\}$ gives a solution of the inverse problem (8.3.1)–(8.3.2) if the element p satisfies equation (8.3.14) and the function u is given by formula (8.3.11) with $f(t) = \Phi(t)\, p + F(t)$ incorporated. In particular, one can prove that the unique solvability of the inverse problem (8.3.1)–(8.3.2) is equivalent to the question whether the operator B is invertible and

$$\mathcal{D}(\, B^{-1}) = \mathcal{D}(\, A^{1/2})\,.$$

Of special interest is one particular case where $\Phi(t) \equiv I$. For this, the integral in (8.3.15) can be calculated without difficulties, thus giving the explicit formula for a solution to equation (8.3.14).

Theorem 8.3.1 *If the operator A is positive, $\Phi(t) \equiv I$ and the inclusions $\Delta(T)^{-1}$ and $\Delta(T/2)^{-1} \in \mathcal{L}(X)$ occur, then for any admissible input data a solution u, p of the inverse problem exists and is unique. In this case the element p is given by the formula*

$$p = -\big(\, I - V(T)\big)^{-1}\, A^{1/2}\, \Delta(T/2)^{-1}\, \Delta(T)\, g\,, \tag{8.3.17}$$

where g is defined by (8.3.16), (8.3.9) *and* (8.3.7).

Proof Recall that the element p can be recovered from equation (8.3.14). From (8.3.7) it follows that

$$\begin{aligned} G_t(0, s) = & - A^{1/2}\, \Delta(T)^{-1} \Big\{ (\, \beta\, I - \alpha\, A^{-1/2}\,) \\ & \times\; V(\, 2\,T - s\,) + (\, \beta\, I + \alpha\, A^{-1/2})\; V(s) \Big\}\,. \end{aligned}$$

The well-known formula from semigroup theory is used to deduce that for any nonnegative numbers t_1 and t_2

$$(8.3.18) \qquad A^{1/2} \int_{t_1}^{t_2} V(s)\, ds = V(t_1) - V(t_2),$$

yielding

$$A^{1/2} \int_0^T V(2T-s)\, ds = A^{1/2} \int_T^{2T} V(s)\, ds = V(T) - V(2T),$$

$$A^{1/2} \int_0^T V(s)\, ds = V(0) - V(T) = I - V(T).$$

Since the operators $A^{1/2}$ and $\Delta(T)$ are commuting, we arrive at the chain of relations

$$\begin{aligned}
A^{1/2} \int_0^T G_t(0,s)\, ds &= -\, A^{1/2} \Delta(T)^{-1} (\beta I - \alpha A^{-1/2}) A^{1/2} \int_0^T V(2T-s)\, ds \\
&\quad - A^{1/2} \Delta(T)^{-1} (\beta I + \alpha A^{-1/2}) A^{1/2} \int_0^T V(s)\, ds \\
&= -\, A^{1/2} \Delta(T)^{-1} (\beta I - \alpha A^{-1/2}) (V(T) - V(2T)) \\
&\quad - A^{1/2} \Delta(T)^{-1} (\beta I + \alpha A^{-1/2}) (I - V(T)) \\
&= -\, A^{1/2} \Delta(T)^{-1} (\beta I - \alpha A^{-1/2}) V(T) (I - V(T)) \\
&\quad - A^{1/2} \Delta(T)^{-1} (\beta I + \alpha A^{-1/2}) (I - V(T)) \\
&= -\, A^{1/2} \Delta(T)^{-1} \big[(\beta I - \alpha A^{-1/2}) V(T) \\
&\quad + (\beta I + \alpha A^{-1/2}) (I - V(T)) \\
&= -\, A^{1/2} \Delta(T)^{-1} \big[\alpha (A^{-1/2} - A^{-1/2} V(T)) \\
&\quad + \beta (I + V(T)) \big] (I - V(T))
\end{aligned}$$

$$= - A^{1/2}\,\Delta(T)^{-1}\,\big[\alpha\,(I - V(T)) + \beta\,A^{1/2}$$

$$\times\ (I + V(T))\big]\,A^{-1/2}\,(I - V(T))$$

$$= - A^{1/2}\,\Delta(T)^{-1}\,\Delta(T/2)\,A^{-1/2}\,(I - V(T))\,,$$

implying that

$$A^{1/2}\int\limits_0^T G_t(0,s)\,ds \;=\; -A^{1/2}\,\Delta(T)^{-1}\,\Delta(T/2)\,A^{-1/2}(I - V(T))\,. \tag{8.3.19}$$

Note that the operator $A^{1/2}$ is invertible and the inclusion $g \in \mathcal{D}(A^{1/2})$ occurs. Because of these facts, equation (8.3.14) is equivalent to the following one:

$$A^{1/2}\,B\,p \;=\; A^{1/2}\,g\,,$$

which leads by (8.3.19) to

$$-A^{1/2}\,\Delta(T)^{-1}\,\Delta(T/2)\,A^{-1/2}\,(I - V(T))\,p \;=\; A^{1/2}\,g\,,$$

whence formula (8.3.17) immediately follows. This completes the proof of the theorem.■

Under several additional assumptions equation (8.3.14) reduces to a **second kind equation**. A key role here is played by the operator

$$\delta(T) \;=\; \alpha\,(I + V(2T)) + \beta\,A^{1/2}\,(I - V(2T))\,, \tag{8.3.20}$$

being still subject to the relation

$$\delta(T)^{-1} \,\in\, \mathcal{L}(X)\,. \tag{8.3.21}$$

It should be noted that for the fulfilment of condition (8.3.21) it is sufficient that either $\alpha = 0$ or $\beta = 0$. For $\beta \neq 0$ the domain of the operator $\delta(T)$ coincides with $\mathcal{D}(A^{1/2})$ and in this case by the Banach theorem on closed graph condition (8.3.21) implies that

$$A^{1/2}\,\delta(T)^{-1} \in\, \mathcal{L}(X)\,. \tag{8.3.22}$$

Lemma 8.3.1 *If the operator A is positive, conditions* (8.3.6) *and* (8.3.21) *hold, the function $\Phi \in C^1([0,T];\,\mathcal{L}(X))$ and the operator*

$$\Phi(0)^{-1} \in \mathcal{L}(X)\,,$$

then for any admissible input data equation (8.3.14) *is equivalent to the following one:*

$$p - B_1 p = g_1 , \tag{8.3.23}$$

where

$$B_1 = \Phi(0)^{-1} \Big[\beta A^{1/2} \delta(T)^{-1} \int_0^T \big[V(2T - s) - V(s) \big] \Phi'(s) \, ds + \alpha \, \delta(T)^{-1} \times \Big(2 V(T) \Phi(T) - \int_0^T \big[V(2T - s) + V(s) \big] \Phi'(s) \, ds \Big) \Big] , \tag{8.3.24}$$

$$g_1 = - \Phi(0)^{-1} A^{1/2} \delta(T)^{-1} \Delta(T) \, g . \tag{8.3.25}$$

Proof As we have mentioned above, if the operator $\mathcal{A}$ generates a strongly continuous semigroup $V(t)$, then for any continuously differentiable function f we get

$$\mathcal{A} \int_0^t V(t-s) f(s) \, ds = \int_0^t V(t-s) f'(s) \, ds + V(t) f(0) - f(t) .$$

By means of the function $f(s) = g(t - s)$ we establish for any continuously differentiable function g one useful relationship:

$$\mathcal{A} \int_0^t V(s) g(s) \, ds = - \int_0^t V(s) g'(s) \, ds + V(t) g(t) - g(0) . \tag{8.3.26}$$

Moreover, from formula (8.3.26) it follows that

$$\mathcal{A} \int_0^t V(2t - s) g(s) \, ds = \int_0^t V(2t - s) g'(s) \, ds + V(2t) g(0) - V(t) g(t) . \tag{8.3.27}$$

The function $g_1(s) = g(2t-s)$ is aimed to justify this and deduce in passing that

$$\int_0^t V(2t-s)\, g(s)\; ds = \int_t^{2t} V(s)\, g_1(s)\; ds$$

$$= \int_0^{2t} V(s)\, g_1(s)\; ds - \int_0^t V(s)\, g_1(s)\; ds$$

and, because of (8.3.26),

$$\mathcal{A} \int_0^t V(2t-s)\, g(s)\; ds = \mathcal{A} \int_0^{2t} V(s)\, g_1(s)\; ds - \mathcal{A} \int_0^t V(s)\, g_1(s)\; ds$$

$$= - \int_0^{2t} V(s)\, g_1'(s)\; ds + V(2t)\, g_1(2t) - g_1(0)$$

$$+ \int_0^t V(s)\, g_1'(s)\; ds - V(t)\, g_1(t) + g_1(0)$$

$$= \int_0^{2t} V(s)\, g'(2t-s)\; ds + V(2t)\, g(0)$$

$$- \int_0^t V(s)\, g'(2t-s)\; ds - V(t)\, g(t)$$

$$= \int_t^{2t} V(s)\, g'(2t-s)\; ds + V(2t)\, g(0) - V(t)\, g(t)$$

$$= \int_0^t V(2t-s)\, g'(s)\; ds + V(2t)\, g(0) - V(t)\, g(t)\,,$$

thereby justifying formula (8.3.27).

As stated above, for any admissible input data the element g defined by (8.3.16) belongs to the manifold $\mathcal{D}(A^{1/2})$ and, since the operator $A^{1/2}$ is invertible, equation (8.3.14) is equivalent to the following one:

$$A^{1/2} B p = A^{1/2} g . \tag{8.3.28}$$

Combination of (8.3.15) and the equality

$$\begin{aligned} G_t(0, s) = & - A^{1/2} \Delta(T)^{-1} \Big\{ \big(\beta I - \alpha A^{-1/2} \big) \\ & \times V(2T - s) + \big(\beta I + \alpha A^{-1/2} \big) V(s) \Big\} \end{aligned} \tag{8.3.29}$$

gives

$$\begin{aligned} A^{1/2} B p & = A^{1/2} \int_0^T G_t(0, s) \Phi(s) p \, ds \\ & = - A^{1/2} \Delta(T)^{-1} \big(\beta I - \alpha A^{-1/2} \big) A^{1/2} \\ & \quad \times \int_0^T V(2T - s) \Phi(s) p \, ds \\ & \quad - A^{1/2} \Delta(T)^{-1} \big(\beta I + \alpha A^{-1/2} \big) \\ & \quad \times A^{1/2} \int_0^T V(s) \Phi(s) p \, ds . \end{aligned}$$

By successively applying formulae (8.3.26) and (8.3.27) to the couple

$\mathcal{A} = -A^{1/2}$ and $g(t) = \Phi(t)\,p$ we arrive at the chain of relations

$$\begin{aligned}
A^{1/2}\,B\,p &= A^{1/2}\,\Delta(T)^{-1}\,\big(\beta\,I - \alpha\,A^{-1/2}\big)\Bigg[\int_0^T V(2T-s)\,\Phi'(s)\,p\;ds \\
&\quad + V(2T)\,\Phi(0)\,p - V(T)\,\Phi(T)\,p\Bigg] \\
&\quad + A^{1/2}\,\Delta(T)^{-1}\,\big(\beta\,I + \alpha\,A^{-1/2}\big) \\
&\quad \times \Bigg[-\int_0^T V(s)\,\Phi'(s)\,p\;ds + V(T)\,\Phi(T)\,p - \Phi(0)\,p\Bigg] \\
&= A^{1/2}\,\Delta(T)^{-1}\Bigg[\beta\int_0^T \big[V(2T-s) - V(s)\big]\,\Phi'(s)\,p\;ds \\
&\quad + \alpha\,A^{-1/2}\Bigg(2\,V(T)\,\Phi(T)\,p \\
&\quad - \int_0^T \big[V(2T-s) + V(s)\big]\,\Phi'(s)\,p\;ds\Bigg)\Bigg] \\
&\quad - A^{1/2}\,\Delta(T)^{-1}\Big(\alpha\,A^{-1/2}\,\big(I + V(2T)\big) + \beta\,\big(I - V(2T)\big)\Phi(0)\,p \\
&= A^{1/2}\,\Delta(T)^{-1}\Bigg[\beta\int_0^T \big[V(2T-s) - V(s)\big]\,\Phi'(s)\,p\;ds \\
&\quad + \alpha\,A^{-1/2}\Bigg(2\,V(T)\,\Phi(T)\,p - \int_0^T \big[V(2T-s) + V(s)\big] \\
&\quad \times \Phi'(s)\,p\;ds\Bigg)\Bigg] - A^{1/2}\,\Delta(T)^{-1}\,\delta(T)\,A^{-1/2}\,\Phi(0)\,p\,.
\end{aligned}$$

We are led by substituting this expression into the left-hand side of (8.3.28) and applying then the operator

$$\Phi(0)^{-1}\,A^{1/2}\,\delta(T)^{-1}\,\Delta(T)\,A^{-1/2}$$

to both sides of the resulting relation to (8.3.23), thereby completing the proof of the lemma.■

It is worth noting here that, in general, a solution of the inverse problem (8.3.1)–(8.3.2) is not obliged to be unique. The inverse problem

$$(8.3.30) \qquad \begin{cases} u''(t) = A\,u(t) + \Phi(t)\,p\,, & 0 \le t \le T\,, \\ u(0) = 0\,, \quad u'(0) = 0\,, \quad u(T) = 0\,, \end{cases}$$

where Φ is a numerical function, complements our study and gives a particular case of the inverse problem (8.3.1)–(8.3.2) with $\alpha = 1$, $\beta = 0$ under the zero input data. Problem (8.3.30) has at least the trivial solution $u(t) \equiv 0$, $p = 0$. If the operator A has an eigenvector e with associated positive eigenvalue λ, then

$$V(t)\; e = \exp\big(-\sqrt{\lambda}\, t\big)\, e\,.$$

The function Φ will be so chosen as to satisfy the condition

$$\int\limits_0^T \Big(\exp\big(-\sqrt{\lambda}\,(2T - s)\big) - \exp\big(-\sqrt{\lambda}\, s\big)\Big)\,\Phi(s)\; ds = 0\,.$$

With the aid of (8.3.15) and (8.3.29) we deduce that $B\,e = 0$ and, therefore, the pair of functions u and p such that

$$u(t) = \int\limits_0^T G(t,s)\,\Phi(s)\,e\; ds\,,$$

$$p = e$$

is just a nonzero solution of the inverse problem (8.3.30). However, under the restrictions imposed above the inverse problem concerned possesses **Fredholm's character**.

Theorem 8.3.2 *Let the operator A be positive,*

$$\Phi \in C^1\big([0,T];\,\mathcal{L}(X)\big)\,, \qquad \Phi(0)^{-1} \in \mathcal{L}(X)$$

and conditions (8.3.6) *and* (8.3.21) *hold. If the semigroup $V(t)$ generated by the operator $-A^{1/2}$ is compact and the element*

$$h = -\Phi(0)^{-1}\, A^{1/2}\, \delta(T)^{-1}\, \Delta(T)\, g\,,$$

where g is defined by (8.3.16), *then the following assertions are valid:*

(1) *for the inverse problem* (8.3.1)–(8.3.2) *to be solvable for any admissible input data it is necessary and sufficient that this problem with zero input data has a trivial solution only;*

(2) *the set of all solutions to problem* (8.3.1)–(8.3.2) *with zero input data forms in the space* $C^2([0,\, T];\, X\,) \times X$ *a finite-dimensional subspace;*

(3) *there exist functionals* $l_1, l_2, \ldots, l_n \in X^*$ *such that the inverse problem* (8.3.1)–(8.3.2) *is solvable if and only if* $l_i(h) = 0,\ 1 \le l \le n$.

Proof Within the framework of Theorem 8.3.2 Lemma 8.3.1 is certainly true and, therefore, the inverse problem of interest reduces to equation (8.3.23). Recall that the semigroup $V(t)$ is compact. Due to this property $V(t)$ is continuous for any $t > 0$ in the operator topology of the space $\mathcal{L}(X)$. Since the set of all compact operators with respect to this topology forms a closed two-sided ideal, the operator

$$B' = \int\limits_0^T V(2T - s)\, \Phi'(s)\; ds = \int\limits_T^{2T} V(s)\, \Phi'(2T - s)\; ds$$

is compact as a limit of the corresponding Riemann sums in the space $\mathcal{L}(X)$. Then so is, for any $\varepsilon > 0$, the operator

$$\int\limits_\varepsilon^T V(s)\, \Phi'(s)\; ds\,. \tag{8.3.31}$$

The function $f(s) = V(s)\, \Phi'(s)$ is bounded in norm on the segment $[0,\, T]$ and, therefore, as $\varepsilon \to 0$, the integral in (8.3.31) converges in the operator topology of the space $\mathcal{L}(X)$. Because of this fact, the operator

$$B'' = \int\limits_0^T V(s)\, \Phi'(s)\; ds$$

is compact. Since the operators B', B'' and $V(T)$ are compact and the operators $\beta\, A^{1/2}\, \delta(T)^{-1}$, $\delta(T)^{-1}$ and $\Phi(0)^{-1}$ are continuous, one can specify by formula (8.3.24) a compact operator and carry out subsequent studies of equation (8.3.23) on the basis of Fredholm's theory.

In preparation for this, we are going to show that the element g_1 defined by (8.3.25) runs over the entire space X for the varying input data. Indeed, as stated before, the element g defined by (8.3.16) runs over the entire manifold $\mathcal{D}(A^{1/2})$. For $\beta \neq 0$ the operator $\Delta(T)$ executes an isomorphism of $\mathcal{D}(A^{1/2})$ onto the space X, while $A^{1/2}\,\delta(T)^{-1}$ and $\Phi(0)^{-1}$ fall into the category of isomorphisms of the space X. It follows from the foregoing that for $\beta \neq 0$ the element g_1 may be arbitrarily chosen in the space X. For $\beta = 0$ both operators $\delta(T)^{-1}$ and $\Delta(T)$ are isomorphisms of the space X, so that we might attempt the element g_1 in the form

$$g_1 = -\Phi(0)^{-1}\,\delta(T)^{-1}\,\Delta(T)\,A^{1/2}\,g,$$

which serves to motivate that the element $A^{1/2}\,g$ may run over the entire space X, because so does the element g_1. Thus, the solvability of the inverse problem (8.3.1)–(8.3.2) with any admissible input data is equivalent to the question whether equation (8.3.23) is solvable for any $g_1 \in X$ and the first desired assertion follows immediately from Fredholm's alternative.

Under the zero input data functions the element $g_1 = 0$ and the set of all solutions to equation (8.2.3) coincides with the characteristic subspace of the operator B_1 for the unit eigenvalue. Since the operator B_1 is compact, this characteristic subspace will be finite-dimensional. A simple observation may be of help in achieving the final aim in item (2). From formula (8.3.11) it follows that the function u and the element p are related by

$$u(t) = \int_0^T G(t,s)\,\Phi(s)\,p\,ds\,.$$

In conformity with Fredholm's alternative equation (8.2.23) is solvable if and only if $l_i(g_1) = 0$, $1 \leq i \leq n$, where $\{\,l_i\}$, $1 \leq i \leq n$, constitute a basis of the finite-dimensional space formed by all solutions to the adjoint homogeneous equation $l - B_1^*\,l = 0$. It remains only to note that the element h coincides with g_1. All this enables us to deduce the statement of item (3), thereby completing the proof of the theorem. ■

We confine ourselves to the case when the operator A is self-adjoint in a Hilbert space X. If so, the operator A is positive if and only if it is positive definite. When treating $\Phi(t)$ involved in problem (8.3.1)–(8.3.2) as a numerical function and identifying the value $\Phi(t)$ with the operator of multiplication by the number $\Phi(t)$ in the space X, we are now in a position to define the function $\varphi(\lambda)$ on the positive semi-axis by

$$\varphi(\lambda) = -\int_0^T \frac{\beta\sqrt{\lambda}\,\mathrm{ch}\,\sqrt{\lambda}\,(T-s) + \alpha\,\mathrm{sh}\,\sqrt{\lambda}\,(T-s)}{\beta\sqrt{\lambda}\,\mathrm{ch}\,T\sqrt{\lambda} + \alpha\,\mathrm{sh}\,T\sqrt{\lambda}}\,\Phi(s)\,ds \tag{8.3.32}$$

under the natural premises

$$\alpha \geq 0\,, \qquad \beta \geq 0\,, \qquad \alpha + \beta > 0\,. \tag{8.3.33}$$

It is clear that the function φ is analytical on the positive semi-axis and all of its zeroes are isolated in the case when $\Phi(t) \not\equiv 0$.

Theorem 8.3.3 *If the operator A is self-adjoint and positive definite in a Hilbert space X, the function Φ with values in the space $\mathbf{R}$ is of Hölder's type on the segment $[0, T]$, $\Phi(t) \not\equiv 0$ and $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$, then the following assertions are valid:*

(1) *a solution of the inverse problem* (8.3.1)–(8.3.2) *exists if and only if*

$$\int\limits_0^{+\infty} |\varphi(\lambda)|^{-2}\, d(\,E_\lambda g,\, g\,) \;<\; \infty\,, \tag{8.3.34}$$

where E_λ stands for the resolution of unity for the operator A, the function φ is defined by (8.3.32) *and the element g is given by formula* (8.3.16)*;*

(2) *if a solution of the inverse problem* (8.3.1)–(8.3.2) *exists, then for this solution to be unique it is necessary and sufficient that the point spectrum of the operator A contains no zeroes of the function $\varphi(\lambda)$.*

Proof Since the operator A is positive definite, there exists $\varepsilon > 0$ such that

$$A = \int\limits_{\varepsilon}^{+\infty} \lambda\, dE_\lambda$$

and on the basis of the theory of operators

$$V(t) \;=\; \int\limits_{\varepsilon}^{+\infty} \exp\left(-\sqrt{\lambda}\, t\,\right)\, dE_\lambda\,, \tag{8.3.35}$$

$$A^{1/2} = \int\limits_{\varepsilon}^{+\infty} \sqrt{\lambda}\; dE_\lambda\,. \tag{8.3.36}$$

With the aid of relations (8.3.35)–(8.3.36) we derive by the multiplicativity of the mapping

$$f(\lambda) \mapsto f(A)$$

the formula

$$\Delta(T) = \int_{\varepsilon}^{+\infty} d(\lambda)\, dE_\lambda\,, \tag{8.3.37}$$

where

$$d(\lambda) = \alpha \left(1 - \exp\left(-2T\sqrt{\lambda}\right)\right) + \beta\left(1 + \exp(-2T\sqrt{\lambda})\right)\sqrt{\lambda}\,. \tag{8.3.38}$$

Because of (8.3.33), the function $d(\lambda)$ is positive on the positive semi-axis. This provides support for the view that the operator $\Delta(T)$ is invertible. For $\beta = 0$ the function $d(\lambda)$ has, as $\lambda \to +\infty$, a non-zero limit equal to the number α. If $\beta \neq 0$, then the function $d(\lambda)$ is equivalent to $\beta\sqrt{\lambda}$ as $\lambda \to +\infty$. This serves to motivate that the function $d(\lambda)^{-1}$ is bounded on $[\varepsilon, +\infty)$ and, therefore, inclusion (8.3.6) occurs, since

$$\Delta(T)^{-1} = \int_{\varepsilon}^{+\infty} d(\lambda)^{-1}\, dE_\lambda\,. \tag{8.3.39}$$

From the theory of self-adjoint operators it follows that

$$\beta I - \alpha A^{-1/2} = \int_{\varepsilon}^{+\infty} \left(\beta - \frac{\alpha}{\sqrt{\lambda}}\right) dE_\lambda\,, \tag{8.3.40}$$

$$\beta I + \alpha A^{-1/2} = \int_{\varepsilon}^{+\infty} \left(\beta + \frac{\alpha}{\sqrt{\lambda}}\right) dE_\lambda\,. \tag{8.3.41}$$

By the multiplicativity of the mapping

$$f(\lambda) \mapsto f(A)$$

in combination with relations (8.3.29), (8.3.36)–(8.3.41) we deduce that the operator $G_t(0, s)$ can be put in correspondence with the function

$$\begin{aligned} g(\lambda, s) = -\sqrt{\lambda}\; d(\lambda)^{-1} \Bigg[&\left(\beta - \frac{\alpha}{\sqrt{\lambda}}\right) \\ &\times \exp\left(-\sqrt{\lambda}\,(2T - s)\right) \\ &+ \left(\beta + \frac{\alpha}{\sqrt{\lambda}}\right) \exp\left(-\sqrt{\lambda}\, s\right) \Bigg]\,. \end{aligned} \tag{8.3.42}$$

Thus, formula (8.3.25) gives the representation

$$B = \int_0^T \int_\varepsilon^{+\infty} g(\lambda, s)\, \Phi(s)\; dE_\lambda\; ds\,.$$

Observe that the function $g(\lambda, s)$ is continuous and bounded on $[\varepsilon, +\infty) \times [0, T]$. By the Fubini theorem we thus have

$$B = \int_\varepsilon^{+\infty} \varphi_1(\lambda)\; dE_\lambda\,, \tag{8.3.43}$$

where

$$\varphi_1(\lambda) = \int_0^T g(\lambda, s)\, \Phi(s)\; ds\,.$$

Upon substituting (8.3.38) into (8.3.42) we find by minor manipulations that

$$g(\lambda, s) = -\frac{\beta\sqrt{\lambda}\,\mathrm{ch}\sqrt{\lambda}\,(T-s) + \alpha\,\mathrm{sh}\sqrt{\lambda}\,(T-s)}{\beta\sqrt{\lambda}\,\mathrm{ch}\sqrt{\lambda}\,T + \alpha\,\mathrm{sh}\sqrt{\lambda}\,T}\,, \tag{8.3.44}$$

thereby justifying the equality

$$\varphi_1(\lambda) = \varphi(\lambda)\,,$$

where the function φ is defined by relation (8.3.32). Hence the inverse problem in hand reduces to equation (8.3.14) taking for now the form

$$\varphi(A)\; p = g\,. \tag{8.3.45}$$

Due to Lemma 8.2.1 of Section 8.2 with regard to equation (8.3.45) we finish the proof of the theorem. ■

The next stage of our study is concerned with discussions of items (1)–(2) of Theorem 8.3.3. First of all it should be noted that, in view of (8.3.44), the function $g(\lambda, s)$ is nonnegative for any $\lambda > 0$ and all values $s \in [0, T]$. This function may equal zero only if $\beta = 0$ and $s = T$ for $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$. If now the function Φ is continuous, nonnegative and does not equal zero identically, then $\varphi(\lambda) < 0$ for any $\lambda > 0$. Since the spectrum of a positive definite operator is located on the semi-axis $\lambda > 0$, the operator A meets the requirements of item (2) of Theorem 8.3.3.

For any $\Phi \in C^1[0, T]$ we establish by integrating by parts in (8.3.32) the relation

(8.3.46)

$$\varphi(\lambda) = \frac{\frac{\alpha}{\sqrt{\lambda}}\, \Phi(T) - \left(\beta \operatorname{sh} \sqrt{\lambda}\, T + \frac{\alpha}{\sqrt{\lambda}} \operatorname{ch} \sqrt{\lambda}\, T \right) \Phi(0)}{\beta \sqrt{\lambda} \operatorname{ch} (T \sqrt{\lambda}) + \alpha \operatorname{sh} (T \sqrt{\lambda})}$$

$$- \int_0^T \Phi'(s)\, \frac{\beta \operatorname{sh} \sqrt{\lambda}\, (T-s) + \frac{\alpha}{\lambda} \operatorname{ch} \sqrt{\lambda}\, (T-s)}{\beta \sqrt{\lambda} \operatorname{ch} T \sqrt{\lambda} + \alpha \operatorname{sh} T \sqrt{\lambda}}\, ds\,,$$

which immediately implies as $\lambda \to +\infty$ that

$$\varphi(\lambda) = \begin{cases} -\dfrac{\Phi(0)}{\sqrt{\lambda}} \operatorname{th} T \sqrt{\lambda} + o\Big(\dfrac{1}{\sqrt{\lambda}} \Big), & \beta \neq 0, \\ -\dfrac{\Phi(0)}{\sqrt{\lambda}} \operatorname{cth} T \sqrt{\lambda} + o\Big(\dfrac{1}{\sqrt{\lambda}} \Big), & \beta = 0. \end{cases}$$

Thus, as $\lambda \to +\infty$, in either of the cases $\beta = 0$ and $\beta \neq 0$ we find that

$$\varphi(\lambda) = -\frac{\Phi(0)}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}} \Big). \tag{8.3.47}$$

If $\Phi(0) \neq 0$, then estimate (8.3.47) ensures the existence of positive constants c and λ_0 such that for all numbers $\lambda \geq \lambda_0$

$$|\varphi(\lambda)| \geq \frac{c}{\sqrt{\lambda}}\,, \tag{8.3.48}$$

yielding

$$\int_{\lambda_0}^{+\infty} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g) \leq c^{-2} \int_{\lambda_0}^{+\infty} \lambda\, d(E_\lambda g, g) \tag{8.3.49}$$

$$\leq c^{-2} \int_0^{+\infty} \lambda\, d(E_\lambda g, g)\,.$$

Observe that the integral on the right-hand side of (8.3.49) is finite if and only if $g \in \mathcal{D}(A^{1/2})$ and, therefore, for all admissible input data of the inverse problem (8.3.1)–(8.3.2) the estimate is valid:

$$\int_{\lambda_0}^{+\infty} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g) < \infty\,.$$

If the function φ does not vanish on the spectrum of the operator A, then the integral

$$\int_{\varepsilon}^{\lambda_0} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g)$$

is finite, because φ is continuous. Thus, we arrive at the following assertion.

Theorem 8.3.3 *If the operator A is self-adjoint and positive definite in a Hilbert space X, the function $\Phi \in C^1[0, T]$ is nonnegative, $\Phi(0) > 0$ and the numbers $\alpha \ge 0$ and $\beta \ge 0$ are such that $\alpha + \beta > 0$, then a solution u, p of the inverse problem (8.3.1)-(8.3.2) exists and is unique for any admissible input data.*

One assumes, in addition, that the operator A has a discrete spectrum. In trying to derive an explicit formula for the element p by the **Fourier method** of separation of variables we rely on an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ composed by the eigenvectors of the operator A and $A e_k = \lambda_k e_k$. With the aid of the expansions

$$g = \sum_{k=1}^{\infty} g_k e_k\,, \qquad p = \sum_{k=1}^{\infty} p_k e_k$$

we might split up equation (8.3.14) into the infinite system of equations

$$\varphi(\lambda_k)\, p_k = g_k\,, \qquad k = 1, 2, \ldots. \tag{8.3.50}$$

To decide for yourself whether a solution is unique, a first step is to check that all the values $\varphi(\lambda_k) \ne 0$. When this is the case, (8.3.50) implies that

$$p_k = \frac{g_k}{\varphi(\lambda_k)}\,,$$

so that

$$\int_0^{+\infty} |\varphi(\lambda)|^{-2}\, d(E_\lambda g, g) = \sum_{k=1}^{\infty} |\varphi(\lambda_k)|^{-2}\, |g_k|^2 \tag{8.3.51}$$

and the sum of the series on the right-hand side of (8.3.51) equals the element p norm squared. The convergence of this series is equivalent to the solvability of equation (8.3.14) and, in view of this, the element p is representable by

$$p = \sum_{k=1}^{\infty} \varphi(\lambda_k)^{-1}\, (g, e_k)\, e_k\,. \tag{8.3.52}$$

The resulting expression may be of help in approximating the element p.

Chapter 9

Applications of the Theory of Abstract Inverse Problems to Partial Differential Equations

9.1 Symmetric hyperbolic systems

We now study the **Cauchy problem**

$$(9.1.1) \qquad a_0(x,t)\,\frac{\partial u}{\partial t} + \sum_{i=1}^{n} a_i(x,t)\,\frac{\partial u}{\partial x_i} = f(x,t,u)\,,$$

$$x \in \mathbf{R}^n, \qquad 0 \le t \le T\,,$$

$$(9.1.2) \qquad u(x,0) = u_0(x)\,, \qquad x \in \mathbf{R}^n,$$

where the functions u, f and u_0 take the values in a Hilbert space H and the coefficients $a_i(x,t)$, $0 \le i \le n$, take the values in the space $\mathcal{L}(H)$.

Assume that the operators $a_i(x,t)$, $0 \le i \le n$, are symmetric and the operator $a_0(x,t)$ is positive definite uniformly in the variables x, t. Of great importance is the case when the space H is finite-dimensional. If this happens, equation (9.1.1) admits the form of the **symmetric t-hyperbolic**

system of first order partial differential equations for the components of the vector-function u. Quite often, in the development of advanced theory that does not matter that the space H is finite-dimensional, while in notations operator style is much more convenient than matrix one.

Common practice involves the following constraints for a given positive integer $s > \frac{n}{2} + 1$:

$$(1) \qquad a_i \in C\big([0, T]; C_b^s\big(\mathbf{R}^n; \mathcal{L}(H)\big)\big), \ 0 \le i \le n,$$

$$(2) \qquad \| a_0(x, t_1) - a_0(x, t_2) \|_{C_b(\mathbf{R}^n; \mathcal{L}(H))} \le L \, | t_1 - t_2 | , \ 0 \le t_1, t_2 \le T.$$

Here the symbol $C_b(\Omega, X)$ designates the set of all continuous bounded on Ω functions with values in the space X; the norm on that space is defined by

$$\| f \|_{C_b(\Omega, X)} = \sup_{x \in \Omega} \| f(x) \|_X .$$

By definition, the space $C_b^s(\Omega, X)$ contains all the functions defined on Ω with values in the space X, all the derivatives of which up to the order s belong to the space $C_b(\Omega, X)$. In turn, the norm on that space is defined by

$$\| f \|_{C_b^s(\Omega, X)} = \sum_{|\alpha| \le s} \| D^\alpha f \|_{C_b(\Omega, X)} .$$

In what follows $X = L_2\big(\mathbf{R}^n; H\big)$ is adopted as a basic space and the Cauchy problem (9.1.1)–(9.1.2) is completely posed in the abstract form as follows:

$$\begin{cases} u'(t) = A(t)\, u(t) + F\big(t, u(t)\big), & 0 \le t \le T, \\ u(0) = u_0 , \end{cases}$$

where

$$(9.1.3) \qquad A(t) = \sum_{i=1}^{n} a_0^{-1}(x, t)\, a_i(x, t) \frac{\partial}{\partial x_i} ,$$

$$F(t, u) = a_0^{-1}(x, t)\, f(x, t, u) .$$

The domain of the operator $A(t)$ corresponds to the **maximal operator** in the space X generated by the differential expression on the right-hand side of (9.1.3). It should be noted that in the spaces

$$X_0 = W_2^{s-1}\big(\mathbf{R}^n; H\big), \qquad X_1 = W_2^s\big(\mathbf{R}^n; H\big)$$

the validity of conditions (S1)–(S2) from Section 5.4 is ensured by assumptions (1)–(2) of the present section (for more detail see Kato (1970, 1973, 1975a), Massey (1972)).

Here and below, in the space $H = \mathbf{R}^m$ with the usual inner product the operators $a_i(x,t)$, $0 \le i \le n$, will be identified with their matrices in a natural basis of the space H. One assumes, in addition, that the right-hand side of equation (9.1.1) is of the structure

$$f(x,t,u) = \big(p(t)\,a(x,t) + b(x,t)\big)\,u + g(x,t)\,, \tag{9.1.4}$$

where the matrix functions a and b of sizes $k \times m$ and $m \times m$, respectively, are known in advance and the unknown $m \times k$-matrix function p is sought. The vector function g with values in the space H is also given. The problem statement of finding the coefficient $p(t)$ necessitates involving solutions to equation (9.1.1) subject to the initial conditions

$$u_j(x,0) = u_{0j}(x)\,, \qquad 1 \le j \le k\,, \qquad x \in \mathbf{R}^n, \tag{9.1.5}$$

and the **overdetermination conditions**

$$u_j(x_j,t) = \psi_j(t)\,, \qquad 1 \le j \le k\,, \qquad 0 \le t \le T\,, \tag{9.1.6}$$

where the points $x_j \in \mathbf{R}^n$, $1 \le j \le k$, are kept fixed.

For the reader's convenience we denote by $U(x,t)$ the $m \times k$-matrix, whose columns are formed by the components of the vectors $u_1(x,t), \ldots,$ $u_k(x,t)$, by $G(x,t)$ – the $m \times k$-matrix, whose columns are formed by the components of the same vector $g(x,t)$, by $U_0(x)$ – the $m \times k$-matrix, whose columns are composed by the components of the vectors $u_{01}(x)$, $\ldots$, $u_{0k}(x)$ and by $\Psi(t)$ – the $m \times k$-matrix, whose columns are formed by the components of the vectors $\psi_1(t)$, $\ldots$, $\psi_k(t)$. Within these notations, one can replace exactly k Cauchy problems associated with equation (9.1.1) by only one problem in the matrix form

$$a_0(x,t)\,\frac{\partial U}{\partial t} + \sum_{i=1}^{n} a_i(x,t)\,\frac{\partial U}{\partial x_i} \tag{9.1.7}$$

$$= \big(p(t)\,a(x,t) + b(x,t)\big)\,U + G(x,t)\,,$$

$$U(x,0) = U_0(x)\,. \tag{9.1.8}$$

In the sequel the symbol H_1 designates the space of all $m \times k$ matrices. The usual inner product in this matrix space is defined by

$$(a,\, b) = \operatorname{tr}(a\,b')\,,$$

where tr means taking a **matrix trace** equal to the sum of its diagonal elements. Obviously, any symmetric $m \times m$-matrix being viewed in the space H_1 as an operator of matrix multiplication from the left describes a symmetric operator in the same space. Moreover, the norm of this operator in the space H coincides with its norm in the space H_1. In these spaces the same is certainly true for the constants of positive definiteness. Because of this, upon substituting the system (9.1.1) and the space H in place of (9.1.7) and H_1, respectively, the governing system does remain symmetric and t-hyperbolic and is still subject to conditions (1)–(2) imposed above. In particular, operator (9.1.3) meets conditions (H1)–(H4) of Section 6.7 as well as conditions (H3.1)–(H4.1) of Section 6.8 with regard to the basic spaces

$$X = L_2(\mathbf{R}^n; H_1), \qquad X_0 = W_2^{s-1}(\mathbf{R}^n; H_1), \qquad X_1 = W_2^s(\mathbf{R}^n; H_1).$$

We begin by introducing an operator B acting from the space X_0 into the space $Y = H_1$. One way of proceeding is to hold a matrix $U \in X_0$ fixed and operate with the vector $u_j(x)$ formed by the j-column of this matrix, $1 \le j \le k$. By definition, the j-column of the matrix BU, $1 \le j \le k$, is equal to $u_j(x_j)$. Since $s-1 > n/2$, Sobolev's embedding theorem implies that the space X_0 is continuously embedded into the space $C_b(\mathbf{R}^n; H_1)$. Therefore, the operator B is well-defined and $B \in \mathcal{L}(X_0; Y)$. By means of the operator B condition (9.1.6) admits an alternative form

$$BU(x,t) = \psi(t), \qquad 0 \le t \le T. \tag{9.1.9}$$

Using the decomposition $F(t,U,p) = a_0^{-1}\big[(pa+b)U + G\big]$ behind we deduce that the inverse problem of recovering the matrix p can be recast in the abstract form as follows:

$$\begin{cases} U'(t) = A(t)U(t) + F\big(t, U(t), p(t)\big), & 0 \le t \le T, \\ U(0) = U_0, & \\ BU(t) = \psi(t), & 0 \le t \le T. \end{cases}$$

which coincides with the inverse problem (6.8.1)–(6.8.3) of Section 6.8. Also, the operator B satisfies condition (6.8.4). All this enables us to apply Theorem 6.8.1 of Section 6.8 in establishing the solvability of the inverse problem at hand. This can be done using the replacements

$$f_1(t,U) = a_0^{-1}(bU + G), \qquad f_2(t,U,p) = a_0^{-1}paU$$

in the counterpart of decomposition (6.8.5):

$$F(t,U,p) = f_1(t,U) + f_2(t,U,p).$$

The function f_3 arising from condition (H6) of Section 6.7 is defined by means of the relation

$$f_3(t, y, p) = B\left(a_0^{-1}\, p\, B(a\, y)\right) .$$

In such a setting we are able to write down explicitly the mapping $p = \Phi(t, z)$ involved in condition (H7) of Section 6.7 as an inverse of the mapping $z = f_3(t, \psi(t), p$ with respect to the variable p. Simple calculations show that

$$\Phi(t, z) = \left(\sum_{j=1}^{k} a_0(x_j, t)\, z\, P_j \right) \left(B\, (a\, \psi\,) \right)^{-1} ,$$

where P_j is a matrix of size $k \times k$, whose element in kth row and jth column is equal to 1 and the remainings ones are equal to zero. For the fulfilment of condition (H7) of Section 6.7 it suffices to require that there is a positive constant δ such that for all $t \in [0, T]$

$$|\det (B\, a)\, \psi\, | \geq \delta . \tag{9.1.10}$$

Denote by $M(k, m)$ the space of all $k \times m$-matrices. As can readily be observed, the validity of conditions (H8)-(H9) of Section 6.7 as they are understood in Theorem 6.8.1 of Section 6.8 is ensured by the inclusions

$$\begin{cases} a \in C([0, T];\, C_b^s(\mathbf{R}^n;\, M(k, m)))\, , \\ b \in C([0, T];\, C_b^s(\mathbf{R}^n;\, M(m, m)))\, , \\ g \in C([0, T];\, C_b^s(\mathbf{R}^n;\, H))\, . \end{cases} \tag{9.1.11}$$

Other conditions of Theorem 6.8.1 from Section 6.8 will be satisfied under the following constraints:

$$\begin{cases} \psi \in C^1([0, T];\, H_1)\, , \\ \psi(0) = B\, U_0\, , \\ U_0 \in W_2^s(\mathbf{R}^n;\, H_1)\, . \end{cases} \tag{9.1.12}$$

Corollary 9.1.1 *Under conditions* (1)-(2) *and* (9.1.10)-(9.1.12) *the inverse problem* (9.1.7)-(9.1.9) *is uniquely solvable for all sufficiently small values T_1 in the class of functions*

$$u \in C^1([0, T_1];\, L_2(\mathbf{R}^n;\, H_1)) \bigcap C([0, T_1];\, W_2^s(\mathbf{R}^n;\, H_1))' ,$$

$$p \in C([0, T_1];\, H_1)\, .$$

Let us stress that the above result guarantees the local existence and no more. In the general case the global existence fails to be true. The next inverse problem is aimed at confirming this statement:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + p(t)\, u\,, \qquad x \in \mathbf{R}\,, \qquad 0 \le t \le T\,,$$

$$u(x,0) = u_0(x)\,, \qquad x \in \mathbf{R}\,,$$

$$u(0,t) = \psi(t)\,, \qquad 0 \le t \le T\,.$$

Via the representation

$$u(x,t) = v(x,t) \exp\Big(\int_0^t p(\xi)\, d\xi\Big)$$

it is not difficult to derive that its solution is given by the formulae

$$u(x,t) = u_0(x+t)\, \psi(t) \,/\, u_0(t)\,,$$

$$p(t) = \Big(u_0(t)\, \psi'(t) - u_0'(t)\, \psi(t)\Big) \,/\, (\, \psi(t)\, u_0(t))\,.$$

All of the solvability conditions will be satisfied if the input data functions are sufficiently smooth, the function u_0 is rapidly decreasing at infinity, $u_0(0) = \psi(0)$ and the function ψ has no zeroes on the segment $[0,\ T]$. In particular, the functions

$$u_0(x) = \exp(-x^2) \cos(\alpha\, x)\,, \qquad \varphi(t) = 1$$

suit us perfectly and appear in later discussions. In this case

$$p(t) = 2t + \alpha \operatorname{tg}(\alpha\, t)$$

and a solution of the inverse problem cannot be extended continuously across the point $t = \pi/(2\,\alpha)$. We always may choose the value α in such a way that the interval of the existence of a solution will be as small as we like.

In further development we may attempt the right-hand side of equation (9.1.1) in the form

$$f(x,t,u) = a(x,t)\, u + b(x,t)\, p(t) + g(x,t)\,, \tag{9.1.13}$$

where, for all fixed variables $\mathbf{x} \in \mathbf{R}^n$, $t \in [0,\ T]$, the matrices $a(x,t)$ and $b(x,t)$ of size $m \times m$ (linear operators in the space H) and the function

$g(x,t) \in H$ are known in advance. Additional information is available in the form of **pointwise overdetermination**

$$u(x_0,t) = \psi(t), \qquad 0 \le t \le T, \tag{9.1.14}$$

where x_0 is a fixed point in the space $\mathbf{R}^n$. By means of the operator

$$B\colon \bar{u} \mapsto u(x_0)$$

relation (9.1.14) can be transmitted in (6.8.3) of Section 6.8. The system (9.1.1) reduces to (6.8.14) of Section 6.8 as the outcome of manipulations with the right-hand side of equation (6.8.1) during which the decompositions take place:

$$L_1(t)\,u = a_0^{-1}(x,t)\,a(x,t)\,u,$$

$$L_2(t)\,p = a_0^{-1}(x,t)\,b(x,t)\,p,$$

$$F(t) = a_0^{-1}(x,t)\,g(x,t).$$

As stated above, conditions (1)–(2) assure us of the validity of assumptions (S1)–(S2) of Section 5.4 with regard to the spaces

$$X = L_2(\mathbf{R}^n;\, H), \qquad X_0 = W_2^{s-1}(\mathbf{R}^n;\, H), \qquad X_1 = W_2^s(\mathbf{R}^n;\, H).$$

Due to Sobolev's embedding theorems the operator B satisfies condition (6.8.4) with the space $Y = H$ involved. To prove the solvability, we make use of Theorem 6.8.2 of Section 6.8 with the following members:

$$\begin{cases} a \in C([0,T];\, C_b^s(\mathbf{R}^n;\, \mathcal{L}(H))), \\ b \in C([0,T];\, W_2^s(\mathbf{R}^n;\, \mathcal{L}(H))), \\ g \in C([0,T];\, W_2^s(\mathbf{R}^n;\, H)), \end{cases} \tag{9.1.15}$$

$$|\det b(x_0,\, t)| \ge \delta > 0, \qquad 0 \le t \le T, \tag{9.1.16}$$

$$\begin{cases} u_0 \in W_2^s(\mathbf{R}^n;\, H), \\ \psi \in C^1([0,\, T];\, H), \\ u_0(x_0) = \psi(0). \end{cases} \tag{9.1.17}$$

Corollary 9.1.2 *If conditions* (1)–(2) *hold together with conditions* (9.1.15)–(9.1.17), *then a solution u, p of the inverse problem* (9.1.1)–(9.1.2), (9.1.13)–(9.1.14) *exists and is unique in the class of functions*

$$u \in C^1([0, T_1]; L_2(\mathbf{R}^n; H)) \bigcap C([0, T_1]; W_2^s(\mathbf{R}^n; H)),$$

$$p \in C([0, T]; H).$$

Granted decomposition (9.1.13), the subsidiary information is provided in the form of **integral overdetermination**

$$\int\limits_{\mathbf{R}^n} u(x,t)\,\omega(x)\,dx = \psi(t), \qquad 0 \le t \le T, \tag{9.1.18}$$

where ω is a continuously differentiable and finite numerical function. Relation (9.1.18) admits the form (6.7.3) since the introduction of the operator B acting in accordance with the rule

$$B\colon\ u(x) \mapsto \int\limits_{\mathbf{R}^n} u(x)\,\omega(x)\,dx.$$

The operator so defined is bounded from the space $X = L_2(\mathbf{R}^n; H)$ into the space Y and smoothing in the sense of condition (6.7.7), which can immediately be confirmed by appeal to the well-known **Ostrogradsky formula**. Indeed, for any $u \in \mathcal{D}(A(t))$ we thus have

$$\begin{aligned} B\,A(t)\,u &= \int\limits_{\mathbf{R}^n} \Big(\sum_{i=1}^n a_0^{-1}(x,t) \\ &\qquad \times a_i(x,t)\,\frac{\partial u}{\partial x_i} \Big)\,\omega(x)\,dx \\ &= -\sum_{i=1}^n \int\limits_{\mathbf{R}^n} \Big[\frac{\partial}{\partial x_i}\, \big(a_0^{-1}(x,t) \\ &\qquad \times a_i(x,t)\,\omega(x)\big) \Big]\,u(x)\,dx \end{aligned} \tag{9.1.19}$$

and the latter formula on the right-hand side of (9.1.19) reflects the operator extension made up to a bounded operator from the space X into the space

Y. From (9.1.19) it follows that relations (6.7.7) hold true if condition (1) is fulfilled for $s = 1$.

To prove the solvability of the inverse problem in a weak sense, we rely on Theorem 6.7.2 of Section 6.7. Because the operator B is of smoothing character, we get rid of the condition $s > n/2 + 1$ and no embedding theorem is needed here. On the same grounds as before, we set

$$X = L_2(\mathbf{R}^n;\, H), \qquad Y = H.$$

For conditions (H1)–(H4) of Section 2.7 to be satisfied it suffices to require the fulfilment of condition (1) for $s = 1$ and condition (2) of the present section (see Kato (1970, 1973, 1975a)). The remaining assumptions of Theorem 6.7.2 are due to the set of restrictions

$$\text{(9.1.20)} \qquad \begin{cases} a \in C([0, T];\, C_b(\mathbf{R}^n;\, \mathcal{L}(H))), \\ b \in C([0, T];\, L_2(\mathbf{R}^n;\, \mathcal{L}(X))), \\ g \in C([0, T];\, L_2(\mathbf{R}^n;\, H)), \end{cases}$$

$$\text{(9.1.21)} \qquad \begin{cases} u_0 \in L_2(\mathbf{R}^n;\, H), \\ \psi \in C^1([0, T];\, H), \\ \int\limits_{\mathbf{R}^n} u_0(x)\, \omega(x)\, dx = \psi(0), \end{cases}$$

$$\text{(9.1.22)} \qquad \left| \det \int\limits_{\mathbf{R}^n} b(x, t)\, \omega(x)\, dx \right| \geq \delta > 0, \qquad 0 \leq t \leq T.$$

Corollary 9.1.3 *If condition* (1) *holds for* $s = 1$ *and conditions* (2), (9.1.20)–(9.1.22) *are fulfilled, then a solution* u, p *of the inverse problem* (9.1.1)–(9.1.2), (9.1.13), (9.1.18) *exists and is unique in the class of functions*

$$u \in C([0, T];\, L_2(\mathbf{R}^n;\, H)), \qquad p \in C([0, T];\, H).$$

Proof Before we undertake the proof, let us recall for justifying the solvability in a strong sense the contents of Theorem 6.7.4 from Section 6.7 with regard to the basic spaces

$$X = L_2(\mathbf{R}^n;\, H), \qquad X_0 = W_2^1(\mathbf{R}^n;\, H), \qquad Y = H.$$

Assumptions (H1)–(H4) imposed in Section 6.7 hold true under condition (2) in combination with condition (1) for $s = 2$ (for more detail see Kato (1970, 1973, 1975a)). The rest of Theorem 6.7.4 from Section 6.7 immediately follows from the set of constraints

$$(9.1.23)\qquad \begin{cases} a \in C([0, T]; C_b^1(\mathbf{R}^n; \mathcal{L}(H))), \\ b \in C([0, T]; W_2^1(\mathbf{R}^n; \mathcal{L}(H))), \\ g \in \dot{C}([0, T]; W_2^1(\mathbf{R}^n; H)), \end{cases}$$

$$(9.1.24)\qquad \begin{cases} u_0 \in W_2^1(\mathbf{R}^n; H), \\ \psi \in C^1([0, T]; H), \\ \int\limits_{\mathbf{R}^n} u_0(x)\, \omega(x)\, dx = \psi(0), \end{cases}$$

$$(9.1.25)\qquad \left| \det \int\limits_{\mathbf{R}^n} b(x, t)\, \omega(x)\, dx \right| \geq \delta > 0, \qquad 0 \leq t \leq T.$$

Corollary 9.1.4 *If condition* (1) *holds with* $s = 2$ *and conditions* (2), (9.1.23)–(9.1.25) *are fulfilled, then a solution* u, p *of the inverse problem* (9.1.1)–(9.1.2), (9.1.13), (9.1.18) *exists and is unique in the class of functions*

$$u \in C^1([0, T]; L_2(\mathbf{R}^n; H)), \qquad p \in C([0, T]; H).$$

9.2 Second order equations of hyperbolic type

Let us consider in the space $\mathbf{R}^n$ a bounded domain Ω, whose boundary is smooth enough. First of all we set up in the domain $D = \Omega \times [0, T]$ the **initial boundary value (direct) problem** for the second order **hyperbolic**

equation which will be involved in later discussions:

$$\begin{aligned}(9.2.1)\qquad &\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} h_i(x,t)\,\frac{\partial^2 u}{\partial x_i\,\partial t} + h(x,t)\,\frac{\partial u}{\partial t}\\ &- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\,\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x,t)\,\frac{\partial u}{\partial x_i}\\ &+ c(x,t)\,u = f\left(x,t,u,\frac{\partial u}{\partial t}\right),\\ &x\in\Omega, \qquad t\in[0,T],\end{aligned}$$

$$(9.2.2)\qquad u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x), \qquad x\in\Omega,$$

$$(9.2.3)\qquad u(x,t) = 0, \qquad x\in\partial\Omega, \qquad t\in[0,T].$$

The problem statement necessitates imposing rather mild restrictions on the coefficients of equation (9.2.1):

$$(9.2.4)\qquad h_i,\ h,\ a_{ij},\ b_i,\ c \in C^2(\bar\Omega\times[0,T]),$$

$$(9.2.5)\qquad \sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\,\xi_j \ge \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \alpha>0,$$

$$(9.2.6)\qquad a_{ij}(x,t) = a_{ji}(x,t).$$

For the purposes of the present section we have occasion to use two differential operators $A_1(t)$ and $A_2(t)$, $0\le t\le T$, acting from the space $\overset{\circ}{W}{}_2^1(\Omega)$ into the space $L_2(\Omega)$ and from the space $W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega)$ into the space $L_2(\Omega)$, respectively, with the values

$$A_1(t)\,u = \sum_{i=1}^{n} h_i(x,t)\,\frac{\partial u}{\partial x_i} + h(x,t)\,u,$$

$$\begin{aligned}A_2(t)\,u = &- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\,\frac{\partial u}{\partial x_j}\right)\\ &+ \sum_{i=1}^{n} b_i(x,t)\,\frac{\partial u}{\partial x_i} + c(x,t)\,u.\end{aligned}$$

By merely setting $v = u_t$ one can reduce equation (9.2.1) to the following **system of the first order**:

$$\begin{cases} u_t = v\,, \\ v_t + A_1\, v + A_2\, u = f\,. \end{cases}$$

In giving it as a single equation of the first order in the Banach space it will be sensible to introduce the matrices

$$U(t) = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

and the operator

$$A(t) = \begin{pmatrix} 0 & I \\ -A_2(t) & -A_1(t) \end{pmatrix}$$

acting in the space $X = \overset{\circ}{W}{}_2^1(\Omega) \times L_2(\Omega)$ and possessing the domain

$$\mathcal{D}(A(t)) = \left(W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega) \right) \times \overset{\circ}{W}{}_2^1(\Omega)\,.$$

Bacause of its form, the direct problem (9.2.1)–(9.2.3) is treated as the **abstract Cauchy problem**

(9.2.7) $$U'(t) = A(t)\, U(t) + F(t, U(t))\,, \qquad 0 \le t \le T\,,$$

(9.2.8) $$U(0) = U_0\,.$$

In the sequel we shall need yet a family of equivalent norms. For any element

$$U(t) = \begin{pmatrix} u \\ v \end{pmatrix}$$

the associated norm will be taken to be

(9.2.9) $$\| U(t) \|_t^2 = \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\, \frac{\partial u}{\partial x_i}\,,\, \frac{\partial u}{\partial x_j} \right)_{L_2(\Omega)} + (u,\, u)_{L_2(\Omega)} + (v,\, v)_{L_2(\Omega)}\,.$$

When conditions (9.2.4)–(9.2.6) imposed above are fulfilled, the direct problem (9.2.7)–(9.2.8) and the family of norms (9.2.9) are covered by the framework of Section 6.10 (see Ikawa (1968)).

In order to set up an inverse problem, the function on the right-hand side of (9.2.1) should be representable by .

$$f(x,t,u,v) = f_1(x,t)\,p(t) + f_2(x,t)\,, \tag{9.2.10}$$

where the unknown coefficient p is sought. Additional information is provided by the condition of **integral overdetermination**

$$\int\limits_{\Omega} u(x,t)\,w(x)\,dx = \psi(t)\,, \qquad 0 \le t \le T\,. \tag{9.2.11}$$

The inverse problem of interest consists of finding a pair of the functions u, p from the system (9.2.1)–(9.2.3), (9.2.10)–(9.2.11).

By appeal to Theorem 6.10.2 of Section 6.10 we define the linear operator B being a functional and acting from the space X into the space $Y = \mathbf{R}$ in accordance with the rule

$$B\begin{pmatrix} u \\ v \end{pmatrix} = \int\limits_{\Omega} v(x)\,w(x)\,dx\,.$$

If the function $w \in L_2(\Omega)$, then $B \in \mathcal{L}(X, Y)$. Observe that for $w \in \overset{\circ}{W}{}_2^1(\Omega)$ the operator B possesses a **smoothing effect** that can be reflected with the aid of relation (6.10.4). Indeed, after integrating by parts it is plain to show that for any element

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A(t))$$

we get

$$\begin{aligned} B\,A(t)\,U &= -\int\limits_{\Omega} \big(\,A_1\,v + A_2\,u\,\big)\,w\,dx \\ &= -\int\limits_{\Omega} \big(\,A_1\,v\,\big)\,w\,dx - \int\limits_{\Omega} \big(\,A_2\,u\,\big)\,w\,dx \\ &= \int\limits_{\Omega} \Big(\,\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\,(\,h_i\,w\,) - h\,w\Big)\,v\,dx \\ &\quad + \int\limits_{\Omega} \Big(\,\sum_{i,j=1}^{n} a_{ij}\,\frac{\partial u}{\partial x_i}\,\frac{\partial w}{\partial x_j}\,\Big)\,dx \\ &\quad - \int\limits_{\Omega} \Big(\,\sum_{i=1}^{n} b_i\,\frac{\partial u}{\partial x_i} + c\,u\Big)\,w\,dx\,, \end{aligned} \tag{9.2.12}$$

which assures us of the validity of the estimate

$$\left| B\,A(t) \begin{pmatrix} u \\ v \end{pmatrix} \right| \le c \left(\| u \|_{W_2^1(\Omega)} + \| v \|_{L_2(\Omega)} \right), \tag{9.2.13}$$

thereby justifying the inclusion

$$\overline{B\,A(t)} \in \mathcal{L}(X,\, Y)\,,$$

which is valid for each $t \in [0,\, T]$. By the same token,

$$\overline{B\,A(t)} \in C\big([0,\, T];\, \mathcal{L}(X,\, Y)\big)\,.$$

We take for granted that

$$f_1,\, f_2 \in C\big([0,\, T];\, L_2(\Omega)\big)\,, \qquad w \in \overset{\circ}{W}{}_2^1(\Omega)\,, \tag{9.2.14}$$

$$u_0 \in \overset{\circ}{W}{}_2^1(\Omega)\,, \quad u_1 \in L_2(\Omega)\,, \quad \psi \in C^2[0,\, T]\,, \tag{9.2.15}$$

$$\int_\Omega u_0(x)\, w(x)\, dx = \psi(0)\,, \qquad \int_\Omega u_1(x)\, w(x)\, dx = \psi'(0)\,, \tag{9.2.16}$$

$$\int_\Omega f_2(x,t)\, w(x)\, dx \ne 0\,, \qquad 0 \le t \le T\,. \tag{9.2.17}$$

Under the conditions imposed above we may refer to Theorem 6.10.2 of Section 6.10. Indeed, by virtue of the first compatibility condition (9.2.16) relation (9.2.11) is equivalent to the following one:

$$B\,U(t) = \psi'(t)\,, \qquad 0 \le t \le T\,. \tag{9.2.18}$$

What is more, from relation (9.2.10) it follows that

$$F = L_1(t)\, U + L_2(t)\, p + F(t)\,, \tag{9.2.19}$$

where

$$L_1(t) \equiv 0\,, \qquad L_2(t)\, p = \begin{pmatrix} 0 \\ f_1(x,t)\, p \end{pmatrix}, \qquad F(t) = \begin{pmatrix} 0 \\ f_2(x,t) \end{pmatrix}.$$

On the strength of conditions (9.2.14)–(9.2.17) the inverse problem (9.2.7)–(9.2.8), (9.2.18)–(9.2.19) is in line with the premises of Theorem 6.10.2 from Section 6.10, whose use permits us to obtain the following result.

Corollary 9.2.1 *If all the conditions* (9.2.4)-(9.2.6) *and* (9.2.14)-(9.2.17) *hold, then a solution* u, p *of the inverse problem* (9.2.1)-(9.2.3), (9.2.10)-(9.2.11) *exists and is unique in the class of functions*

$$u \in C([0, T]; W_2^1(\Omega)) \bigcap C^1([0, T]; L_2(\Omega)), \qquad p \in C[0, T].$$

If we assume, in addition, that

$$f_1, f_2 \in C([0, T]; \overset{\circ}{W}{}_2^1(\Omega)),, \tag{9.2.20}$$

$$u_0 \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega), \qquad u_1 \in \overset{\circ}{W}{}_2^1(\Omega), \tag{9.2.21}$$

then Theorem 6.10.4 of Section 6.10 applies equally well, due to which a solution U of the inverse problem (9.2.7)-(9.2.8), (9.2.18)-(9.2.19) satisfies the condition $U \in C^1([0, T]; X)$. Thus, we arrive at the following assertion.

Corollary 9.2.2 *If conditions* (9.2.4)-(9.2.6), (9.2.14)-(9.2.17), (9.2.20)-(9.2.21) *hold, then a solution* u, p *of the inverse problem* (9.2.1)-(9.2.3), (9.2.10)-(9.2.11) *exists and is unique in the class of functions*

$$u \in C^2([0, T]; L_2(\Omega)) \bigcap C^1([0, T]; W_2^1(\Omega)) \bigcap C([0, T]; W_2^2(\Omega)),$$

$$p \in C[0, T].$$

A similar approach may be of help in investigating **Neumann's boundary condition**

$$\frac{\partial u}{\partial n} + \sigma(x)\, u = 0, \qquad x \in \partial\Omega, \qquad t \in [0, T], \tag{9.2.22}$$

where n denotes, as usual, a **conormal vector** and

$$\frac{\partial u}{\partial n} = \sum_{i,j=1}^{n} a_{ij}(x,t)\, \nu_i\, \frac{\partial u}{\partial x_j}.$$

Here $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ refers to a unit external normal to the boundary $\partial\Omega$ at point x and the function σ is sufficiently smooth on the boundary $\partial\Omega$. The preceding methodology provides proper guidelines for choosing the space $X = W_2^1(\Omega) \times L_2(\Omega)$ and the domain of the operator $A(t)$

$$\mathcal{D}(A(t)) = \left\{ U(t) = \begin{pmatrix} u \\ v \end{pmatrix} \in W_2^2(\Omega) \times W_2^1(\Omega) \colon \left(\frac{\partial u}{\partial n} + \sigma u \right)\bigg|_{\partial\Omega} = 0 \right\}.$$

Under the extra restriction saying that

(H) $h_i(x,t) \equiv 0$, $1 \le i \le n$, *and* $a_{ij}(x,t)\Big|_{\partial\Omega}$ *does not depend on* t, the domain of the operator $A(t)$ does not depend on the variable t.

Furthermore, a family of equivalents norms on the space X is introduced for each element

$$U(t) = \begin{pmatrix} u \\ v \end{pmatrix}$$

by merely setting

$$(9.2.23) \quad \| U(t) \|_t^2 = \sum_{i,j=1}^{n} \left(a_{ij}(x,t) \, \frac{\partial u}{\partial x_i} \, , \, \frac{\partial u}{\partial x_j} \right)_{L_2(\Omega)}$$

$$+ \int\limits_{\partial \Omega} \sigma(x) \, | u(x) |^2 \, dS + \beta \, \| u \|_{L_2(\Omega)}^2 + \| v \|_{L_2(\Omega)}^2 \, .$$

Provided that conditions (9.2.4)–(9.2.6) and assumption (H) hold, the Cauchy problem (9.2.7)–(9.2.8) and the family of norms (9.2.23) meet assumptions (HH1)–(HH4) of Section 6.10 if $\beta > 0$ is sufficiently large (for more detail see Ikawa (1968)).

Before proceeding to deeper study of the inverse problem (9.2.10)–(9.2.11), it is worth mentioning here that Theorem 6.10.2 of Section 6.10 could be useful in such a setting under conditions (9.2.14), (9.2.16)–(9.2.17) in combination with condition (9.2.15) in a slightly weakened sense:

$$(9.2.24) \qquad u_0 \in W_2^1(\Omega) \, , \qquad u_1 \in L_2(\Omega) \, , \qquad \psi \in C^2[0, T] \, .$$

But in this respect a profound result has been obtained with the following corollary.

Corollary 9.2.3 *If conditions* (9.2.4)–(9.2.6), *(H)*, (9.2.14), (9.2.24) *and* (9.2.16)–(9.2.17) *hold, then a solution u, p of the inverse problem* (9.2.1)–(9.2.2), (9.2.10)–(9.2.11), (9.2.22) *exists and is unique in the class of functions*

$$u \in C([0, T]; \, W_2^1(\Omega)) \bigcap C^1([0, T]; \, L_2(\Omega)) \, , \qquad p \in C[0, T] \, .$$

In concluding this section it should be noted that Theorem 6.10.4 of Section 6.10 may be of help in verifying the solvability of the inverse problem concerned in a strong sense under the additional constraints:

$$(9.2.25) \qquad \begin{cases} f_i \in C([0, T]; \, W_2^1(\Omega)) \, , \\ (\partial f_i / \partial n + \sigma f_i) \Big|_{\partial \Omega} = 0, \; i = 1, \, 2 \, ; \end{cases}$$

$$(9.2.26) \qquad \begin{cases} u_0 \in W_2^2(\Omega) \, , \qquad u_1 \in W_2^1(\Omega) \, , \\ (\partial u_0 / \partial n + \sigma u_0) \Big|_{\partial \Omega} = 0 \, . \end{cases}$$

This type of situation is covered by the following assertion.

Corollary 9.2.4 *If conditions* (9.2.4)–(9.2.6), (H), (9.2.14), (9.2.16)–(9.2.17), (9.2.24), (9.2.25)–(9.2.26) *hold, then a solution* u, p *of the inverse problem* (9.2.1)–(9.2.2), (9.2.10)–(9.2.11), (9.2.22) *exists and is unique in the class of functions*

$$u \in C^2([0,T];\, L_2(\Omega)) \bigcap C^1([0,T];\, W_2^1(\Omega)) \bigcap C([0,T];\, W_2^2(\Omega))\,,$$

$$p \in C[0,T]\,.$$

9.3 The system of equations from elasticity theory

We now consider a bounded domain $\Omega \subset \mathbf{R}^3$, whose boundary is sufficiently smooth. It is supposed that Ω is occupied by an **elastic body** with a **density** $\rho(x)$, where $x = (x_1,\, x_2,\, x_3)$ denotes a point of the domain Ω. From the linear elasticity theory there arises the governing system of equations

$$\rho\, \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i\,, \tag{9.3.1}$$

$$x \in \Omega\,, \qquad t \in [0,T]\,, \qquad 1 \le i \le 3\,,$$

where $u(x,t) = \big(u_1(x,t),\, u_2(x,t),\, u_3(x,t)\big)$ is the **displacement vector**, $\sigma_{ij}(x,t)$ is the **stress tensor** and $f(x,t) = \big(f_1(x,t),\, f_2(x,t),\, f_3(x,t)\big)$ is the **density of the external force**. In dealing with equations (9.3.1) as well as throughout the entire section, we will use the standard summation convention.

The main idea behind setting of a well-posed problem is that the subsidiary information is related to the system (9.3.1). Assume that the displacement vector satisfies the initial conditions

$$u(x,0) = u_0(x)\,, \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x)\,, \qquad x \in \Omega\,, \tag{9.3.2}$$

and the **Dirichlet boundary condition**

$$u(x,t) = 0\,, \qquad x \in \partial\Omega\,, \qquad 0 \le t \le T\,. \tag{9.3.3}$$

The rheology of an elastic body is described by **Hook's law** as follows:

$$\sigma_{ij} = a_{ijmn}(x)\, e_{mn}(u)\,, \tag{9.3.4}$$

where

$$e_{mn}(u) = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)$$

are the components of the **deformation tensor**. In the general case the **elasticity coefficients** a_{ijmn} and the density $\rho(x)$ of the body are essentially bounded and measurable. Also, the coefficients a_{ijmn} are supposed to satisfy the conditions of symmetry and positive definiteness

$$a_{ijmn} = a_{ijnm} = a_{jimn} = a_{mnij} , \tag{9.3.5}$$

$$a_{ijmn}(x)\,\xi_{ij}\,\xi_{mn} \geq c\,\xi_{ij}\,\xi_{ij} , \qquad c > 0 , \tag{9.3.6}$$

and the density $\rho(x)$ has the bound

$$\rho(x) \geq \rho_0 > 0 . \tag{9.3.7}$$

Further treatment of the system (9.3.1)–(9.3.4) as an abstract Cauchy problem is connected with introduction of the Lebesgue space with weights $X = \big(L_{2,\rho}(\Omega)\big)^3$. In so doing the symbols $u(t)$ and $f(t)$ will refer to the same functions $u(x,t)$ and $f(x,t)$ but being viewed as abstract functions of the variable t with values in the space X. The symbols u_0 and u_1 will be used in treating the functions $u_0(x)$ and $u_1(x)$ as the elements of the space X. With these ingredients, the system (9.3.1)–(9.3.4) reduces to the **abstract Cauchy problem** in the Banach space X for the second order equation

$$u''(t) = A\,u(t) + f(t) , \qquad 0 \leq t \leq T , \tag{9.3.8}$$

$$u(0) = u_0 , \qquad u'(0) = u_1 , \tag{9.3.9}$$

where A denotes a self-adjoint operator in the space X (for more detail see Duvaut and Lions (1972), Fikera (1974), Godunov (1978), Sanchez-Palencia (1980)). When the density of the body and the elasticity coefficients are well-characterized by smooth functions, the operator A coincides with an extension of the differential operator defined for smooth functions and corresponding to the system (9.3.1), (9.3.4).

We are now in a position to set up the inverse problem of finding the external force function via a prescribed **regime of oscillations** at a fixed point

$$u(x_0,t) = \psi(t) , \qquad 0 \leq t \leq T , \tag{9.3.10}$$

under the agreement that the function f is representable by

$$f(x,t) = g(x,t)\,p(t) + h(x,t) . \tag{9.3.11}$$

Here the matrix $g(x,t)$ of size 3×3 and the vector-valued function $h(x,t)$ are known in advance for all $x \in \Omega$, $t \in [0, T]$, while the unknown vector $p(t)$ is sought.

The conditions enabling to resolve the inverse problem (9.3.1)–(9.3.4), (9.3.10)–(9.3.11) uniquely can be derived from Theorem 8.2.1 of Section 8.1. Indeed, when $\partial\,\Omega \in C^3$ and the density ρ and the elasticity coefficients a_{ijmn} belong to the space $C^3(\bar{\Omega})$, it will be sensible to introduce the operator B acting from X into $Y = \mathbf{R}^3$ in accordance with the rule

$$B\,u \;=\; u(x_0)\,.$$

By means of B it is possible to recast (9.3.10) as relation (8.1.3). From the well-known results concerning the **regularity of the elliptic system solutions** it is clear that

$$\mathcal{D}(A) \;=\; \big(W_2^2(\Omega)\big)^3 \bigcap \big(\overset{\circ}{W}{}_2^1(\Omega)\big)^3 .$$

Furthermore, Sobolev's embedding theorems imply that the operator B satisfies condition (8.1.18) with $m = 1$. The nonhomogeneous term of equation (9.3.8) can be written as

$$f(t) \;=\; L_1(t)\,u + L_2(t)\,u' + L_3(t)\,p + F(t)\,,$$

where $L_1(t) \equiv 0$, $L_2(t) \equiv 0$, $L_3(t)\,p = g(x,t)\,p$ and $F(t) = h(x,t)$. Being self-adjoint and negative definite (see Sanchez-Palencia (1980)), the operator A is involved in the definition of the space E from Section 8.1. Recall that we have used relation (8.1.6). We claim that the space thus obtained coincides with $\mathcal{D}(\sqrt{-A}\,)$, that is, $E = \big(\overset{\circ}{W}{}_2^1(\Omega)\big)^3$. Provided the additional conditions

(9.3.12) $$g\,,\,h \,\in\, C([0,\,T];\,\overset{\circ}{W}{}_2^3(\Omega))\,,$$

$$\det g(x_0,t) \,\neq\, 0\,,$$

(9.3.13) $$u_0 \in W_2^4(\Omega) \bigcap \overset{\circ}{W}{}_2^3(\Omega)\,,$$

$$u_1 \in \overset{\circ}{W}{}_2^3(\Omega)\,, \qquad \psi \,\in\, C^2\big([0,\,T];\,\mathbf{R}^3\big)\,,$$

(9.3.14) $$u_0(x_0) \;=\; \psi(0)\,, \qquad u_1(x_0) \;=\; \psi'(0)$$

hold, we establish on account of Theorem 8.1.2 from Section 8.1 the following result.

Corollary 9.3.1 *Let conditions* (9.3.5)–(9.3.7) *hold together with conditions* (9.3.12)–(9.3.14) *and let the inclusions*

$$a_{ijmn} \in C^3(\bar{\Omega}), \qquad \rho \in C^3(\bar{\Omega})$$

and $\partial\Omega \in C^3$ *occur. Then a solution* u, p *of the inverse problem* (9.3.1)–(9.3.4), (9.3.10)–(9.3.11) *exists and is unique in the class of functions*

$$u \in C^2\big([0, T];\ \big(L_{2,\rho}(\Omega)\big)^3\big), \qquad p \in C\big([0, T];\ \mathbf{R}^3\big).$$

Before proceeding to the next case, we adopt the same structure (9.3.11) of the function f in the situation when the subsidiary information is provided by the condition of **integral overdetermination**

$$\int_\Omega u(x,t)\, w(x)\, dx = \psi(t), \qquad 0 \le t \le T. \tag{9.3.15}$$

We note in passing that condition (9.3.15) can be written in the form (8.1.3) if the operator B is defined by the relation

$$B u = \int_\Omega u(x)\, w(x)\, dx .$$

In such a setting it is possible to weaken the restrictions on the smoothness of the boundary and the coefficients. To be more specific, it suffices to require that $\partial\Omega \in C^2$, $\rho \in C^2(\bar{\Omega})$ and $a_{ijmn} \in C^2(\bar{\Omega})$, retaining the domain $\mathcal{D}(A)$ of the operator A and accepting for each $u \in \mathcal{D}(A)$ the set of relations

$$(A u)_i = \rho^{-1} \frac{\partial}{\partial x_j} \left(a_{ijmn}(x) \frac{\partial u_m}{\partial x_n} \right), \qquad 1 \le i \le 3 .$$

By the well-known Ostrogradsky formula it is plain to derive that

$$(B A u)_i = - \int_\Omega a_{ijmn}(x) \frac{\partial u_m}{\partial x_n} \frac{\partial}{\partial x_j} \left(\frac{w(x)}{\rho(x)} \right) dx, \qquad 1 \le i \le 3,$$

for any function $w(x)$ being subject to the relation

$$w \in \overset{\circ}{W}{}_2^1(\Omega). \tag{9.3.16}$$

This serves as a basis for decision-making that the operator B satisfies condition (8.1.27) of Section 8.1. To prove the solvability of the inverse problem at hand, we refer to Corollary 8.1.2 of Section 8.1, all the conditions of which will be satisfied once we impose the set of constraints

$$g\,,\, h \in C([0,T];\, L_2(\Omega))\,, \qquad \det \int_\Omega g(x,t)\, w(x)\, dx \neq 0\,, \tag{9.3.17}$$

$$u_0 \in \overset{\circ}{W}{}_2^1(\Omega)\,, \quad u_1 \in L_2(\Omega)\,, \quad \psi \in C^2([0,T];\, \mathbf{R}^3)\,, \tag{9.3.18}$$

$$\int_\Omega u_0(x)\, w(x)\, dx = \psi(0)\,, \qquad \int_\Omega u_1(x)\, w(x)\, dx = \psi'(0)\,. \tag{9.3.19}$$

As we have mentioned above, Corollary 8.1.2 applies equally well to related problems arising in elasticity theory. As a final result we get the following assertion.

Corollary 9.3.2 *Let conditions* (9.3.5)-(9.3.7) *hold together with conditions* (9.3.17)-(9.3.19) *and let the inclusions*

$$a_{ijmn} \in C^2(\bar\Omega)\,, \qquad \rho \in C^2(\bar\Omega)$$

and $\partial\Omega \in C^2$ *occur. Then a solution* u, p *of the inverse problem* (9.3.1)-(9.3.4), (9.3.11), (9.3.15) *exists and is unique in the class of functions*

$$u \in C^1([0,T];\, (L_2(\Omega))^3) \bigcap C([0,T];\, (W_2^1(\Omega))^3), \qquad p \in C([0,T];\, \mathbf{R}^3).$$

If there is a need for imposing the conditions under which a weak solution becomes differentiable, it suffices to require that

$$\begin{cases} g\,,\, h \in C^1([0,T];\, L_2(\Omega))\,, \\ \psi \in C^3([0,T];\, \mathbf{R}^3)\,, \end{cases} \tag{9.3.20}$$

$$\begin{cases} u_0 \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)\,, \\ u_1 \in \overset{\circ}{W}{}_2^1(\Omega)\,, \end{cases} \tag{9.3.21}$$

and apply Corollary 8.1.3 of Section 8.1 with these members. Under such an approach we deduce the following result.

Corollary 9.3.3 *Let the conditions of Corollary* 9.3.2 *hold and inclusions* (9.3.20)–(9.3.21) *occur. Then a solution* u, p *of the inverse problem* (9.3.1)–(9.3.4), (9.3.11), (9.3.15) *exists and is unique in the class of functions*

$$u \in C^2([0,T];\, L_2(\Omega)) \bigcap C^1([0,T];\, W_2^1(\Omega)) \bigcap C([0,T];\, W_2^2(\Omega))\,,$$

$$p \in C^1([0,T];\, \mathbf{R}^3)\,.$$

In concluding this section we are interested in learning more about one particular case where the external force function $f(x,t)$ is representable by

$$f(x,t) = \Phi(t)\, p(x) + F(x,t)\,, \tag{9.3.22}$$

where the coefficient p is unknown. The subsidiary information is provided by the condition of **final overdetermination**

$$u(x,T) = u_2(x)\,, \qquad x \in \Omega\,. \tag{9.3.23}$$

With the aid of relations (8.2.1)–(8.2.3) the inverse problem at hand may also be posed in an abstract form. Since the operator A is self-adjoint, we make use of Corollary 8.2.7 from Section 8.2, the validity of which is ensured by the following restrictions:

$$F \in C^1([0,T];\, L_2(\Omega)) + C([0,T];\, W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))\,, \tag{9.3.24}$$

$$u_0,\, u_2 \in W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega)\,, \qquad u_1 \in \overset{\circ}{W}{}_2^1(\Omega)\,, \tag{9.3.25}$$

$$\begin{cases} \Phi \in C^1[0,T]\,; & \Phi(t) \geq 0\,, \\ \Phi'(t) > 0\,, & 0 \leq t \leq T\,. \end{cases} \tag{9.3.26}$$

Corollary 9.3.5 *Let conditions* (9.3.5)–(9.3.7) *hold together with conditions* (9.3.24)–(9.3.26) *and let the inclusions*

$$a_{ijmn} \in C^2(\bar{\Omega})\,, \qquad \rho \in C^2(\bar{\Omega})$$

and $\partial\Omega \in C^2$ *occur. Then a solution* u, p *of the inverse problem* (9.3.1)–(9.3.4), (9.3.22)–(9.3.23) *exists and is unique in the class of functions*

$$u \in C^2([0,T];\, L_2(\Omega))\,, \qquad p \in L_2(\Omega)\,.$$

9.4 Equations of heat transfer

This section is devoted to the inverse problem in the domain $D = \Omega \times [0, T]$ of finding a pair of the functions $u(x,t)$ and $p(x)$ from the set of relations

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + b(x)\, u + \Phi(x,t)\, p(x) + F(x,t), \quad x \in \Omega, \quad 0 \le t \le T, \tag{9.4.1}$$

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x), \qquad x \in \Omega, \tag{9.4.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \qquad 0 \le t \le T, \tag{9.4.3}$$

where Ω is a bounded domain in the space $\mathbf{R}^n$ with boundary $\partial\Omega \in C^2$.

For the purposes of the present section we impose the following restrictions on its ingredients:

(1) $a_{ij} \in C^1(\bar{\Omega})$, $a_{ij} = a_{ji}$, $\sum\limits_{i,j=1}^{n} a_{ij}(x)\, \xi_i\, \xi_j \ge c \sum\limits_{i=1}^{n} \xi^2$, $\quad c > 0$,

(2) $b \in C(\bar{\Omega})$, $b \le 0$,

(3) Φ, $\dfrac{\partial \Phi}{\partial t} \in C(\bar{D})$, $\Phi > 0$, $\dfrac{\partial \Phi}{\partial t} > 0$,

(4) $F \in C^1([0, T];\, L_2(\Omega)) + C([0, T];\, W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$,

(5) $u_0,\, u_1 \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$.

Adopting $X = L_2(\Omega)$ as a basic space we introduce the differential operator

$$A\, u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + b(x)\, u$$

with the domain

$$\mathcal{D}(A) = W_2^2(\Omega) \bigcap \overset{\circ}{W}{}_2^1(\Omega).$$

The operator A is self-adjoint and negative definite and its inverse A^{-1} is compact (see Gilbarg and Trudinger (1983)). These properties serve to motivate that the operator A generates a strongly continuous **compact semigroup** $V(t)$ (see Fattorini (1983)).

When the partial ordering relation is established in the Hilbert space $X = L_2(\Omega)$, we will write $f \leq g$ if and only if $f(x) \leq g(x)$ almost everywhere in Ω. The space X with ordering of such a kind becomes a **Hilbert lattice** and the relevant semigroup $V(t)$ appears to be **positive**. Therefore, the inverse problem (9.4.1)–(9.4.3) can be set up in the abstract form well-characterized by relations (7.3.3)–(7.3.5). Corollary 7.3.1 of Section 7.3 provides sufficient background for justifying the following statement.

Corollary 9.4.1 *If conditions* (1)–(5) *hold, then a solution* u, p *of the inverse problem* (9.4.1)–(9.4.3) *exists and is unique in the class of functions*

$$u \in C^1([0,T];\, L_2(\Omega)) \bigcap C([0,T];\, W_2^2(\Omega)),$$

$$p \in L_2(\Omega).$$

In this context, one thing is worth noting: the condition $b \leq 0$ is not strong, since we can be pretty sure upon substituting

$$u(x,t) = v(x,t)\exp(\lambda t)$$

that the function v satisfies the same system of relations but with the functions $b(x)-\lambda$, $\Phi(x,t)\exp(-\lambda t)$ and $u_1(x)\exp(-\lambda T)$ in place of $b(x)$, $\Phi(x,t)$ and $u_1(x)$, respectively. The value λ can always be chosen in such a way that the inequality $b(x)-\lambda \leq 0\,(b \in C(\bar{\Omega}))$ should be valid almost everywhere in the domain Ω. In view of this, it is necessary to replace the relation $\partial\Phi/\partial t > 0$ involved in condition (3) by another relation $\partial\Phi/\partial t - \lambda\Phi > 0$, which ensures that the inverse problem at hand will be uniquely solvable even if the bound $b \leq 0$ fails to be true.

One more example can add interest and aid in understanding in which it is required to determine a pair of the functions $u(x,t)$, $p(t)$ from the set of relations

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x,t)\,\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} a_i(x,t)\,\frac{\partial u}{\partial x_i} + a(x,t)\,u + g(x,t), \tag{9.4.4}$$

$$x \in \Omega, \qquad 0 \leq t \leq T,$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{9.4.5}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad 0 \leq t \leq T, \tag{9.4.6}$$

$$(9.4.7) \qquad a(x,t) = \alpha_1(x,t) + \alpha_2(x,t)\, p(t)\,, \quad x \in \Omega\,, \quad 0 \le t \le T\,,$$

$$(9.4.8) \qquad u(x_0,t) = \psi(t)\,, \qquad 0 \le t \le T\,,$$

where x_0 is a fixed point in the domain Ω. The solvability of this inverse problem will be proved on account of Theorem 6.7.1 from Section 6.7 if we succeed in setting up the inverse problem (9.4.1)–(9.4.8) in an abstract form. This can be done using the system of relations (6.7.1)–(6.7.3) and approving $X = L_r(\Omega)$, $r \ge 1$, and $Y = \mathbf{R}$ as the basic spaces. After that, we introduce in the space X a differential operator with the domain $\mathcal{D}_1 = W_r^2(\Omega) \bigcap \overset{\circ}{W}{}_r^1(\Omega)$ that assigns the values

$$(9.4.9) \qquad A(t)\, u = \sum_{i,j=1}^{n} a_{ij}(x,t)\, \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\, \frac{\partial u}{\partial x_i} + \alpha_1(x,t)\, u\,.$$

Furthermore, we impose the following restrictions on the coefficients of operator (9.4.9):

(A) *all the functions* a_{ij}, $\partial a_{ij}/\partial x_i$, a_i, $\alpha_1 \in C(\bar{D})$ *satisfy Hölder's condition in* t *with exponent* $\alpha \in (0,\, 1]$ *and constant not depending on* x; $a_{ij} = a_{ji}$; *there is a constant* $c > 0$ *such that for all* $(x,t) \in D$ *and* $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\, \xi_i\, \xi_j \ge c \sum_{i=1}^{n} \xi_i^2\,;$$

(B) *there exists a constant* $a_0 > 0$ *such that* $\alpha_1(x,t) \ge a_0$ *for each* $(x,t) \in D$.

It is clear that condition (A) describes not only the smoothness of the coefficients, but also the **parabolic character** of equation (9.4.4). Condition (B) is not essential for subsequent studies, since the substitution

$$u(x,t) = v(x,t)\, \exp\big(\lambda t\big)$$

if λ is sufficiently large, provides its validity in every case. Under conditions (A)–(B) propositions (PP1)–(PP2) of Section 6.7 are certainly true (for more detail see Sobolevsky (1961)).

Let the values $r > n/(2\alpha)$ and $\beta \in (0,\, \alpha)$ be related by $r > n/(2\beta)$. Due to the well-known properties of the **fractional powers** of elliptic operators the manifold $D(A^{\beta}(0))$ with $\beta > 0$ is continuously embedded into the space $W_r^{2\beta}(\Omega)$. In turn, Sobolev's space $W_r^{2\beta}(\Omega)$ is continuously embedded into the space $C(\bar{\Omega})$. From such reasoning it seems clear that the manifold $D(A^{\beta}(0))$ with the fixed values r and β is continuously embedded into the space $C(\bar{\Omega})$, making it possible to specify the operator B by the relation

$$B u = u(x_0).$$

The operator B so defined satisfies condition (6.7.7) and may be of help in reducing relation (9.4.8) to equality (6.7.7).

Via the decomposition $f = -\alpha_2\, p\, u + g$ approved we write down equation (9.4.4) in the form (6.7.1) obtained in Section 6.7. Observe that representation (6.7.8) is valid with $f_1 = g$ and $f_2 = -\alpha_2\, p\, u$. The function f_3 built into condition (PP4) of Section 6.7 takes now the form

$$f_3(t, z, p) = -\alpha_2(x_0, t)\, p\, z,$$

which implies that condition (PP5) of Section 6.7 is met if for each $t \in [0, T]$ the relation

$$\alpha_2(x_0, t)\, \psi(t) \neq 0$$

holds true. In that case the function Φ arising from equality (6.7.12) is given by the formula

$$\Phi(t, z) = -\big(\alpha_2(x_0, t)\, \psi(t)\big)^{-1} z,$$

which assures us of the validity of condition (PP7) from Section 6.7. For the fulfilment of the last condition (PP6) from Section 6.7 it suffices to require that

$$g \in C([0, T];\, W_r^4(\Omega) \cap \overset{\circ}{W}{}_r^3(\Omega)), \qquad \alpha_2 \in C([0, T];\, C^4(\Omega))$$

and

$$\left.\frac{\partial \alpha_2}{\partial x_i}\right|_{\partial\Omega} = 0, \qquad 1 \leq i \leq n.$$

Conditions (6.7.9) are equivalent to the inclusion

$$\psi \in C^1[0, T]$$

supplied by the accompanying equality

$$u_0(x_0) = \psi(0).$$

Finally, the inclusion

$$u_0 \in \mathcal{D}(A^{1+\beta}(0))$$

is valid under the constraint

$$u_0 \in \mathcal{D}(A^2(0)) = \left\{ u \in W_r^4(\Omega):\ u\Big|_{\partial\Omega} = A(0)\, u\Big|_{\partial\Omega} = 0 \right\}.$$

Therefore, applying Theorem 6.7.1 of Section 6.7 yields the following result.

Corollary 9.4.2 *Let conditions (A)-(B) hold for* $r > n/(2\alpha)$, $\psi \in C^1[0,T]$, $\alpha_2(x_0,t) \neq 0$ *and* $\psi(t) \neq 0$ *for each* $t \in [0,T]$. *If* $\alpha_2 \in C([0,T]; C^4(\Omega))$,

$$\frac{\partial \alpha_2}{\partial x_i}\bigg|_{\partial\Omega} = 0, \qquad 1 \le i \le n,$$

$g \in C([0,T]; W_r^4(\Omega) \cap \overset{\circ}{W}{}_r^3(\Omega))$, $u_0 \in W_r^4(\Omega)$, $u_0|_{\partial\Omega} = A(0)\,u_0|_{\partial\Omega} = 0$ *and* $u_0(x_0) = \psi(0)$, *then there exists a value* $T_1 > 0$ *such that a solution* u, p *of the inverse problem* (9.4.4)-(9.4.8) *exists and is unique in the class of functions*

$$u \in C^1([0,T_1]; L_r(\Omega)) \bigcap C([0,T_1]; W_r^2(\Omega)),$$

$$p \in C[0,T_1].$$

We now focus our attention on the inverse problem for the **quasilinear equation** in which it is required to find a pair of the functions $u(x,t)$ and $p(t)$ from the set of relations

$$\frac{\partial u}{\partial u} = \sum_{i,j=1}^{n} a_{ij}(x,t,u)\,\frac{\partial^2 u}{\partial x_i\,\partial x_j} + f_1\left(x,t,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) + f_2(x,t,u,p(t)), \quad x \in \Omega, \quad 0 \le t \le T, \tag{9.4.10}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{9.4.11}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad 0 \le t \le T, \tag{9.4.12}$$

$$\int_\Omega u(x,t)\,w(x)\,dx = \psi(t), \quad 0 \le t \le T. \tag{9.4.13}$$

Its solvability is ensured by Theorem 6.6.1 of Section 6.6 because we are still in the abstract framework of Section 6.6.

When operating in the spaces $X = L_r(\Omega)$ and $Y = \mathbf{R}$, we introduce in the space X the differential operator

$$A(t,u)\colon\ v \mapsto \sum_{i,j=1}^{n} a_{ij}(x,t,u)\,\frac{\partial^2 v}{\partial x_i\,\partial x_j}$$

with the domain $\mathcal{D}_1 = W_r^2(\Omega) \cap \overset{\circ}{W}{}_r^1(\Omega)$. The symbol B stands for an operator from the space X into the space Y being a functional and acting in accordance with the rule

$$B\,u = \int_\Omega u(x)\,w(x)\,dx\,.$$

Also, we take for granted that

$$(9.4.14)\qquad u_0 \in W_r^2(\Omega) \bigcap \overset{\circ}{W}{}_r^1(\Omega)\,, \qquad a_{ij} \in C^1\big(\bar\Omega \times [0,T] \times \mathbf{R}\big)\,,$$

$$(9.4.15)\qquad a_{ij}\big(x,0,u_0(x)\big) = a_{ji}\big(x,0,u_0(x)\big)\,,$$

$$(9.4.16)\qquad \sum_{i,j=1}^n a_{ij}(x,0,u_0(x))\;\xi_i\,\xi_j \geq c \sum_{i=1}^n \xi_i^2\,, \qquad c>0\,,$$

which assure us of the validity of condition (P1) from Section 6.6. In this view, it is meaningful to insert $\alpha = \frac{1}{2}$ in conditions (P2)–(P4) of Section 6.6. From the well-known properties of the fractional powers of elliptic operators it follows that the operator $A_0^{-1/2}$ carries out the space $L_r(\Omega)$ into the space $\overset{\circ}{W}{}_r^1(\Omega)$ as an isomorphism such that the norm $\|\,A_0^{1/2}\,u\,\|$ is equivalent to the norm of the space $W_r^1(\Omega)$. We note in passing that the space $W_r^1(\Omega)$ with $r>n$ is embedded into the space $C(\bar\Omega)$. Because of this, the estimate

$$(9.4.17)\qquad \Big|\big(A_0^{-1/2}\,u\big)\,(x)\Big| \leq k\,\|\,u\,\|$$

is valid with constant $k>0$.

Let a number R be so chosen as to satisfy the bound

$$\|\,A_0^{1/2}\,u_0\,\| < R\,.$$

Moreover, it is supposed that for all values $x \in \Omega$, $t \in [0,T]$ and $v \in [-k\,R,\,k\,R]$

$$(9.4.18)\qquad a_{ij}(x,t,v) = a_{ji}(x,t,v)\,,$$

$$(9.4.19)\qquad \sum_{i,j=1}^n a_{ij}(x,t,v)\;\xi_i\,\xi_j \geq c \sum_{i=1}^n \xi_i^2\,, \qquad c>0\,.$$

If $\| u \| \le R$, then by inequality (9.4.17) $v = A_0^{-1/2} u$ takes on the values from the segment $[-k R, k R]$ as a function of the arrgument x. By virtue of (9.4.18)–(9.4.19), $A\big(t, A_0^{-1/2} u\big)$ is a uniformly elliptic operator with domain $\mathcal{D}_1$. Since the partial derivatives of the function a_{ij} are bounded in the bounded domain $\bar{\Omega} \times [0, T] \times [-k R, k R]$, this function satisfies in the same domain the Lipschitz condition with respect to the variables t and v. This serves to motivate that condition (P2) with $\beta = 1$ is satisfied. Let us stress that the first two items of condition (P3) of Section 6.6 are automatically fulfilled and the following restriction is needed in the sequel for its validity:

$$\psi \in C^2[0, T] . \tag{9.4.20}$$

We now proceed to discussions of conditions (P5)–(P9) of Section 6.6. The nonhomogeneous term of the equation is available here in the form (6.6.7) we have adopted in Section 6.6. We restrict ourselves to the simple cases when

$$f_2(x, t, u, p) = G(x, t)\, p , \tag{9.4.21}$$

$$f_2(x, t, u, p) = p\, u , \tag{9.4.22}$$

in which the function f_3 arising from condition (P6) of Section 6.6 becomes, respectively,

$$f_3(t, z, p) = \bigg(\int_{\Omega} G(x, t)\, w(x)\, dx \bigg)\, p$$

or

$$f_3(t, z, p) = z\, p .$$

It is easily seen that condition (P8) of Section 6.6 with regard to the function f_1 holds true for $\alpha = \frac{1}{2}$ and $\beta = 1$, since the function f_1 satisfies on the manifold $\bar{\Omega} \times [0, T] \times [-k R, k R] \times \mathbf{R}^n$ the Lipschitz condition, the norm of the element u in the space $\overset{\circ}{W}{}_r^1(\Omega)$ is equivalent to the norm $\| A_0^{1/2} u \|$ and the bound (9.4.17) is obviously attained. In particular, its fulfilment is ensured by the continuity and boundedness of all first derivatives of the function f_1 on the manifold $\bar{\Omega} \times [0, T] \times [-k R, k R] \times \mathbf{R}^n$. For the function given by formula (9.4.22) the validity of condition (P8) from Section 6.6 with regard to the function f_2 is clear, whereas for the function given by formula (9.4.21) it is stipulated by the inclusion

$$G \in C^1\big([0, T];\, L_r(\Omega)\big) . \tag{9.4.23}$$

Because of (9.4.21) and (9.4.23), conditions (P5) and (P7) of Section 6.6 will be satisfied once we require that for all $t \in [0, T]$

$$\int_{\Omega} G(x,t)\, w(x)\, dx \neq 0\,. \tag{9.4.24}$$

When decomposition (9.4.22) is approved, these conditions follow from the relation

$$\psi(t) \neq 0\,, \qquad 0 \leq t \leq T\,. \tag{9.4.25}$$

At the next stage condition (P4) of Section 6.6 is of interest. Since the operator B does not depend on the variable t, relations (6.6.6) of Section 6.6 take place under the constraints

$$w \in L_q(\Omega)\,, \qquad \frac{1}{r} + \frac{1}{q} = 1\,. \tag{9.4.26}$$

To verify the rest of condition (P4), it suffices to show that the inequality

$$\begin{aligned} &\left| \int_{\Omega} \left(\sum_{i,j=1} \big(a_{ij}(x,t,u) - a_{ij}(x,s,v)\big) \, \frac{\partial^2 w}{\partial x_i\, \partial x_j} \right) w(x)\, dx \right| \\ &\qquad\qquad \leq c \Big(|\, t - s\,| + \|\, u - v\,\|_{W_r^1(\Omega)} \Big) \,\|\, w\,\|_{W_r^1(\Omega)} \end{aligned} \tag{9.4.27}$$

holds true for all $u,\ v \in \overset{\circ}{W}{}_r^1(\Omega)$ with $\|\, A_0^{1/2}\, u\,\| \leq R$ and $\|\, A_0^{1/2}\, v\,\| \leq R$ and each $w \in W_r^2(\Omega) \cap \overset{\circ}{W}{}_r^1(\Omega)$. Indeed, let $r > 2$ and

$$w \in \overset{\circ}{W}{}_s^1(\Omega)\,, \qquad s = r/(r-2)\,. \tag{9.4.28}$$

After integrating by parts we finally get

$$\begin{aligned} &\int_{\Omega} \left(\sum_{i,j=1}^{n} \big(a_{ij}(x,t,u) - a_{ij}(x,s,v)\big) \, \frac{\partial^2 w}{\partial x_i\, \partial x_j} \right) w(x)\, dx \\ &\qquad = - \int_{\Omega} \sum_{i,j=1}^{n} \left(\frac{\partial}{\partial x_i} \Big(\big[a_{ij}(x,t,u(x)) \right. \\ &\qquad\qquad \left. - a_{ij}(x,s,v(x)) \big]\, w(x) \Big) \right) \frac{\partial w}{\partial x_j}\, dx\,. \end{aligned}$$

Hölder's inequality yields that the left hand-side of (9.4.27) has the upper estimate

$$\left(\sum_{i,j=1}^{n} \left\| \frac{\partial}{\partial x_i} \left[\left(a_{ij}(x,t,u(x)) - a_{ij}(x,s,v(x)) \right) w(x) \right] \right\|_{L_q(\Omega)} \right) \| w \|_{W_r^1(\Omega)} ,$$

by means of which it remains to find that

$$\left\| \frac{\partial}{\partial x_i} \left[\left(a_{ij}(x,t,u(x)) - a_{ij}(x,s,v(x)) \right) w(x) \right] \right\|_{L_q(\Omega)} \le c\left(|t-s| + \| u - v \|_{W_r^1(\Omega)} \right) . \tag{9.4.29}$$

If we agree to consider

$$a_{ij} \in C^2(\bar{\Omega} \times [0,T] \times [-kR, kR]) , \tag{9.4.30}$$

then

$$\begin{aligned} \frac{\partial}{\partial x_i} \left[\left(a_{ij}(x,t,u) - a_{ij}(x,s,v) \right) w \right] = & \left(\frac{\partial a_{ij}}{\partial x_i}(x,t,u) - \frac{\partial a_{ij}}{\partial x_i}(x,s,v) \right) w \\ & + \left[\frac{\partial a_{ij}}{\partial u}(x,t,u) - \frac{\partial a_{ij}}{\partial u}(x,s,v) \right] \\ & \times \left(\frac{\partial u}{\partial x_i} \right) w + \frac{\partial a_{ij}}{\partial u}(x,s,v) \\ & \times \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) w \\ & + \left(a_{ij}(x,t,u) - a_{ij}(x,s,v) \right) \frac{\partial w}{\partial x_i} . \end{aligned} \tag{9.4.31}$$

Since $\| A_0^{1/2} u \| \le R$ and $\| A_0^{1/2} v \| \le R$, relation (9.4.17) implies the bounds $|u(x)| \le kR$ and $|v(x)| \le kR$, valid for each $x \in \Omega$. Having stipulated condition (9.4.30), the collection of estimates

$$\left| \frac{\partial a_{ij}}{\partial x_i}(x,t,u) - \frac{\partial a_{ij}}{\partial x_i}(x,s,v) \right| \le L\left(|t-s| + |u(x) - v(x)| \right) , \tag{9.4.32}$$

$$\left| \frac{\partial a_{ij}}{\partial u}(x,t,u) - \frac{\partial a_{ij}}{\partial u}(x,s,v) \right| \le L\left(|t-s| + |u(x) - v(x)| \right) , \tag{9.4.33}$$

$$\left| \frac{\partial a_{ij}}{\partial u} (x, s, v) \right| \le L \tag{9.4.34}$$

and

$$\big| a_{ij}(x, t, u) - a_{ij}(x, s, v) \big| \le L \big(|t - s| + |u(x) - v(x)| \big) \tag{9.4.35}$$

becomes true. Since

$$\frac{1}{s} + \frac{1}{r} = \frac{1}{q},$$

we are able to use the **generalized Hölder inequality**

$$\| f g \|_{L_q(\Omega)} \le \| f \|_{L_s(\Omega)} \cdot \| g \|_{L_r(\Omega)}, \tag{9.4.36}$$

which in combination with relations (9.4.31)–(9.4.35) serves as a basis for the desired estimate (9.4.29).

Observe that $q = r/(r-1) < s$, due to which relation (9.4.28) implies (9.4.26). Thus, condition (P4) of Section 6.6 holds true if $r > \max\{ n, 2 \}$ and conditions (9.4.28), (9.4.30) are satisfied. Summarizing, we deduce the following corollary.

Corollary 9.4.3 *Let conditions* (9.4.18)–(9.4.19) *and* (9.4.30) *hold, all of the partial derivatives of the first order of the function f_1 be continuous and bounded on the manifold $\bar{\Omega} \times [0, T] \times [-k R, k R] \times \mathbf{R}^n$ and let representation* (9.4.21) *and conditions* (9.4.23)–(9.4.24), (9.4.28) *be valid. If*

$$u_0 \in W_r^2(\Omega) \bigcap \overset{\circ}{W}{}_r^1(\Omega), \quad r > \max\{ n, 2 \}, \qquad \psi \in C^2[0, T]$$

and the compatibility condition

$$\int_\Omega u_0(x) \, w(x) \, dx = \psi(0)$$

holds, then for any $\sigma < \frac{1}{2}$ there exists a value $T_1 > 0$ such that a solution u, p of the inverse problem (9.4.10)–(9.4.13) *exists and is unique in the class of functions*

$$u \in C^1([0, T_1]; L_r(\Omega)) \bigcap C([0, T_1]; W_r^2(\Omega)),$$

$$p \in C^\sigma[0, T_1].$$

Corollary 9.4.4 *Let conditions* (9.4.18)–(9.4.19) *and* (9.4.30) *be fulfilled, all of the partial derivatives of the first order of the function* f_1 *be continuous and bounded on the manifold* $\bar{\Omega} \times [0, T] \times [-k\,R,\, k\,R] \times \mathbf{R}^n$ *and let representation* (9.4.22) *and conditions* (9.4.20), (9.4.25), (9.4.28) *hold. If*

$$u_0 \in W_r^2(\Omega) \bigcap \overset{\circ}{W}{}_r^1(\Omega)\,, \qquad r > \max\{\, n,\, 2\,\}\,,$$

and the compatibility condition

$$\int\limits_\Omega u_0(x)\, w(x)\, dx \;=\; \psi(0)$$

holds, then for any $\sigma < \frac{1}{2}$ *there exists a value* $T_1 > 0$ *such that a solution* u, p *of the inverse problem* (9.4.10)–(9.4.13) *exists and is unique in the class of functions*

$$u \in C^1\big([0,\, T_1];\, L_r(\Omega)\big) \bigcap C\big([0,\, T_1];\, W_r^2(\Omega)\big)\,,$$
$$p \in C^\sigma[0,\, T_1]\,.$$

9.5 Equation of neutron transport

Let us consider in the space $\mathbf{R}^3$ a strictly convex, closed and bounded domain Ω with a smooth boundary $\partial\,\Omega$ and a closed bounded domain D. We proceed to special investigations of the **transport equation** in the domain $G = \Omega \times D \times [0, T]$ for the function $u = u(x, v, t)$

$$\text{(9.5.1)} \quad \frac{\partial u}{\partial t} + \big(v,\, \mathrm{grad}_x\, u\big) + \sigma(x, v, t)\, u = \int\limits_D K(x, v, v', t)\, u(x, v', t)\; dv' + f(x, v, t)\,,$$

supplied by the initial and boundary conditions

$$\text{(9.5.2)} \qquad u(x, v, 0) \;=\; u_0(x, v)\,, \quad x \in \Omega\,, \quad v \in D\,,$$

$$\text{(9.5.3)} \qquad u(x, v, t) \;=\; 0\,, \qquad x \in \partial\,\Omega\,,$$
$$(\, v,\, n_x\,) \;<\; 0\,, \qquad 0 \le t \le T\,,$$

where n_x refers to a unit external normal to the boundary $\partial\Omega$ at a point x. The monographs and original papers by Case and Zweifel (1972), Cercignani (1975), Germogenova (1986), Hejtmanek (1984), Richtmyer (1978), Shikhov (1973), Vladimirov (1961), Wing (1962) are devoted to this type of equations. From a physical point of view the meaning of the function u is the **density of the neutron distribution** over the phase space $\Omega \times D$ at time t.

Common practice involves by means of the source density f the **emission of neutrons** caused by the process of **fission**. In this case the function f may, in general, depend on u and is actually unknown. We are interested in recovering both functions u and f under the agreement that the source density f is representable by

$$f(x,v,t) = \Phi(x,v,t)\,p(t) + F(x,v,t)\,, \tag{9.5.4}$$

where the functions Φ and F are known in advance, while the unknown coefficient p is sought. In trying to find the function p the subsidiary information is provided in the form of **integral overdetermination**

$$\int\limits_{\Omega\times D} u(x,v,t)\,w(x,v)\;dx\;dv = \psi(t)\,, \qquad 0 \le t \le T\,. \tag{9.5.5}$$

Accepting $r > 1$ and $q = r/(r-1)$ we take for granted that

(1) $\omega \in L_q(\Omega \times D)$;

(2) $\sigma \in C^1\big([0,T];\, L_\infty(\Omega \times D)\big)$;

(3) *the operator*

$$K(t)\colon\; u(x,v) \mapsto \int\limits_D K(x,v,v',t)\,u(x,v')\;dv'$$

complies with the inclusion $K \in C^1\big([0,T];\, \mathcal{L}\big(L_r(\Omega \times D)\big)\big)$;

(4) Φ, $\big(v,\, \mathrm{grad}_x\,\Phi\big) \in C\big([0,T];\, \mathcal{L}\big(L_r(\Omega \times D)\big)\big)$, *the equality*

$$\Phi(x,v,t) = 0$$

holds for all $x \in \partial\Omega$, $v \in D$ *such that* $(v,\, n_x) < 0$ *and*

$$\int\limits_{\Omega\times D} \Phi(x,v,t)\,w(x,v)\;dx \ne 0$$

at every moment $t \in [0, T]$;

(5) $F = F_1 + F_2$, *where*

$$F_1 \in C^1([0, T]; L_r(\Omega \times D)) ,$$

$$F_2, \ (v, \operatorname{grad}_x F_2) \in C([0, T]; L_r(\Omega \times D)) ,$$

$$F_2(x, v, t) = 0 \quad \text{for all} \quad x \in \partial\Omega, \ v \in D, \ t \in [0, T], \ (v, n_x) < 0 ;$$

(6) $u_0, \ (v, \operatorname{grad}_x u_0) \in L_r(\Omega \times D), \ u_0(x, v) = 0$ *for all* $x \in \partial\Omega, \ v \in D, \ (v, n_x) < 0$;

(7) $\psi \in C^1[0, T]$ and $\psi(0) = \int\limits_{\Omega \times D} u_0(x, v)\, w(x, v)\, dx\, dv$.

Theorem 9.5.1 *Let conditions* (1)–(7) *hold for* $r > 1$ *and* $q = r/(r-1)$. *Then a solution* u, p *of the inverse problem* (9.5.1)–(9.5.5) *exists and is unique in the class of functions*

$$u \in C^1([0, T]; L_r(\Omega \times D)) , \qquad p \in C[0, T] .$$

Proof In order to apply in such a setting Theorem 6.5.5 of Section 6.5, we begin by adopting the basic spaces $X = L_r(\Omega \times D)$ and $Y = \mathbf{R}$ and introducing the operator (functional) B with the values

$$B\, u = \int\limits_{\Omega \times D} u(x, v)\, w(x, v)\, dx\, dv .$$

From condition (1) it is clear that $B \in \mathcal{L}(X, Y)$ as well as equality (6.5.3) is an abstract form of relation (9.5.5). It will be sensible to introduce the operator

$$A\, u = -(v, \operatorname{grad}_x u) ,$$

whose domain $\mathcal{D}(A)$ consists of all functions $u \in L_r(\Omega \times D)$ such that $(v, \operatorname{grad}_x u) \in L_r(\Omega \times D)$ and $u(x, v) = 0$ at all points $x \in \partial\Omega$ and $v \in D$, complying with the inequality $(v, n_x) < 0$. The operator A falls into the category of generators of strongly continuous semigroups in the space $L_r(\Omega \times D)$ (for more detail see Hejtmanek (1984), Jorgens (1968), Shikhov

(1973)). For the purposes of the present section we keep in representation (6.5.3) the following members:

$$L_1(t)\,u = -\sigma\,u + K(t)\,u\,,$$

$$L_2(t)\,p = \Phi\,p\,,$$

$$F(t) = F(x,v,t)\,,$$

which assure us of the validity of the conditions of Theorem 6.5.5 from Section 6.5 and lead to the desired assertion. ■

Let us find out when condition (3) will be satisfied. Since either of the functions

$$c_1(t) = \sup_{(x,\,v)} \int_D \big|\,K(x,v,w,t)\big|\,dw\,, \qquad c_2(t) = \sup_{(x,\,w)} \int_D \big|\,K(x,v,w,t)\big|\,dv$$

is finite, the operator $K(t)$ is bounded in the space $X = L_r(\,\Omega\times D\,)$, so that the estimate is valid:

$$\|\,K(t)\| \leq c_1(t)^{1/r}\,c_2(t)^{1-1/r}.$$

Therefore, condition (3) holds true if the function $K = K(x,v,v',t)$ and its derivative meet the requirements

$$K\,,\ \frac{\partial K}{\partial t} \in C(\bar\Omega\times[0,T];\ L_\infty(D)\times L_1(D))\,,$$

$$K\,,\ \frac{\partial K}{\partial t} \in C(\bar\Omega\times[0,T];\ L_1(D)\times L_\infty(D))\,.$$

By imposing the stronger restriction on the function ω that

(1′) *w, $\big(v,\ \mathrm{grad}_x\,w\big) \in L_q(\,\Omega\times D\,)$ and $w(x,v) = 0$ at all points $x\in\partial\,\Omega$, $v\in D$, satisfying the condition $(v,\ n_x) < 0$;*

we are able to show by integrating by parts that the operator B is in line with condition (6.4.5). To ensure the solvability of the inverse problem at hand under rather mild restrictions on the coefficients of the equations and the input data, we take for granted, in addition, that

(2′) $\sigma\in C\big([0,T];\ L_\infty(\,\Omega\times D\,)\big)$;

(3′) $K(t)\in C\big([0,T];\ \mathcal{L}\big(\,L_r(\,\Omega\times D\,)\big)\big)$;

(4′) $\Phi \in C([0,T];\ L_r(\Omega \times D))$;

$$\int\limits_{\Omega\times D} \Phi(x,v,t)\,w(x,v)\;dx\;dv \neq 0\,, \qquad 0 \le t \le T\,,$$

(5′) $F \in C([0,T];\ L_r(\Omega \times D))$;

(6′) $u_0 \in L_r(\Omega \times D)$;

(7′) $\psi \in C^1[0,T]\,, \qquad \psi(0) = \int\limits_{\Omega\times D} u_0(x,v)\,w(x,v)\;dx\;dv\,.$

After that, applying Theorem 6.4.1 of Section 6.4 yields the following proposition.

Corollary 9.5.1 *Let conditions* (1′)-(2′) *hold for* $r > 1$ *and* $q = r/(r-1)$. *Then a solution* u, p *of the inverse problem* (9.5.1)-(9.5.5) *exists and is unique in the class of functions*

$$u \in C([0,T];\ L_r(\Omega \times D))\,, \qquad p \in C[0,T]\,.$$

When the additional conditions (2)-(3), (6) and

(8) $\Phi \in C^1([0,T];\ L_r(\Omega\times D))$, $F \in C^1([0,T];\ L_r(\Omega\times D))$, $\psi \in C^2[0,T]$

are imposed, Theorem 6.4.2 of Section 6.4 ensures the inverse problem solution to be continuously differentiable.

Corollary 9.5.2 *Let conditions* (1′)-(7′) *hold for* $r > 1$ *and* $q = r/(r-1)$. *Under the additional conditions* (2)-(3), (6) *and* (8) *a solution* u, p *of the inverse problem* (9.5.1)-(9.5.5) *exists and is unique in the class of functions*

$$u \in C^1([0,T];\ L_r(\Omega \times D))\,, \qquad p \in C^1[0,T]\,.$$

Likewise, the inverse problem of finding the source f is meaningful for the case when

(9.5.6) $$f(x,v,t) = \Phi(x,v,t)\,p(v,t) + F(x,v,t)\,,$$

where the coefficient $p(v,t)$ is unknown, while the functions $\Phi(x,v,t)$ and $F(x,v,t)$ are available. The subsidiary information is that

$$(9.5.7) \qquad \int_D u(x,v,t)\, w(x)\, dx = \psi(v,t)\,, \qquad v \in D\,, \qquad 0 \le t \le T\,.$$

The well-founded choice of the spaces for this problem is connected with $X = L_r(\Omega \times D)$ and $Y = L_r(D)$. We confine ourselves to the case when

$$(9.5.8) \qquad w \in \overset{\circ}{W}{}_q^1(\Omega)\,, \qquad q = r/(r-1)\,, \qquad r > 1\,.$$

With this relation in view, the operator

$$B\, u = \int_\Omega u(x,v)\, w(x)\, dx$$

satisfies condition (6.4.5). For this reason it suffices to require for further reference to Theorem 6.4.1 that

$$(9.5.9) \qquad \begin{cases} \sigma \in C([0,T];\, L_\infty(\Omega \times D))\,, \\ K(t) \in C([0,T];\, \mathcal{L}(L_r(\Omega \times D)))\,, \end{cases}$$

$$(9.5.10) \quad \Phi \in C([0,T];\, L_{r,\infty}(\Omega \times D))\,, \qquad \int_\Omega \Phi(x,v,t)\, w(x)\, dx \ge \delta > 0\,,$$

$$(9.5.11) \qquad F \in C([0,T];\, L_r(\Omega \times D))\,,$$

$$(9.5.12) \qquad \begin{cases} u_0 \in L_r(\Omega \times D)\,, \\ \psi \in C^1([0,T];\, L_r(D))\,, \\ \psi(v,0) = \int_\Omega u_0(x,v)\, w(x)\, dx\,. \end{cases}$$

All this enables us to obtain the following result.

Corollary 9.5.3 *Under conditions* (9.5.8)–(9.5.12) *a solution u, p of the inverse problem* (9.5.1)–(9.5.3), (9.5.6)–(9.5.7) *exists and is unique in the class of functions*

$$u \in C([0,T];\, L_r(\Omega \times D)), \qquad p \in C([0,T];\, L_r(D)).$$

If one assumes, in addition, that

$$\left\{\begin{array}{l} \sigma \in C^1([0,T];\, L_\infty(\Omega \times D)), \\ K(t) \in C^1([0,T];\, \mathcal{L}(L_r(\Omega) \times D)), \end{array}\right. \tag{9.5.13}$$

$$\left\{\begin{array}{l} \Phi \in C^1([0,T];\, L_{r,\infty}(\Omega \times D)), \\ \int\limits_\Omega \Phi(x,v,t)\, w(x)\, dx \geq \delta > 0, \end{array}\right. \tag{9.5.14}$$

$$F \in C^1([0,T];\, L_r(\Omega \times D)), \tag{9.5.15}$$

$$\left\{\begin{array}{l} u_0,\ (v,\, \operatorname{grad}_x u_0) \in L_r(\Omega \times D), \\ \psi \in C^2([0,T];\, L_r(D)), \end{array}\right. \tag{9.5.16}$$

$$\psi(v,0) = \int\limits_\Omega u_0(x,v)\, w(x)\, dx, \qquad v \in D, \tag{9.5.17}$$

$$u_0(x,v) = 0 \tag{9.5.18}$$

for all $x \in \partial\Omega$ and $v \in D$, satisfying the condition

$$(v,\, n_x) < 0,$$

then Theorem 6.4.2 applies equally well, due to which this solution is continuously differentiable in t.

Corollary 9.5.4 *Let conditions* (9.5.8) *and* (9.5.13)–(9.5.18) *hold. Then a solution u, p of the inverse problem* (9.5.1)–(9.5.3), (9.5.6)–(9.5.7) *exists and is unique in the class of functions*

$$u \in C^1([0,T];\, L_r(\Omega \times D)), \qquad p \in C^1([0,T];\, L_r(D)).$$

9.6 Linearized Bolzman equation

A key role in the kinetic theory of gases is played by the function $f(x, v, t)$ characterizing the **distribution of particles** with respect to the coordinates $x \in \Omega$, $\Omega \subset \mathbf{R}^3$, and velocity $v \in \mathbf{R}^3$. The meaning of this function is the distribution density of particles in the phase space $\Omega \times \mathbf{R}^3$ at time t. Once we pass to the limit under small densities, known as the Bolzman limit, the function f will satisfy the **Bolzman equation**

$$\frac{\partial f}{\partial t} + \big(v, \operatorname{grad}_x f\big) = Q(f, f),$$

where the symbol $Q(f, f)$ designates the **integral of collisions** given at a point (x, v, t) by the collection of formulae

$$Q(f, f) = \int\limits_{S \times \mathbf{R}^3} K(v, v_1, e) \left[f_1' f' - f_1 f\right] dS\, dv_1 ,$$

$$S = \left\{ e \in \mathbf{R}^3 \colon |e| = 1 \right\},$$

$$f = f(x, v, t), \qquad f_1 = f(x, v_1, t),$$

$$f' = f(x, v', t), \qquad f_1' = f(x, v_1', t),$$

$$v' = \frac{1}{2}\big(v + v_1 - |v_1 - v|\, e\big),$$

$$v_1' = \frac{1}{2}\big(v + v_1 + |v_1 - v|\, e\big),$$

$$K(v, v_1, e) = |v - v_1|\, \sigma\big(|v - v_1|\, e\big)$$

(for more detail see Carleman (1960), Cercignani (1975), Maslova (1978, 1985), Lanford et al. (1983)). The **differential scattering cross-section** $\sigma(r, e)$ is defined on $(0, +\infty) \times S$ and is uniquely determined by the law of the interaction between particles. For any constant values of parameters ρ, k, T the **Maxwell distribution**

$$M(v) = \rho\,\big(2\pi k T\big)^{-3/2} \exp\big(-v^2/2kT\big)$$

is the exact solution to the Bolzman equation and describes an equilibria with density ρ and temperature T. Here k stands for the universal gas constant. In the case of smallness of the relative deviation of the distribution density from the equilibrium state the Bolzman equation can be linearized by merely setting

$$f(x, v, t) = M(v)\,\big(1 + u(x, v, t)\big)\,.$$

A final result of omitting the nonlinear terms is the so-called **linearized Bolzman equation**

$$\frac{\partial u}{\partial t} + \big(v,\ \mathrm{grad}_x\, u\big) = \Gamma(u)\,,$$

where $\Gamma(u)$ refers to the **linearized integral of collisions**

$$\Gamma(u) = \int\limits_{S\times\mathbf{R}^3} M(v_1)\,K(v, v_1, e)\,\big[\,u_1' + u' - u_1 - u\,\big]\ dS\ dv_1\,.$$

The function $\Gamma(u)$ can be expressed by the difference

$$\Gamma(u) = L\,u - \nu(v)\,u\,,$$

where

$$L\,u = \int\limits_{S\times\mathbf{R}^3} M(v_1)\,K(v, v_1, e)\,\big[\,u_1' + u' - u_1\,\big]\ dS\ dv_1\,,$$

$$\nu(v) = \int\limits_{S\times\mathbf{R}^3} M(v_1)\,K(v, v_1, e)\ dS\ dv_1\,.$$

When the differential cross-section of scattering is known in advance, the operator L and the **collision frequency** $\nu(v)$ can uniquely be determined. However, the law of the interaction between particles is actually unknown and the functions L and ν will be involved in modeling problems as further developments occur. In this context, there is a need for statements of inverse problems relating to the Bolzman equation and, in particular, there arises naturally the problem of determining the frequency of collisions. By standard techniques the question of uniqueness of recovering the coefficients of the equation reduces to the problem of determining the nonhomogeneous term of this equation and, in view of this, necessitates involving the **nonhomogeneous Bolzman equation**

$$\frac{\partial u}{\partial t} + \big(v,\ \mathrm{grad}_x\, u\big) = L\,u - \nu(v)\,u + F(x, v, t)\,. \tag{9.6.1}$$

In what follows careful analysis of the Bolzman equation is based on the **invariants of collisions**

$$\psi_0(v) = 1\,, \quad \psi_1(v) = v_1\,, \quad \psi_2(v) = v_2\,, \quad \psi_3(v) = v_3\,, \quad \psi_4(v) = v^2.$$

We note in passing that the integral of collisions $\Gamma(u)$ with the specified functions becomes zero.

Let Ω be a strictly convex, bounded and closed domain with Lyapunov's boundary in the space $\mathbf{R}^3$. Set $D = \partial\,\Omega \times \mathbf{R}^3$, where n_x is a unit internal normal to the boundary $\partial\,D$ at point x, and introduce the manifolds

$$D^+ = \big\{(x,\, v) \in D\colon\ (v,\, n_x) > 0\big\}\,,$$

$$D^- = \big\{(x,\, v) \in D\colon\ (v,\, n_x) < 0\big\}\,.$$

In the sequel the symbols u^+ and u^- will stand for the traces of the function u on D^+ and D^-, respectively. For the purposes of the present section we have occasion to use $H = L_{2,\,M(v)}\big(\Omega \times \mathbf{R}^3\big)$, $H^\pm = L_{2,\,\pm M(v)\,(v\,,\,n_x)}\big(D^\pm\big)$, $\mathcal{H} = L_{2,\,M(v)}(\mathbf{R}^3)$ and in the space of velocities the **operator of reflection** J acting in accordance with the rule

$$J\colon\ u(x,v) \ \mapsto\ u(x,-v)\,.$$

Let P be the orthogonal projection in H^- onto the orthogonal complement to the subspace of all functions depending only on the variable x.

In further development our starting point is the direct problem. Aside from equation (9.6.1), the determination of the function u necessitates imposing the initial and boundary conditions

$$u(x,v,0) = u_0(x,v)\,, \qquad x \in \Omega\,, \qquad v \in \mathbf{R}^3, \tag{9.6.2}$$

$$u^+ = R\,u^-, \tag{9.6.3}$$

where R is a bounded linear operator from the space H^- into the space H^+.

Also, we take for granted that

(A1) *the operator L is compact and self-adjoint in the space H and*

$$\nu(v) \geq 0\,, \qquad\qquad \nu(v) \in \mathcal{H}\,.$$

If u, $\nu\, u \in \mathcal{H}$, then $\big(\Gamma(u),\, u\big) \leq 0$. In that case

$$\big(\Gamma(u),\, u\big) = 0$$

if and only if u gives a linear combination of the collision invariants;

(A2) *the boundary operator R meets either of the following requirements:*

(a) $\| R \| < 1$;

(b) $R\,1^+ = 1^-$; *for some* $\varepsilon > 0$ *and all elements* $u^- \in H^-$ *the inequality*

$$\| R u^- \|^2 + \varepsilon \, \| P u^- \| \leq \| u^- \|^2$$

holds and the operator $R J$ *is self-adjoint in the space* H^+.

Note that condition (A1) will be satisfied, for example, for the models of hard spheres and power potentials with angular cut-off. Item (a) of condition (A2) confirms the absence of the conservation laws in the interaction between particles with the boundary of the domain, while item (b) reflects the situation in which these laws are met in part. To be more specific, the equality $R\,1^- = 1^+$ expresses the condition of **thermodynamic equilibrium** of the boundary $\partial\,\Omega$, the **reciprocity law** is ensured by self-adjointness of the operator $R J$ and the relation

$$\| R u^- \|^2 + \varepsilon \, \| P u^- \|^2 \leq \| u^- \|^2$$

reveals the dissipative nature of the scattering process of particles on the boundary $\partial\,\Omega$ (for more detail see Cercignani (1975) and Guirand (1976)).

It will be sensible to introduce in the space H the operator

$$A u = -\big(v, \, \mathrm{grad}_x \, u \big) + \Gamma(u) \, ,$$

whose domain is defined by the **resolvent equation** of the transport operator

$$\text{(9.6.4)} \qquad \begin{cases} \big(v, \, \mathrm{grad}_x \, u \big) + \big(\nu(v) + \lambda \big) \, u = \varphi \, , \\ u^+ = \psi \, , \end{cases}$$

where $\lambda > 0$, $\varphi \in H$ and $\psi \in H^+$. It is known from the works of Cercignani (1975) and Maslova (1985) that a solution u of problem (9.6.4) exists and is representable by

$$u = U \varphi + W \psi \, ,$$

where $U \in \mathcal{L}(H)$ and $W \in \mathcal{L}\big(H^+, \, H \big)$. By the same token,

$$\big(\nu(v) + \lambda \big)^{-1/2} \big(v, \, \mathrm{grad}_x \, u \big) \in H \, ,$$

$$\big(\nu(v) + \lambda \big)^{1/2} u \in H \, ,$$

$$\| \big(U \varphi + W \psi \big)^- \| \leq \frac{1}{\lambda} \, \| \varphi \| + \| \psi \| \, .$$

We need a linear manifold in the space H defined as follows:

$$H_0 = \{ u\colon\ u = U\varphi + W\psi\,,\ \varphi \in H\,,\ \psi \in H^+ \}\,.$$

Being a solution of problem (9.6.4), the function $u = u(\lambda, \varphi, \psi)$ is subject to the relation

$$u(\mu, \varphi, \psi) = u(\lambda,\ \varphi + (\lambda - \mu)\, u(\mu, \varphi, \psi),\ \psi)\,,$$

whence it follows that the manifold H_0 does not depend on λ. Furtheremore, set

$$\mathcal{D}(A) = \{ u \in H_0\colon\ u^+ = R\,u^- \}\,.$$

Under these agreements the operator A appears to be maximal dissipative and, therefore, generates a strongly continuous contraction semigroup $V(t)$ in the space H (for more detail see Guirand (1976)).

Thus, if the system (9.6.1)–(9.6.3) is treated as an **abstract Cauchy problem** of the form

$$\begin{cases} u_t = A\,u + F(t)\,, & 0 \le t \le T\,, \\ u(0) = u_0\,, \end{cases}$$

under the constraints

$$u_0 \in \mathcal{D}(A)\,, \qquad F \in C^1([0, T];\ H) + C([0, T];\ \mathcal{D}(A))\,,$$

then a solution u of the initial boundary value problem associated with Bolzman equation exists and is unique in the class of functions

$$u \in C^1([0, T];\ H)\,.$$

Moreover, this solution is given by the formula

$$u(t) = V(t)\,u_0 + \int\limits_0^t V(t - s)\,F(s)\,ds\,.$$

In this direction, we should raise the question of recovering the nonhomogeneous term of equation (9.6.1) provided that one of the following decompositions takes place:

$$F(x, v, t) = \Phi(x, v, t)\,p(t) + \mathcal{F}(x, v, t)\,, \tag{9.6.5}$$

$$F(x, v, t) = \Phi(x, v, t)\,p(v, t) + \mathcal{F}(x, v, t)\,, \tag{9.6.6}$$

$$F(x, v, t) = p(x, v) + \mathcal{F}(x, v, t)\,, \tag{9.6.7}$$

where, respectively, the functions $p(t)$, $p(v,t)$ and $p(x,v)$ are unknown. The subsidiary information is needed to recover them and is provided by the **overdetermination conditions**

$$\int\limits_{\Omega\times\mathbf{R}^3} u(x,v,t)\, w(x,v)\; dx\; dv \;=\; \psi(t)\,, \qquad 0\leq t\leq T\,, \tag{9.6.8}$$

$$\int\limits_{\Omega} u(x,v,t)\, w(x)\; dx \;=\; \psi(v,t)\,, \quad v\in\mathbf{R}^3, \quad 0\leq t\leq T\,, \tag{9.6.9}$$

$$u(x,v,T) \;=\; u_1(x,v)\,, \quad x\in\Omega\,, \quad v\in\mathbf{R}^3. \tag{9.6.10}$$

Let us first dwell on the inverse problem with the supplementary conditions (9.6.5) and (9.6.8). Theorem 6.5.5 of Section 6.5 is used in the spaces $X = H$ and $Y = \mathbf{R}$ with further reference to the operator B being now a functional and acting in accordance with the rule

$$B\,u \;=\; \int\limits_{\Omega\times\mathbf{R}^3} u(x,v)\, w(x,v)\; dx\; dv\,.$$

The operator B so defined is bounded from the space X into the space Y under the agreement that the function

$$w\;\in\; L_{2,\,M(v)^{-1}}\big(\,\Omega\times\mathbf{R}^3\big)\,.$$

Representation (6.5.35) includes the following members:

$$L_1 \;=\; 0\,,$$

$$L_2(t)\,p \;=\; \Phi(x,v,t)\;p\,,$$

$$F(t) \;=\; \mathcal{F}(x,v,t)\,.$$

It is plain to show that the remaining conditions of Theorem 6.5.5 from Section 6.5 are a corollary of the relations

$$\left\{\begin{array}{l} \Phi\,,\;\big(\,v,\;\mathrm{grad}_x\,\Phi\,\big)+\nu\;\Phi\;\in\;C^1\big([0,\,T];\;H\,\big)\,,\\ \mathcal{F}\;\in\;C^1\big([0,\,T];\;H\,\big)\,, \end{array}\right. \tag{9.6.11}$$

$$u_0\,,\;\big(\,v,\;\mathrm{grad}_x\,u_0\,\big)+\nu\,u_0\;\in\;H\,, \qquad \psi\;\in\;C^1[0,\,T]\,, \tag{9.6.12}$$

$$\int\limits_{\Omega \times \mathbf{R}^3} u_0(x,v)\, w(x,v)\, dx\, dv = \psi(0)\,, \quad \Phi^+ = R\,\Phi^-\,, \tag{9.6.13}$$

$$\int\limits_{\Omega \times \mathbf{R}^3} \Phi(x,v,t)\, w(x,v)\, dx\, dv \neq 0\,, \quad 0 \le t \le T\,. \tag{9.6.14}$$

In accordance with what has been said, the following assertion is established.

Corollary 9.6.1 *Let conditions* (9.6.11)–(9.6.14) *hold and*

$$w \in L_{2,\, M(v)^{-1}}(\Omega \times \mathbf{R}^3)\,.$$

One assumes, in addition, that conditions ($A1$)–($A2$) *are fulfilled. Then a solution* u, p *of the inverse problem* (9.6.1)–(9.6.3), (9.6.5), (9.6.8) *exists and is unique in the class of functions*

$$u \in C^1([0,T];\, H)\,, \qquad p \in C[0,T]\,.$$

The inverse problem with the supplementary conditions (9.6.6), (9.6.9) can be resolved in the same manner as before. This amounts to adopting an abstract scheme for further reasoning and applying Theorem 6.5.5 of Section 6.5 to the spaces $X = H$ and $Y = \mathcal{H}$. Setting, by definition,

$$B\,u = \int\limits_{\Omega} u(x,v)\, w(x)\, dx$$

and letting $w \in L_2(\Omega$, we might involve the Cauchy–Schwartz inequality, whose use permits us to derive the estimate

$$\| B \| \le \| w \|_{L_2(\Omega)}\,,$$

giving the inclusion $B \in \mathcal{L}(H, \mathcal{H})$. On account of Theorem 6.5.5 from Section 6.5 it remains to require that

$$\begin{cases} \Phi,\ (v,\, \mathrm{grad}_x\, \Phi) + \nu\, \Phi \in C^1([0,T];\, H)\,, \\ \mathcal{F} \in C^1([0,T];\, H)\,, \end{cases} \tag{9.6.15}$$

$$\begin{cases} u_0\,,\ (v,\, \mathrm{grad}_x\, u_0) + \nu\, u_0 \in H\,, \\ \psi \in C^1([0,T];\, \mathcal{H})\,, \end{cases} \tag{9.6.16}$$

$$\begin{cases} \int\limits_{\Omega} u_0(x,v)\, w(x)\, dx = \psi(v,0)\,, \\ v \in \mathbf{R}^3, \qquad \Phi^+ = R\,\Phi^-\,, \end{cases} \tag{9.6.17}$$

$$\begin{cases} \left| \int\limits_{\Omega} \Phi(x,v,t)\, w(x)\, dx \right| \ge \delta > 0\,, \\ v \in \mathbf{R}^3, \qquad 0 \le t \le T\,, \end{cases} \tag{9.6.18}$$

which are needed below to motivate one useful result.

Corollary 9.6.1 *Let relations* (9.6.15)–(9.6.18) *occur, the function* $w \in L_2(\Omega)$ *and conditions* $(A1)$–$(A2)$ *hold. Then a solution* u, p *of the inverse problem* (9.6.1)–(9.6.3), (9.6.6), (9.6.9) *exists and is unique in the class of functions*

$$u \in C^1([0,T];\, H)\,, \qquad p \in C([0,T];\, \mathcal{H})\,.$$

As stated above, the question of uniqueness in the inverse problem of recovering the coefficients of the equation reduces to the problem of determining the nonhomogeneous terms of this equation. We are going to show this passage in one possible example of recovering the collision frequency $\nu(v)$. The subsidiary information here is provided by (9.6.9). Given two solutions $u_1(x,v,t)$, $\nu_1(v)$ and $u_2(x,v,t)$, $\nu_2(v)$ of the same inverse problem (9.6.1)–(9.6.3), (9.6.9), the main goal of subsequent manipulations is to derive the governing equation with the supplementary conditions for the new functions $u = u_1 - u_2$ and $p = \nu_2 - \nu_1$, which complement our study. This can be done writing equations (9.6.1) twice for the pairs $\{u_1,\, \nu_1\}$ and $\{u_2,\, \nu_2\}$, respectively, and then subtracting one from another. The outcome of this is

$$\frac{\partial u}{\partial t} + \big(v,\, \mathrm{grad}_x\, u\big) = L\,u - \nu_1(v)\,u + u_2(x,v,t)\,p(v)\,. \tag{9.6.19}$$

A similar procedure will work for the derivation of the initial and boundary conditions

$$u(x,v,0) = 0\,, \qquad u^+ = R\,u^- \tag{9.6.20}$$

and the subsidiary information

$$\int\limits_{\Omega} u(x,v,t)\,w(x)\,dx = 0\,. \tag{9.6.21}$$

It is worth noting here that relations (9.6.19)–(9.6.21) with regard to the unknowns u and p constitute the inverse problem of the type (9.6.19)–(9.6.21) with zero input data. If the inverse problem thus obtained has no solutions other than a trivial solution, then we acquire the clear indication that a solution of the inverse problem of recovering the collision frequency is unique. Conditions (9.6.15)–(9.6.18) immediately follow from the set of equalities

$$\mathcal{F} = 0\,, \qquad \Phi = u_2(x,v,t)\,, \qquad u_0 = 0\,, \qquad \psi = 0\,.$$

Observe that the inclusions

$$\Phi, \ \big(v, \ \mathrm{grad}_x \, \Phi \big) + \nu \, \Phi \in \mathcal{C}\big([0, \, T]; \ H \big)$$

and the relation

$$\Phi^{+} = R \, \Phi^{-}$$

occur. This is due to the fact that the function $\Phi = u_2$ gives a solution of the boundary value problem (9.6.1)–(9.6.3). So, it remains to verify the fulfilment of condition (9.6.18). Indeed, by assumption, the pair $\{ u_2, \, \nu_2 \}$ solves the inverse problem at hand and, therefore, the function u_2 satisfies condition (9.6.9), thereby establishing the relationships

$$\int\limits_{\Omega} \Phi(x, v, t) \, w(x) \; dx \; = \; \int\limits_{\Omega} u_2(x, v, t) \, w(x) \; dx \; = \; \psi(v, t)$$

and the equivalence between (9.6.18) and the bound

$$(9.6.22) \qquad | \, \psi(v, t)| \, \geq \, \delta \, > \, 0 \, , \qquad v \in \mathbf{R}^3, \qquad 0 \leq t \leq T \, .$$

Corollary 9.6.3 *Let conditions* (A1)–(A2) *hold, the function* $w \in L_2(\Omega)$ *and let estimate* (9.6.22) *be valid. Then the inverse problem* (9.6.1)–(9.6.3), (9.6.9) *of recovering a pair of the functions* u *and* ν *has no more than one solution in the class of functions*

$$u \, \in \, \mathcal{C}^1 \big([0, \, T]; \, H \big) \, , \qquad \nu \geq 0 \, , \qquad \nu \, \in \, \mathcal{H} \, .$$

We proceed to the study of the inverse problem at hand with the supplementary conditions (9.6.7) and (9.6.10) as usual. This amounts to treating it as an abstract one posed completely by relations (7.1.1)–(7.1.4) with $\Phi(t) \equiv I$ incorporated. In spite of the fact that such problems have been already under consideration in Section 7.1, the final conclusions are not applicable to this case directly. The obstacle involved is connected with the integral of collisions possessing an eigenvalue equal to zero. If this happens, the characteristic subspace consists of the invariants of collisions

$$u(x, v) = \sum_{m=0}^{4} c_m(x) \, \psi_m(x) \, ,$$

where $c_m(x) \in L_2(\Omega)$ for $0 \leq m \leq 4$. If the boundary operator complies with item (b) of condition (A2), then the function $\psi_0(x, v) = 1$ gives the operator A eigenfunction associated with zero eigenvalue. If the semigroup $V(t)$ is generated by the operator A, then

$$V(t)\,\psi_0 = \psi_0$$

for all $t \geq 0$. It is worth noting here that on this semi-axis the value β in estimate (7.1.24) cannot be negative. What is more, item (a) of condition (A2) is out of the question. Although in this case zero is none of the eigenvalues of the operator A, it may enter its continuous spectrum. Therefore, by the theorem on the spectrum mapping the number $\lambda = 1$ is contained in the set of elements of the spectrum of the operator A. Hence estimate (7.1.24) with constants $M = 1$ and $\beta < 0$ fails to hold, since the inequality $\| V(t)\| < 1$ yields the inclusion $1 \in \rho(V(t))$. In order to overcome these difficulties, it is reasonable to impose the extra restriction that the collision frequency is bounded away from zero:

$$(A3) \quad \nu(v) \geq \nu_0 > 0\,, \quad v \in \mathbf{R}^3.$$

This condition characterizes the hard interactions and is satisfied in various models of hard spheres and **power potentials** $U(x) = k/|\,x\,|^n$ with angular cut-off having an exponent $n > 4$ (for more detail see Cercignani (1975)).

We are first interested in the case when conditions (A1)–(A2) hold together with item (a) of condition (A2). Then all the elements of the spectrum of the operator A belong to the half-plane $\left\{\operatorname{Re}\lambda \leq -\nu_0\right\}$ and the semigroup $V(t)$ obeys estimate (7.1.24) with $M = 1$ and $\beta < 0$ (see Guirand (1976)), due to which it is possible to apply Corollary 7.1.3 of Section 7.1. Further passage to formulae (7.1.33) permits one to obtain a solution in the explicit form

$$p = \big(V(T) - I\big)^{-1} A\, g\,,$$

where the element g is defined by relation (7.1.11). The input data become admissible for the inverse problem (9.6.1)–(9.6.3), (9.6.7), (9.6.10) under the constraints

$$(9.6.23) \qquad u_0\,,\ u_1\,,\ \big(v,\, \operatorname{grad}_x u_0\big) + \nu\, u_0\,,\ \big(v,\, \operatorname{grad}_x u_1\big) + \nu\, u_1 \in H\,,$$

$$(9.6.24) \qquad u_0^+ = R\,u_0^-\,, \qquad u_1^+ = R\,u_1^-\,,$$

$$(9.6.25) \qquad \mathcal{F} \in C^1\big([0, T];\, H\big)\,.$$

These investigations allow to formulate the following corollary.

Corollary 9.6.4 *Let conditions* (A1)–(A3) *and* (9.6.23)–(9.6.25) *hold together with item* (a) *of condition* (A2). *Then a solution* u, p *of the inverse problem* (9.6.1)–(9.6.3), (9.6.7), (9.6.10) *exists and is unique in the class of functions*

$$u \in C^1([0, T]; H), \qquad p \in H.$$

The second case of interest is connected with the fulfilment of conditions (A1), (A3) and item (b) of condition (A2), due to which the operator spectrum in its part within the half-plane $\operatorname{Re}\lambda > -\nu_0$ consists of the unique point $\lambda = 0$, which is a simple eigenvalue. The characteristic subspace $\mathbf{N}$, the kernel of the operator A, contains only constants and reduces the operator A. The restriction $V_1(t)$ of the semigroup $V(t)$ on the subspace H_1 being an orthogonal complement of $\mathbf{N}$ in the space H is exponentially decreasing as follows:

$$\| V_1(t)\| \leq \exp(-w_1 t), \qquad w_1 > 0,$$

(see Guirand (1976)). We note in passing that equation (7.1.9) acquires the form

$$\int_0^T V(T-s)\, p\, ds = g. \tag{9.2.26}$$

A simple observation may be of help in justifying that the subspaces $\mathbf{N}$ and H_1 reduce the operator A and thereby the semigroup $V(t)$. In view of this, we are able to split up equation (9.6.26) by merely setting $p = p_0 + p_1$ and $g = g_0 + g_1$ with $p_0, g_0 \in \mathbf{N}$ and $p_1, g_1 \in H_1$ into the following couple:

$$\int_0^T V(T-s)\, p_0\, ds = g_0, \tag{9.6.27}$$

$$\int_0^T V(T-s)\, p_1\, ds = g_1. \tag{9.6.28}$$

The unique solvability of these equations is equivalent to that of equation (9.6.26).

In later discussions equation (9.6.27) comes first. Since $A p_0 = 0$, we obtain for each $t \geq 0$

$$V(t)\, p_0 = p_0.$$

This provides support for the view that (9.6.27) is equivalent to the equality

$$T\,p_0 = g_0\,.$$

Therefore, equation (9.6.27) is uniquely solvable and in this case $p_0 = g_0/T$. Passing now to equation (9.6.28) we write down it in the form

$$\int_0^T V_1(T-s)\,p_1\,ds = g_1\,. \tag{9.6.29}$$

If the inclusions $g \in \mathcal{D}(A)$ and $g_0 \in \mathcal{D}(A)$ occur, then the relation

$$g_1 \in \mathcal{D}(A)$$

takes place. For each $t > 0$ the estimate

$$\|V_1(t)\| < 1$$

holds and, therefore,

$$1 \in \rho\big(V_1(t)\big)$$

and the theorem on the spectrum mapping yields the inclusion $0 \in \rho(A_1)$, where A_1 refers to a part of the operator A acting in the space H_1 and generating the semigroup $V_1(t)$. By Corollary 7.1.4 of Section 7.1 with regard to the semigroup $V_1(t)$ and the function $\Phi(t) \equiv I$ both equation (9.6.29) is uniquely solvable. Thus, we arrive at the following statement.

Corollary 9.6.5 *Let conditions* (A1), (A3), (9.6.23)–(9.6.25) *hold together with item* (b) *of condition* (A2). *Then a solution* u, p *of the inverse problem* (9.6.1)–(9.6.3), (9.6.7), (9.6.10) *exists and is unique in the class of functions*

$$u \in C^1([0, T];\, H)\,, \qquad p \in H\,.$$

Of importance is the Cauchy problem for the Bolzman equation (9.6.1) in which $\Omega = \mathbf{R}^3$ and the boundary conditions are omitted. All this enables one to simplify its statement a little bit. A good choice of the spaces H and $\mathcal{H}$ is carried out in just the same way as we did before. Provided condition (A1) holds, it is reasonable to introduce in the space H the operator

$$A\,u = -\big(v,\, \operatorname{grad}_x u\big) + \Gamma(u)$$

with the domain

$$\mathcal{D}(A) = \big\{\, u \in H\colon\ \big(v,\, \operatorname{grad}_x u\big) \in H,\ \nu(v)\,u \in H \,\big\}\,.$$

Theorem 9.6.1 *The operator A so defined is the generator of a strongly continuous semigroup.*

Proof The main idea behind proof is to involve a pair of the operators

$$A_1\, u \;=\; -\big(\, v,\, \operatorname{grad}_x u\,\big)\,, \qquad \mathcal{D}(A_1) \;=\; \big\{\, u \in H\colon\; \big(\, v,\, \operatorname{grad}_x u\,\big) \in H\,\big\}\,,$$

$$A_2\, u \;=\; -\nu(v)\, u\,, \qquad \mathcal{D}(A_2) \;=\; \big\{\, u \in H\colon\; \nu(v)\, u \in H\,\big\}\,,$$

with some inherent properties: the operator A_1 generates a unitary **group of translations**

$$\big[\, V_1(t)\, u\,\big]\,(x,\, v) \;=\; u\big(\, x - v\, t,\, v\,\big)\,,$$

while the operator A_2 is a generator of the **multiplication semigroup**

$$\big[\, V_2(t)\, u\,\big]\,(x,\, v) \;=\; u(x, v)\, \exp\big(-\nu(v)\, t\,\big)\,.$$

From such reasoning it seems clear that the operators $V_1(t)$ and $V_2(s)$ are commuting for all $t \geq 0$, $s \geq 0$. By **Trotter's theorem** their product

$$\big[\, V(t)\, u\,\big]\,(x,\, v) \;=\; u\big(\, x - v\, t,\, v\,\big)\, \exp\big(-\nu(v)\, t\,\big)$$

is just a strongly continuous semigroup generated by the closure of the operator $A_3 = A_1 + A_2$.

The next step is to examine the **resolvent equation** for the operator A_3 of the form

$$\big(\, v,\, \operatorname{grad}_x u\,\big) + \big(\, \nu(v) + \lambda\,\big)\, u \;=\; \varphi\,, \tag{9.6.30}$$

whose solution is given for $\lambda > 0$ by the formula

$$u(x, v) \;=\; \int\limits_{-\infty}^{0} \varphi(x + v\, t,\, v)\, \exp\big(\big(\, \nu(v) + \lambda\,\big)\, t\,\big)\; dt\,,$$

which is followed on the basis of Young's inequality by the estimate

$$\|\, u(\,\cdot\,,\, v\,)\|_{L_2(\mathbf{R}^3)} \leq \big(\, \nu(v) + \lambda\,\big)^{-1}\, \|\, \varphi(\,\cdot\,,\, v\,)\|_{L_2(\mathbf{R}^3)}\,.$$

Let us multiply both sides of the preceding estimate by $M(v)^{1/2}\, \nu(v)$ and square then the resulting expressions. In light of the obvious bounds

$$0 \;\leq\; \nu\,(\nu + \lambda\,)^{-1} \leq\; 1$$

we can integrate the inequality obtained over $\nu \in \mathbf{R}^3$. As a final result we get

$$\|\, \nu\, u\,\|_H \;\leq\; \|\, \varphi\,\|_H\,,$$

which serves to motivate that $\nu u \in H$ for any function $\varphi \in H$ involved in equation (9.6.30) and

$$(v, \operatorname{grad}_x u) \in H .$$

This provides support for the view that the operator A_3 is closed and generates the semigroup $V(t)$. The domain of the operator A_3 is equal to $\mathcal{D}(A_1) \cap \mathcal{D}(A_2)$, thereby coinciding with $\mathcal{D}(A)$. Since the operator A differs from A_3 by the bounded operator L, it is straightforward to verify that A generates a strongly continuous semigroup as well. This completes the proof of Theorem 9.6.1. ■

Theorem 9.6.1 may be of help in setting up the Cauchy problem for the Bolzman equation

$$\text{(9.6.31)} \qquad \frac{\partial u}{\partial t} + \big(v, \operatorname{grad}_x u\big) = L u - \nu u + F ,$$

$$x \in \mathbf{R}^3, \quad v \in \mathbf{R}^3, \quad 0 \le t \le T ,$$

$$\text{(9.6.32)} \qquad u(x, v, 0) = u_0(x, v) ,$$

$$x \in \mathbf{R}^3, \quad v \in \mathbf{R}^3,$$

as an **abstract Cauchy problem**. This can be done joining (9.6.5), (9.6.8) (with $\Omega = \mathbf{R}^3$) and (9.6.31)–(9.6.32). The solvability in question can be established on account of Theorem 6.5.5 from Section 6.5. For the validity of this theorem it is necessary to impose several conditions similar in form to conditions (9.6.11)–(9.6.14) for the above boundary value problem with only difference that the second relation (9.6.13) assigning the boundary values of the function Φ should be excluded from further consideration.

Corollary 9.6.6 *Let condition (A1) hold, the function*

$$w \in L_{2, M(v)^{-1}}(\mathbf{R}^3 \times \mathbf{R}^3)$$

and

$$\Phi , \big(v, \operatorname{grad}_x \Phi\big) + \nu \Phi \in C^1\big([0, T]; H\big), \mathcal{F} \in C^1\big([0, T]; H\big) ;$$

$$u_0 , \big(v, \operatorname{grad}_x u_0\big) + \nu u_0 \in H , \qquad \psi \in C^1[0, T] ,$$

$$\int\limits_{\mathbf{R}^3 \times \mathbf{R}^3} u_0(x, v)\, w(x, v)\, dx\, dv = \psi(0) , \quad \int\limits_{\mathbf{R}^3 \times \mathbf{R}^3} \Phi(x, v, t)\, w(x, v)\, dx\, dv \neq 0 .$$

Then a solution u, p of the inverse problem (9.6.31)–(9.6.32), (9.6.5), (9.6.8) *exists and is unique in the class of functions*

$$u \in C^1([0, T]; H), \qquad p \in C[0, T].$$

By exactly the same reasoning as before it is possible to set up an inverse problem with the aid of relations (9.6.6) and (9.6.9). In that case the validity of Theorem 6.5.5 from Section 6.5 is ensured by conditions (9.6.15)–(9.6.18) with no restrictions on the boundary values of the function Φ.

Corollary 9.6.7 *Let condition* ($A1$) *hold together with conditions* (9.6.15)–(9.6.16), (9.6.18) *and*

$$\int\limits_{\mathbf{R}^3} u_0(x, v)\, w(x)\, dx = \psi(v, 0), \qquad v \in \mathbf{R}^3.$$

Then a solution u, p of the inverse problem (9.6.31)–(9.6.32), (9.6.6), (9.6.9) *exists and is unique in the class of functions*

$$u \in C^1([0, T]; H), \qquad p \in C([0, T]; \mathcal{H}).$$

In concluding this section it is worth mentioning that the assertion established in Corollary 9.7.3 can be carried over to cover on the same footing the Cauchy problem, leading to the following proposition.

Corollary 9.6.8 *Let condition* ($A1$) *hold together with condition* (9.6.22), $w \in L_2(\mathbf{R}^3)$. *Then the inverse problem* (9.6.31)–(9.6.32), (9.6.9) *of recovering a pair of the functions u, ν has no more than one solution in the class of functions*

$$u \in C^1([0, T]; H), \qquad \nu \geq 0, \qquad \nu \in \mathcal{H}.$$

9.7 The system of Navier–Stokes equations

In this section we deal with inverse problems for the system of Navier–Stokes equations. Let Ω be a bounded domain in the space $\mathbf{R}^n$ with boundary $\partial\Omega \in C^2$. We focus our attention on the system consisting of the **linearized Navier–Stokes equations**

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\,\Delta\,\mathbf{u} + \operatorname{grad} p = \mathbf{f}\,, \qquad x \in \Omega\,, \qquad 0 < t < T\,, \tag{9.7.1}$$

and the **incompressibility equation**

$$\operatorname{div} \mathbf{u} = 0\,, \qquad x \in \Omega\,, \qquad 0 < t < T\,. \tag{9.7.2}$$

The direct problem here consists of finding a vector-valued function

$$\mathbf{u}\colon\ \Omega \times [0,\, T] \mapsto \mathbf{R}^n$$

and a scalar-valued function

$$p\colon\ \Omega \times [0,\, T] \mapsto \mathbf{R}\,,$$

satisfying the system (9.7.1)–(9.7.2) with the supplementary boundary and initial conditions

$$\mathbf{u}(x,t) = 0\,, \qquad x \in \partial\Omega\,, \qquad 0 \le t \le T\,, \tag{9.7.3}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x)\,, \qquad x \in \Omega\,, \tag{9.7.4}$$

if the functions $\mathbf{f}$ and $\mathbf{u}_0$ and the coefficient $\nu = \text{const} > 0$ are known in advance. The system (9.7.1)–(9.7.4) permits one to describe the motion of a **viscous incompressible fluid** in the domain Ω, where the velocity of the fluid is well-characterized by the function $\mathbf{u}$, while the function p is associated with the **pressure**. The coefficient ν is called the **coefficient of kinematic viscosity.** The fluid is supposed to be homogeneous with **unit density**. The books and papers by Fujita and Kato (1964), Ladyzhenskaya (1970), Temam (1979), Kato and Fujita (1962) give an account of recent developments in this area.

Observe that the reader may feel free, in a certain sense, in recovering the pressure p from the system (9.7.1)–(9.7.4). This is due to the fact that the function p can be added by an arbitrary function $g(t)$ without breaking the equality in equation (9.7.1). With this arbitrariness in view, the meaning of "unique solvability" in this context is that the function p is determined up to a summand regardless of x. In trying to distinguish a unique solution in a standard sense it is fairly common to impose a **normalization condition** of the pressure values. For example, one way of proceeding is to require that

$$\int\limits_{\Omega} p(x,t)\, dx = 0 \tag{9.7.5}$$

and then put the system (9.7.1)–(9.7.4) together with condition (9.7.5).

Let us introduce some functional spaces which will be needed in subsequent studies of Navier–Stokes equations. The space

$$\mathbb{L}_2(\Omega) = \left[L_2(\Omega) \right]^n$$

contains all the vector-valued functions $\mathbf{u} = \left(u_1, \ldots, u_n \right)$, the elements of which belong to the space $L_2(\Omega)$. In so doing, the norm on that space is defined by

$$\| \mathbf{u} \|_{\mathbb{L}_2(\Omega)} = \Big(\sum_{i=1}^{n} \| u_i \|_{L_2(\Omega)}^2 \Big)^{1/2} .$$

Along similar lines, the space

$$\mathbb{W}_2^s(\Omega) = \left[W_2^s(\Omega) \right]^n$$

comprises all vector-valued functions $\mathbf{u} = \left(u_1, \ldots, u_n \right)$, the elements of which belong to the space $W_2^s(\Omega)$. In what follows the new space $\mathbb{W}_2^s(\Omega)$ is equipped with the norm

$$\| \mathbf{u} \|_{\mathbb{W}_2^s(\Omega)} = \Big(\sum_{i=1}^{n} \| u_i \|_{W_2^s(\Omega)}^2 \Big)^{1/2} .$$

The symbol $\mathbb{D}(\Omega)$ stands for the set of all vector fields that are infinitely differentiable and finite in the domain Ω. The symbol $D(\Omega)$ is used for the set of all scalar-valued functions which are infinitely differentiable and finite in the domain Ω. From such reasoning it seems clear that $\mathbb{D}(\Omega) = \left[D(\Omega) \right]^n$. Denote by $\overset{\circ}{W}{}_2^s(\Omega)$ the closure of $D(\Omega)$ in the norm of the space $W_2^s(\Omega)$. Set, by definition,

$$\overset{\circ}{\mathbb{W}}{}_2^s(\Omega) = \left[\overset{\circ}{W}{}_2^s(\Omega) \right]^n$$

and introduce

$$\mathbb{D}_0(\Omega) = \left\{ \mathbf{u} \in \mathbb{D}(\Omega) \colon \operatorname{div} \mathbf{u} = 0 \right\} .$$

The closures of $\mathbb{D}_0(\Omega)$ in the norms of the spaces $\mathbb{L}_2(\Omega)$ and $\overset{\circ}{\mathbb{W}}{}_2^1(\Omega)$ are denoted by $\mathbb{H}$ and $\mathbb{V}$, respectively.

The divergence of any element from $\mathbb{H}$ equals zero in a sense of distributions. However, the space $\mathbb{H}$ so defined does not coincide with the set of all **solenoidal vector fields** from the space $\mathbb{L}_2(\Omega)$. To clarify this fact, let us denote by $\mathbf{n}_x$ a unit external normal to the boundary $\partial\Omega$ at point x. It is plain to show that the component $(\mathbf{u}, \mathbf{n}_x)$ vanishes on the boundary $\partial\Omega$ if $\mathbf{u}$ is a smooth function from the space $\mathbb{H}$. On the other hand, there is a natural way of defining the **normal component** $(\mathbf{u}, \mathbf{n}_x)$ on the boundary $\partial\Omega$ for any function $\mathbf{u}$ from the space $\mathbb{H}$. Moreover, one can prove that

$$\mathbb{H} = \left\{ \mathbf{u} \in \mathbb{L}_2(\Omega):\ \operatorname{div}\mathbf{u} = 0,\ \left(\mathbf{u}, \mathbf{n}_x\right)\Big|_{\partial\Omega} = 0 \right\}.$$

The orthogonal complement $\mathbb{H}^{\perp}$ to $\mathbb{H}$ in the space $\mathbb{L}_2(\Omega)$ comprises all **potential vector fields**, that is,

$$\mathbb{H}^{\perp} = \left\{ \mathbf{u} \in \mathbb{L}_2(\Omega):\ \mathbf{u} = \operatorname{grad} p,\ p \in W_2^1(\Omega) \right\}. \tag{9.7.6}$$

Because of the structure of the spaces $\mathbb{H}$ and $\mathbb{H}^{\perp}$, a flexible and widespread approach to solving the direct problems for Navier–Stokes equations is connected with further elimination of the pressure p from equation (9.7.1) by means of **orthogonal projection** on the subspace $\mathbb{H}$.

In turn, the space $\mathbb{V}$ is simply characterized by

$$\mathbb{V} = \left\{ \mathbf{u} \in \overset{\circ}{\mathbb{W}}{}_2^1(\Omega):\ \operatorname{div}\mathbf{u} = 0 \right\}.$$

The theory and methods of abstract differential equations in Banach spaces are much applicable in studying Navier–Stokes equations. This is connected with further treatment of $\mathbf{f}$ as an **abstract function** of the variable t with values in the Hilbert space $\mathbb{L}_2(\Omega)$. Moreover, the function $\mathbf{u}$ is viewed as an abstract function of the variable t with values in the space $\mathbb{H}$. Finally, the function p is considered as an abstract function of the variable t with values in the space $W_2^1(\Omega)$, while u_0 is an element of the space $\mathbb{H}$. With these ingredients, equation (9.7.2) is immediately satisfied and condition (9.7.3) should be included in the domain of the **Laplace operator** Δ. By definition, set

$$\mathcal{D}(\Delta) = \mathbb{W}_2^2(\Omega) \bigcap \mathbb{V}.$$

Let us agree to write $\mathbf{P}$ for the operator of orthogonal projection on the subspace $\mathbb{H}$ in the space $\mathbb{L}_2(\Omega)$. By applying the projector $\mathbf{P}$ to equation (9.7.1) we derive the following system:

$$\mathbf{u}' - \nu\,\mathbf{P}\,\Delta\,\mathbf{u} = \mathbf{P}\,\mathbf{f}, \tag{9.7.7}$$

$$\operatorname{grad} p = \left(I - \mathbf{P} \right) \mathbf{f}. \tag{9.7.8}$$

Before giving further motivations, careful analysis of equation (9.7.8) is needed. Since

$$\left(I - \mathbf{P} \right) \mathbf{f} \in \mathbb{H}^{\perp},$$

for any fixed value t equation (9.7.8) is solvable in the space $W_2^1(\Omega)$. Denote by W^* the subspace of $W_2^1(\Omega)$ that contains all the functions satisfying condition (9.7.5). It is clear that an operator like

$$p \mapsto \operatorname{grad} p$$

executes an **isomorphism** of W^* onto $\mathbb{H}^{\perp}$. This type of situation is covered by the following results.

Lemma 9.7.1 *Let the function*

$$f \in \mathcal{C}([0, T]; \mathbb{L}_2(\Omega)) .$$

Then a solution p *to equation* (9.7.8) *exists and is unique in the class of functions*

$$p \in \mathcal{C}([0, T]; W^*) .$$

Lemma 9.7.2 (Temam (1979)) *Let the function*

$$f \in \mathcal{C}^1([0, T]; \mathbb{L}_2(\Omega)) .$$

Then a solution p *to equation* (9.7.8) *exists and is unique in the class of functions*

$$p \in \mathcal{C}^1([0, T]; W^*) .$$

Consequently, the direct problem at hand reduces to the **Cauchy problem** in the space $\mathbb{H}$ for the abstract differential equation

$$\text{(9.7.9)} \qquad \begin{cases} \mathbf{u}'(t) - \nu\, \mathbf{P}\, \Delta\, \mathbf{u} = \mathbf{P}\, \mathbf{f}(t), & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0 . \end{cases}$$

Here $\mathbb{S} = \nu\, \mathbf{P}\, \Delta$ is the **Stokes operator** with the domain

$$\mathcal{D}(\mathbb{S}) = \mathbb{W}_2^2(\Omega) \cap \mathbb{V} .$$

The operator $\mathbb{S}$ is self-adjoint and negative definite in the Hilbert space $\mathbb{H}$ (see Temam (1979)). In view of this, $\mathbb{S}$ generates a strongly continuous semigroup $V(t)$, satisfying the estimate

$$\text{(9.7.10)} \qquad \| V(t) \| \leq e^{-\omega t}, \qquad t \geq 0, \qquad \omega > 0 .$$

Applying the well-known results concerning the solvability of the abstract Cauchy problem and taking into account Lemmas 9.7.1–9.7.2, we arrive at the following assertions.

Theorem 9.7.1 *Let the function* $\mathbf{f} \in \mathcal{C}([0, T]; \mathbb{L}_2(\Omega))$ *and the element* $\mathbf{u}_0 \in \mathbb{H}$. *Then a solution* $\mathbf{u}, p$ *of the direct problem* (9.7.1)–(9.7.5) *exists and is unique in the class of functions*

$$\mathbf{u} \in \mathcal{C}([0, T]; \mathbb{H}) , \qquad p \in \mathcal{C}([0, T]; W_2^1(\Omega)) .$$

Moreover, the function $\mathbf{u}$ *is given by the formula*

$$\text{(9.7.11)} \qquad \mathbf{u}(t) = V(t)\, \mathbf{u}_0 + \int_0^t V(t-s)\, \mathbf{P}\, \mathbf{f}(s)\, ds .$$

Theorem 9.7.2 *Let the function* $\mathbf{f} \in C^1([0, T]; \mathbb{L}_2(\Omega))$ *and the element* $\mathbf{u}_0 \in \mathbb{W}_2^2 \cap \mathbb{V}$. *Then a solution* $\mathbf{u}, p$ *of the direct problem* (9.7.1)–(9.7.5) *exists and is unique in the class of functions*

$$\mathbf{u} \in C^1([0, T]; \mathbb{H}) \cap C([0, T]; \mathbb{W}_2^1(\Omega)), \qquad p \in C^1([0, T]; W_2^1(\Omega)).$$

Moreover, the function $\mathbf{u}$ *is given by formula* (9.7.11).

We proceed further by completely posing the **inverse problem** for the Navier–Stokes equation assuming that the external force function $\mathbf{f}$ is unknown. On the same grounds as before, we might attempt $\mathbf{f}$ in the form

$$\text{(9.7.12)} \qquad \mathbf{f}(x,t) = \mathbf{\Phi}(x,t)\, g(t) + \mathbf{F}(x,t),$$

where the vector-valued functions $\mathbf{\Phi}(x,t)$ and $\mathbf{F}(x,t)$ are available, while the unknown scalar-valued coefficient $g(t)$ is sought. To ensure the inverse problem concerned to be well-posed, the subsidiary information is prescribed in the form of **integral overdetermination**

$$\text{(9.7.13)} \qquad \int_\Omega \big(\mathbf{u}(x,t), \mathbf{w}(x)\big)\, dx = \varphi(t), \qquad 0 \le t \le T,$$

where the vector-valued function $\mathbf{w}(x)$ and the scalar-valued function $\varphi(t)$ are given. To write (9.7.12)–(9.7.13) in the abstract form, let us use the symbol $\Phi(t)$ for the operator of multiplication by the function $\mathbb{P}\,\mathbf{\Phi}$. The operator $\Phi(t)$ so defined acts from the space $\mathbf{R}$ into the space $\mathbb{H}$. In so doing, the function $\mathbf{F}(x,t)$ is treated as an abstract function $\mathbf{F}(t)$ of the variable t with values in the space $\mathbb{L}_2(\Omega)$. In this line, it is sensible to introduce the linear operator B acting from the space $\mathbb{H}$ into the space $\mathbf{R}$ in accordance with the rule

$$B\mathbf{u} = \int_\Omega \big(\mathbf{u}(x), \mathbf{w}(x)\big)\, dx.$$

With these ingredients, the inverse problem at hand acquires the **abstract form**

$$\text{(9.7.14)} \qquad \begin{cases} \mathbf{u}'(t) = \mathbb{S}\,\mathbf{u}(t) + \mathbf{\Phi}(t)\, g(t) + \mathbf{P}\,\mathbf{F}(t), & 0 \le t \le T, \\ \mathbf{u}(0) = \mathbf{u}_0, & \\ B\,\mathbf{u}(t) = \varphi(t), & 0 \le t \le T, \end{cases}$$

where $\mathbb{S}$ is the Stokes operator. The system (9.7.14) is joined with equation (9.7.8) written in the abstract form, making it possible to determine the pressure p with the aid of the relation

$$\text{(9.7.15)} \qquad \operatorname{grad} p = \big(I - \mathbf{P}\big)\big[\Phi(t)\, g(t) + \mathbf{F}(t)\big], \qquad 0 \le t \le T,$$

and the normalizatioin condition (9.7.5). As a final result we get the following assertion.

Theorem 9.7.3 *Let the functions*

$$\mathbf{\Phi},\, \mathbf{F} \in C([0,\, T],\, \mathbb{L}_2(\Omega)),$$

$\mathbf{w} \in W_2^2(\Omega) \cap \mathbb{V}$, $u_0 \in H$, $\varphi \in C^1[0,\, T]$, *the compatibility condition*

$$\int_\Omega \big(\mathbf{u}_0(x),\, \mathbf{w}(x)\big)\, dx \;=\; \varphi(0) \tag{9.7.16}$$

hold and

$$\psi(t) \;=\; \int_\Omega \big(\mathbf{P}\,\mathbf{\Phi}(x,t),\, \mathbf{w}(x)\big)\, dx \;\neq\; 0 \tag{9.7.17}$$

for all $t \in [0,\, T]$. *Then a solution* $\mathbf{u}$, p, g *of the inverse problem* (9.7.1)–(9.7.5), (9.7.12)–(9.7.13) *exists and is unique in the class of functions*

$$\mathbf{u} \,\in\, C\big([0,\, T];\, \mathbb{H}\big)\,, \qquad p \,\in\, C\big([0,\, T];\, W_2^1(\Omega)\big)\,, \qquad g \,\in\, C[0,\, T]\,.$$

Proof The abstract inverse problem (9.7.14) will be of special investigations on the basis of the general theory well-developed in Section 6.2. Applying the operator B to an element $\mathbf{u} \in \mathbb{H}$ yields the relation

$$B\,\mathbf{u} \;=\; \big(\mathbf{u},\, \mathbf{w}\big)_{\mathbb{H}}\,.$$

As far as the operator $\mathbb{S}$ is self-adjoint, condition (6.2.7) holds true, since the function

$$\mathbf{w} \in \mathcal{D}(\mathbb{S}) = \mathbb{W}_2^2(\Omega) \bigcap \mathbb{V}\,.$$

Moreover,

$$\overline{B\,\mathbb{S}}\,\mathbf{u} \;=\; \big(\mathbf{u},\, \mathbb{S}\,\mathbf{w}\big)_{\mathbb{H}}\,.$$

In the sequel we make use of Theorem 6.2.1. The operator $(B\,\Phi)^{-1}$ coincides with the operator of division by the function defined by (9.7.17). The premises of Theorem 9.7.3 provide the validity of Theorem 6.2.1, due to which the inverse problem (6.7.14) is solvable in the class of functions

$$\mathbf{u} \,\in\, C\big([0,\, T];\, \mathbb{H}\big)\,, \qquad\qquad g \in C[0,\, T]\,.$$

Moreover, this solution is unique. Let us consider equation (9.7.15) with the supplementary condition (9.7.5). Observe that the function

$$\mathbf{\Phi}(t)\, g(t) + \mathbf{F}(t)$$

is continuous with respect to t in the norm of the space $\mathbb{L}_2(\Omega)$. So, on account of Lemma 9.7.1 equation (9.7.15) with the supplementary condition (9.7.5) is uniquely solvable in the class of functions

$$p \,\in\, C\big([0,\, T];\, W_2^1(\Omega)\big)\,.$$

This leads to the desired assertion of Theorem 9.7.3. ■

Theorem 9.7.4 *If the functions* $\mathbf{\Phi}, \mathbf{F} \in C^1([0,T]; \mathbb{L}_2(\Omega))$, $\mathbf{u}_0 \in \mathbb{W}_2^2(\Omega) \cap \mathbb{V}$, $\varphi \in C^2[0,T]$, $\mathbf{w} \in \mathbb{W}_2^2(\Omega) \cap \mathbb{V}$, *the compatibility condition* (9.7.16) *holds and the function* ψ *defined by* (9.7.17) *does not vanish at each point* $t \in [0,T]$, *then a solution* $\mathbf{u}, p, g$ *of the inverse problem* (9.7.1)–(9.7.5), (9.7.12)–(9.7.13) *exists and is unique in the class of functions*

$$\mathbf{u} \in C^1([0,T]; \mathbb{H}) \bigcap C([0,T]; \mathbb{W}_2^2(\Omega)),$$

$$p \in C^1([0,T]; W_2^1(\Omega)), \qquad g \in C^1[0,T].$$

Proof Recall that the inverse problem at hand can be posed in the abstract form (9.7.14). As stated above, the operator B satisfies condition (9.2.7). This provides reason enough to refer to Theorem 6.2.3, whose use permits one to conclude that a solution of the abstract problem (9.7.14) exists and is unique in the class of functions

$$\mathbf{u} \in C^1([0,T]; \mathbb{H}) \bigcap C([0,T]; \mathcal{D}(\mathbb{S})), \qquad g \in C^1[0,T].$$

Equation (9.7.15) comes second. The function $\mathbf{\Phi}(t)\, g(t) + \mathbf{F}(t)$ is continuously differentiable with respect to t in the norm of the space $\mathbb{L}_2(\Omega)$. Therefore, Lemma 5.7.2 asserts the unique solvability of the inverse problem (9.7.15), (9.7.5) in the class of functions

$$p \in C^1([0,T]; \mathbb{W}_2^1(\Omega))$$

and, therefore, it remains to take into account that

$$\mathcal{D}(\mathbb{S}) = \mathbb{W}_2^2(\Omega) \bigcap \mathbb{V},$$

thereby completing the proof. ■

In concluding this section we are interested in the inverse problem with the final overdetermination under the agreement that the external force function $\mathbf{f}(x,t)$ built into equation (9.7.1) admits the form

$$\mathbf{f}(x,t) = \Phi(t)\, \mathbf{g}(x) + \mathbf{F}(x,t), \tag{9.7.18}$$

where the scalar-valued function $\Phi(t)$ and the vector-valued function $\mathbf{F}(t)$ are given, while the unknown vector-valued function $\mathbf{g}(x)$ is sought in the space $\mathbb{H}$. A well-posed statement necessitates imposing the condition of final overdetermination

$$\mathbf{u}(x,T) = \mathbf{u}_1(x), \qquad x \in \Omega. \tag{9.7.19}$$

The inverse problem posed above can be written in the **abstract form**

$$\left\{\begin{aligned} \mathbf{u}'(t) &= \mathbb{S}\,\mathbf{u}(t) + \Phi(t)\,\mathbf{g} + \mathbf{P}\ \mathbf{F}(t)\,, \quad 0 \le t \le T\,, \\ \mathbf{u}(0) &= \mathbf{u}_0\,, \\ \mathbf{u}(T) &= \mathbf{u}_1\,, \end{aligned}\right. \tag{9.7.20}$$

with the governing equation (9.7.15) and the supplementary condition (9.7.5).

Theorem 9.7.5 *Let* $\mathbf{F} \in C^1([0,T];\mathbb{L}_2(\Omega))$, $\Phi \in C^1[0,T]$, $\Phi(t) > 0$ *and* $\Phi'(t) \ge 0$ *for all* $t \in [0,T]$ *and the functions* $\mathbf{u}$, $\mathbf{u}_1 \in \mathbb{W}_2^2(\Omega) \cap \mathbb{V}$. *Then a solution* $\mathbf{u}$, p, $\mathbf{g}$ *of the inverse problem* (9.7.1)–(9.7.5), (9.7.18)–(9.7.19) *exists and is unique in the class of functions*

$$\mathbf{u} \in C^1([0,T];\mathbb{H}) \bigcap C([0,T];\mathbb{W}_2^2(\Omega))\,,$$

$$p \in C^1([0,T];W_2^1(\Omega))\,, \qquad \mathbf{g} \in \mathbb{H}\,.$$

Proof Consider the abstract inverse problem (9.7.20). From relation (9.7.10) and the conditions imposed above it follows that Theorem 9.7.5 is fitted into the present framework and ensures the unique solvability of problem (9.7.20) in the class of functions

$$\mathbf{u} \in C^1([0,T];\mathbb{H}) \bigcap C([0,T];\mathcal{D}(\mathbb{S}))\,, \qquad \mathbf{g} \in \mathbb{H}\,.$$

Since $\mathcal{D}(\mathbb{S}) = \mathbb{W}_2^2(\Omega) \cap \mathbb{V}$, the functions $\mathbf{u}$ and $\mathbf{g}$ just considered belong to the needed classes. The function $\Phi(t)\,\mathbf{g} + \mathbf{F}(t)$ involved in equation (9.7.15) is continuously differentiable with respect to t in the norm of the space $\mathbb{L}_2(\Omega)$. Consequently, in agreement with Lemma 9.7.2, the inverse problem (9.7.15), (9.7.5) is uniquely solvable in the class of functions

$$p \in C^1([0,T];W_2^1(\Omega))\,.$$

This completes the proof of Theorem 9.7.5. ■

9.8 The system of Maxwell equations

The final goal of our studies is the system of **Maxwell equations** in a bounded domain $\Omega \subset \mathbf{R}^3$:

$$\left\{\begin{aligned} \operatorname{rot} E &= -\frac{\partial B}{\partial t}\,, \\ \operatorname{rot} H &= \frac{\partial D}{\partial t} + J\,, \end{aligned}\right. \tag{9.8.1}$$

where E is the vector of the **electric field strength**, H is the vector of the **magnetic field strength**, D and B designate the **electric and magnetic induction** vectors, respectively. In what follows we denote by J the **current density**.

In the sequel we deal with a **linear medium** in which the vectors of strengthes are proportional to those of inductions in accordance with the governing laws:

$$\left\{ \begin{aligned} D &= \varepsilon E , \\ B &= \mu H , \end{aligned} \right. \tag{9.8.2}$$

We assume, in addition, that **Ohm's law**

$$J = \sigma E + I \tag{9.8.3}$$

is satisfied in the domain Ω, where ε is the **dielectric permeability** of the medium and μ is the **magnetic permeability**. We denote by σ the **electric conductance** and use the symbol I for the **density of the extraneous current**.

The functions ε, μ and σ are supposed to be continuous on $\bar{\Omega}$. The density of the extraneous current I is viewed as a continuous function of the variables $(x,\, t)$, where $x \in \bar{\Omega}$ and $0 \leq t \leq T$.

The boundary conditions for the Maxwell system usually reflect some physical peculiarities of the boundary $\partial\Omega$. When $\partial\Omega$ happens to be a **perfect conductor**, it is reasonable to impose one more condition

$$\big[n_x \times D \big] = 0 , \qquad x \in \partial\Omega , \tag{9.8.4}$$

where n_x is a unit external normal to the boundary $\partial\Omega$ at point x.

A problem statement necessitates assigning, in addition to the boundary condition, the supplementary initial data. In particular, we are able to prescribe by means of relation (9.8.2) either the strengthes or the inductions. Further development is connected with the initilal conditions for the vectors of the electric and magnetic inductions:

$$\left\{ \begin{aligned} D(x,0) &= D_0(x) , \\ B(x,0) &= B_0(x) , \end{aligned} \right. \tag{9.8.5}$$

The direct problem here consists of finding the functions E, D, H, B from the system (9.8.1)–(9.8.5) for the given functions ε, μ, σ, I, D_0 and B_0 involved.

As far as the statement of the direct problem is concerned, some remarks help motivate what is done. By convention, the Maxwell system includes also the following equations:

$$\operatorname{div} D = \rho\,, \tag{9.8.6}$$

$$\operatorname{div} B = 0\,, \tag{9.8.7}$$

where ρ is the density of the **electric charge distribution**. Equation (9.8.6) can be treated as a relation, by means of which the function ρ is well-defined. Indeed, while studying problem (9.8.1)–(9.8.5) the density ρ is simply calculated by formula (9.8.6). On the other hand, equation (9.8.1) yields the equality

$$\operatorname{div} B(x,t) = 0$$

for all $t > 0$ under the natural premise $\operatorname{div} B(x,0) \equiv 0$. As a matter of fact, relation (9.8.7) amounts to some constraint on the initial data. What is more, if, for instance, $\operatorname{div} B_0 = 0$, then equality (9.8.7) is automatically fulfilled. With these ingredients, there is no need for the occurrence of relations (9.8.6)–(9.8.7) in further development and so we might confine ourselves to the system (9.8.1)–(9.8.5) only. For more detail we refer the reader to Birman and Solomyak (1987), Bykhovsky (1957), Duvant and Lions (1972).

It will be sensible to introduce some functional spaces related to the Maxwell system. Let Ω be a bounded domain with boundary $\partial\,\Omega \in C^2$. We initiate the construction of the set

$$H(\operatorname{rot},\,\Omega) = \left\{ u \in L_2(\Omega)^3\colon\ \operatorname{rot} u \in L_2(\Omega)^3 \right\}, \tag{9.8.8}$$

which becomes a Hilbert space equipped with the norm

$$\|\,u\,\|_{\mathrm{rot}} = \left(\|\,u\,\|^2_{L_2(\Omega)^3} + \|\operatorname{rot} u\,\|^2_{L_2(\Omega)^3} \right)^{1/2} \tag{9.8.9}$$

Denote by $H_0(\operatorname{rot},\,\Omega)$ a closure of all vector fields which are infinite differentiable and have compact support in the domain Ω. The subspace $H_0(\operatorname{rot},\,\Omega)$ so constructed will be closed in the space $H(\operatorname{rot},\,\Omega)$ and admits a clear description. Indeed, it turns out that for any element $u \in H(\operatorname{rot},\,\Omega)$ we might reasonably try to preassign a value of the function $[n_x \times u]$ on the boundary $\partial\,\Omega$. The boundary value will belong to a certain Sobolev space with a negative differential exponent, so there is some reason to be concerned about this. More specifically, we have

$$H_0(\operatorname{rot},\,\Omega) = \Big\{ u \in H(\operatorname{rot},\Omega)\colon\ [n_x \times u]\Big|_{\partial\,\Omega} = 0 \Big\}. \tag{9.8.10}$$

It is worth noting here that for any vector field $u \in H(\mathrm{rot}, \Omega)$ and any function $v \in \overset{\circ}{W}{}_2^1(\Omega)$ the integral relation

$$\int_\Omega v \,\mathrm{rot}\, u \, dx = \int_\Omega \left[u \times \mathrm{grad}\, v \right] dx \tag{9.8.11}$$

is the outcome of integrating by parts. Also, it is plain to show that the operator

$$A\colon\ u \mapsto \mathrm{rot}\, u$$

with the domain $\mathcal{D}(A) = H_0(\mathrm{rot}, \Omega)$ is self-adjoint in the space $L_2(\Omega)^3$.

An alternative form of the system (9.8.1)–(9.8.5) is based on (9.8.2) with excluding the vectors of the electric and magnetic field strengthes and substituting (9.8.3) into (9.8.1). We are led by expressing the t-derivatives of the induction vectors via all other functions to the statement of the direct problem

$$\frac{\partial D}{\partial t} = \mathrm{rot}\, \frac{B}{\mu} - \frac{\sigma}{\varepsilon}\, D - I, \tag{9.8.12}$$

$$\frac{\partial B}{\partial t} = -\mathrm{rot}\, \frac{D}{\varepsilon}, \tag{9.8.13}$$

$$\left[n_x \times D \right] = 0, \qquad x \in \partial\Omega, \tag{9.8.14}$$

$$D(x,0) = D_0(x), \qquad B(x,0) = B_0(x). \tag{9.8.15}$$

In what follows it seems worthwhile to consider only the system (9.8.12)–(9.8.15) as an alternative form of writing the problem posed initially for the Maxwell system.

Let a basic functional space will be taken to be

$$\mathbb{H} = L_2(\Omega)^6 = L_2(\Omega)^3 \times L_2(\Omega)^3 \tag{9.8.16}$$

and the system of equations (9.8.12)–(9.8.13) will be of special investigations in that space. The inner product in the space $\mathbb{H}$ is introduced by means of weight functions: for (D_1, B_1), $(D_2, B_2) \in \mathbb{H}$ set, by definition,

$$\big((D_1, B_1), (D_2, B_2)\big)_{\mathbb{H}} = \left(\frac{D_1}{\varepsilon}, D_2 \right)_{L_2(\Omega)^3} \left(\frac{B_1}{\mu}, B_2 \right)_{L_2(\Omega)^3}. \tag{9.8.17}$$

Observe that the inner product (9.8.17) with the functions ε and μ being still subject to the initial restrictions becomes equivalent to the usual inner product like

$$\left(D_1, D_2 \right)_{L_2(\Omega)^3} + \left(B_1, B_2 \right)_{L_2(\Omega)^3}.$$

In trying to achieve an abstract form in the space $\mathbb{H}$ it will be reasonable to refer to an unbounded operator $\mathcal{A}$ such that

$$\mathcal{A}\colon (D,\, B) \mapsto \left(\operatorname{rot} \frac{B}{\mu}\,,\, -\operatorname{rot} \frac{D}{\varepsilon} \right), \tag{9.8.18}$$

whose domain is

$$\mathcal{D}(\mathcal{A}) = \big\{ (D,\, B)\colon\; D \in H_0(\operatorname{rot},\, \Omega),\, B \in H(\operatorname{rot},\, \Omega) \big\}\,. \tag{9.8.19}$$

If $\mu,\, \varepsilon \in C^1$, which will be true in the sequel, then the operator $\mathcal{A}$ acts from the space $\mathbb{H}$ into the space $\mathbb{H}$. Let us stress that the domain of the operator $\mathcal{A}$ is formed in such a way that the boundary condition (9.8.14) will be covered by (9.8.19).

Several basic properties of the operator $\mathcal{A}$ are quoted in the following assertion (see Duvant and Lions (1972)).

Theorem 9.8.1 *Let Ω be a bounded domain in the space $\mathbf{R}^3$ with boundary $\partial\Omega \in C^2$ and let the functions ε and μ belong to the space $C^1(\bar{\Omega})$ and will be positive in $\bar{\Omega}$. Then the domain of the operator $\mathcal{A}$ is dense in the space $\mathbb{H}$. Moreover, the operator $\mathcal{A}$ is closed and skew-Hermitian, that is,*

$$\mathcal{D}(A^*) = \mathcal{D}(A) \qquad \textit{and} \qquad \mathcal{A}^* = -\mathcal{A}\,.$$

We need to rearrange problem (9.8.12)–(9.8.15) in an abstract form. With this aim, the pair $(D,\, B)$ is now treated as an abstract function of the variable t with values in the space $\mathbb{H}$. Let us introduce one more operator

$$\mathcal{B}\colon (D,\, B) \mapsto \left(-\frac{\sigma}{\varepsilon}\, D,\, 0 \right), \tag{9.8.20}$$

which is bounded in the space $\mathbb{H}$ due to the inclusion $\sigma \in C(\bar{\Omega})$. Set, by definition, $\mathcal{F} = (-I,\, 0)$, which is viewed as a function of the variable t with values in the space $\mathbb{H}$. With respect to $u = (D,\, B)$ and $u_0 = (D_0,\, B_0)$ the direct problem (9.8.12)–(9.8.15) acquires the abstract form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{A}\, u + \mathcal{B}\, u + \mathcal{F}(t)\,, & 0 < t < T\,, \\ u(0) = u_0\,. \end{cases} \tag{9.8.21}$$

Theorem 9.8.1 implies that the operator $\mathcal{A}$ generates a strongly continuous group. In conformity with perturbation theory the property to generate a strongly continuous group will be retained once we add a bounded operator $\mathcal{B}$ to the operator $\mathcal{A}$. That is why the sum of the operators $\mathcal{A}+\mathcal{B}$ generates a strongly continuous group $V(t)$, making it possible to exploit here the results of Section 5.2 (Theorems 5.2.2 and 5.2.3). This reference enables us to prove several assertions.

Theorem 9.8.2 *Let all the conditions of Theorem* 9.8.1 *hold and the inclusions*

$$\sigma \in C(\bar{\Omega}), \qquad I \in C([0, T]; L_2(\Omega)^3), \qquad D_0, B_0 \in L_2(\Omega)^3$$

occur. Then a solution D, B of the direct problem (9.8.12)–(9.8.15) *exists and is unique in the class of functions*

$$D, B \in C([0, T]; L_2(\Omega)^3).$$

Theorem 9.8.3 *Let under the premises of Theorem* 9.8.1 *the inclusions*

$$\sigma \in C(\bar{\Omega}), \qquad I \in C^1([0, T]; L_2(\Omega)^3),$$

$$D_0 \in H_0(\text{rot}, \Omega), \qquad B_0 \in H(\text{rot}, \Omega)$$

occur. Then a solution D, B of the direct problem (9.8.12)–(9.8.15) *exists and is unique in the class of functions*

$$D, B \in C^1([0, T]; L_2(\Omega)^3) \bigcap C([0, T]; H(\text{rot}, \Omega)).$$

The statement of an inverse problem involves the density of the extraneous current as an unknown of the structure

$$I(x,t) = \Phi(x,t)\, p(t) + g(x,t), \tag{9.8.22}$$

where the matrix Φ of size 3×3 and the vector-valued function g are known in advance, while the unknown vector-valued function p is sought. To complete such a setting of the problem, we take the **integral overdetermination** in the form

$$\int_\Omega D(x,t)\, w(x)\, dx = \varphi(t), \qquad 0 \le t \le T, \tag{9.8.23}$$

where the function $w(x)$ is known in advance. The system of equations (9.8.12)–(9.8.15), (9.8.22)–(9.8.23) is treated as the inverse problem for the Maxwell system related to the unknown functions D, B and p.

Theorem 9.8.4 *Let under the conditions of Theorem* 9.8.1

$$\sigma \in C(\bar{\Omega}), \qquad D_0, B_0 \in L_2(\Omega)^3$$

and the elements Φ_{ij} of the matrix Φ belong to the space $C([0, T]; L_2(\Omega))$. If, in addition,

$$g \in C([0, T]; L_2(\Omega)^3), \qquad w \in \overset{\circ}{W}{}_2^1(\Omega), \qquad \varphi \in C^1([0, T]; \mathbf{R}^3),$$

the compatibility condition

$$\int_\Omega D_0(x)\, w(x)\; dx \;=\; \varphi(0)$$

holds and for all $t \in [0,\, T]$

$$\det \int_\Omega \Phi(x,t)\, w(x)\; dx \neq 0\,,$$

then a solution D, B, p *of the inverse problem* (9.8.12)–(9.8.15), (9.8.22), (9.8.25) *exists and is unique in the class of functions*

$$D,\, B \in C\big([0,\, T];\, L_2(\Omega)^3\big)\,, \qquad\qquad p \in C\big([0,\, T];\, \mathbf{R}^3\big)\,.$$

Proof Our starting point is the definition of the operator $\Phi(t)$ acting from the space $\mathbf{R}^3$ into the space $\mathbb{H}$ in accordance with the rule

$$\Phi(t)\colon\; p \mapsto \big(\Phi(x,t)\, p,\, 0\big)\,.$$

When the function G is specified by means of the relation

$$G = (-g,\, 0)\,,$$

one can treat it as a function of the variable t with values in the space $\mathbb{H}$. With respect to $\mathcal{F} = (-I,\, 0)$ equality (9.8.22) takes the form

$$\mathcal{F}(t) \;=\; \Phi(t)\, p + G(t)\,.$$

Let the operator B from the space $\mathbb{H}$ into $\mathbf{R}^3$ assign the values

$$B \begin{pmatrix} D \\ H \end{pmatrix} \;=\; \int_\Omega D(x)\, w(x)\; dx\,,$$

by means of which condition (9.8.23) acquires the abstract form

$$B\, u(t) \;=\; \varphi(t)\,.$$

Finally, the abstract counterpart of the inverse problem (9.8.12)–(9.8.15), (9.8.22)–(9.8.23) is as follows:

$$\text{(9.8.24)} \qquad \begin{cases} \dfrac{du}{dt} \;=\; A\, u + \Phi(t)\, p + G(t)\,, & 0 < t < T\,, \\ u(0) \;=\; u_0\,, & \\ B\, u(t) \;=\; \varphi(t)\,, & 0 \le t \le T\,, \end{cases}$$

where the operator $A = \mathcal{A} + \mathcal{B}$ generates a strongly continuous group.

The abstract inverse problem (9.8.24) is fitted into the framework of Section 6.2 provided condition (6.2.7) holds. It is straightforward to verify its validity by plain calculations

$$B\,A \begin{pmatrix} D \\ B \end{pmatrix} = \int\limits_{\Omega} \left(\operatorname{rot} \frac{B}{\mu} \right) w(x)\,dx - \int\limits_{\Omega} \frac{\sigma}{\varepsilon}\,D\,w(x)\,dx$$

under the natural premise $(D,\,B) \in \mathcal{D}(A)$. What is more, by formula (9.8.11) we find that

$$B\,A \begin{pmatrix} D \\ B \end{pmatrix} = \int\limits_{\Omega} \left[\, B \times \operatorname{grad} w \,\right] \frac{dx}{\mu(x)} - \int\limits_{\Omega} \frac{\sigma(x)}{\varepsilon(x)}\,D(x)\,w(x)\,dx\,. \tag{9.8.25}$$

In light of the emerging constraints on the functions ε, μ, σ and w the right-hand side of (9.8.25) is a continuous linear operator from the space $\mathbb{H}$ into the space $\mathbf{R}^3$ and, as a matter of fact, specifies the operator $\overline{B\,A}$. Therefore, condition (6.2.7) holds true as required. For this reason the statement in question immediately follows from Theorem 6.2.1, thereby completing the proof of this theorem. ■

Theorem 9.8.5 *Let under the premises of Theorem* 9.8.4 *the inclusions*

$$D_0 \in H_0(\operatorname{rot},\,\Omega)\,, \qquad B_0 \in H(\operatorname{rot},\,\Omega)$$

occur. One assumes, in addition, that the elements Φ_{ij} of the matrix Φ belong to the space $C^1([0,\,T];\,L_2(\Omega))$ and

$$g \in C^1([0,\,T];\,L_2(\Omega)^3)\,, \qquad \varphi \in C^2([0,\,T];\,\mathbf{R}^3)\,.$$

Then a solution D, B, p of the inverse problem (9.8.12)–(9.8.15), (9.8.22), (9.8.25) *exists and is unique in the class of functions*

$$D,\,B \in C^1([0,\,T];\,L_2(\Omega)^3) \bigcap C([0,\,T];\,H(\operatorname{rot},\,\Omega)),$$

$$p \in C^1([0,\,T];\,\mathbf{R}^3)\,.$$

Proof To prove the above assertion, we refer once again to the results of Section 6.2 concerning the existence and uniqueness of a strong solution of the abstract inverse problem (9.8.24). In particular, Theorem 6.2.3 fits our purposes, since its conditions are stipulated by the boundedness of the operator $B\,A$. Recall that we have substantiated this reference in proving Theorem 9.8.4. Finally, the statement of Theorem 9.8.5 immediately follows from Theorem 6.2.3 and thereby completes the proof of the theorem. ■

Chapter 10

Concluding Remarks

In conclusion we give a brief commentary regarding the results set forth in this book. Chapter 1 covers inverse problems for partial differential equations of parabolic type. The detailed outline of related direct problems is available in many textbooks and monographs. In particular, we refer the reader to the books by A. Bitsadze (1976), R. Courant and D. Hilbert (1953, 1962), A. Friedman (1964), S. Godunov (1971), O. Ladyzhenskaya (1973), O. Ladyzhenskaya et al. (1968), J.-L. Lions (1970), V. Mikhailov (1976), S. Mikhlin (1977), S. Sobolev (1954), A. Tikhonov and A. Samarskii (1963), V. Vladimirov (1971) and the paper of A. Il'in et al. (1962).

Research of inverse problems for the heat conduction equation was initiated by the famous paper of A. Tikhonov (1935) in which the problem of recovering the initial data of the Cauchy problem with the final overdetermination was properly posed and carefully explored. The coefficient inverse problem for the heat conduction equation was first investigated by B. Jones (1962), J. Douglas and B. Jones (1962) in the situation when the subsidiary information is the value of a solution at a fixed point of space variables.

The availability of integral overdetermination within the framework of inverse problems for parabolic equations owes a debt to V. Soloviev (1985). The main idea behind the problem statement here is to take the integral in the overdetermination condition over the domain of space variables. An alternative integral overdetermination was proposed in the papers of

A. Prilepko and A. Kostin (1992a,b), where integration is accomplished during some period of time.

Section 1.2 focuses on the inverse problem involving the source function of the form (1.2.1). It is worth noting here that such a structure of the source arose for the first time in the work of A. Prilepko (1966b) concerning elliptic equations. In the original setting the inverse problem (1.2.34)–(1.2.37) is much applicable in theory and practice. In the simplest case when the operator L is self-adjoint and $h(x,t) \equiv 1$, A. Iskenderov and R. Tagiev (1979) derived an explicit formula for a solution of the posed problem by appeal to the operator semigroup generated by the operator L as an infinitesimal operator. Further development of this formula was independently repeated by W. Rundell (1980). The proof of the uniqueness of a solution of the inverse problem (1.2.34)–(1.2.37) was carried out by V. Isakov (1982a) within the classical framework. Later this inverse problem was extensively investigated by means of various methods in several functional spaces. The existence and uniqueness theorems for a solution of the inverse problem concerned have been proved by A. Prilepko and V. Soloviev (1987c), V. Soloviev (1989), A. Prilepko and A. Kostin (1992a), V. Isakov (1991b). Being concerned with a self-adjoint operator L and a function $h(x,t) \equiv h(x)$ (not depending on t), D. Orlovsky (1990) established necessary and sufficient conditions for the existence of a solution of the inverse problem (1.2.34)–(1.2.37) as well as necessary and sufficient conditions of the solution uniqueness in terms of the spectral function of the operator L. The results we have cited are based on a semigroup approach to the inverse problem well-developed by A. Prilepko and D. Orlovsky (1989), A. Prilepko and A. Kostin (1992a), Yu. Eidelman (1983, 1987, 1990, 1993a,b), D. Orlovsky (1988), A. Prilepko and I. Tikhonov (1992, 1993). Let us stress that the inverse problem (1.2.14)–(1.2.17) with the integral overdetermination is a natural generalization of the inverse problem with the final overdetermination which has been under consideration earlier by A. Prilepko and A. Kostin (1992a), A. Prilepko and I. Tikhonov (1994), A. Prilepko, A. Kostin and I. Tikhonov (1992), I. Tikhonov and Yu. Eidelman (1994).

In Section 1.3 the property of Fredholm's type solvability of the inverse problem (1.2.2)–(1.2.5) is revealed. The first results in connection with this property are available in A. Prilepko and V. Soloviev (1987a), V. Soloviev (1988), where solutions to equation (1.2.2) are sought in Hölder's classes of functions. Later this property was analysed from various viewpoints and in various functional spaces by D. Orlovsky (1988, 1990), A. Prilepko and A. Kostin (1992a), A. Lorenzi and A. Prilepko (1993), A. Prilepko and I. Tikhonov (1994).

In Section 1.4 the inverse problem of recovering a coefficient at the unknown function in a parabolic equation was completely posed and resolved under the agreement that the subsidiary information is prescribed in the form of final overdetermination. This simplest statement of the nonlinear coefficient inverse problem was extensively investigated by many scientists. Recent years have seen the publication of numerous papers on this subject. In Section 1.4 we quote mainly the results obtained by A. Prilepko, A. Kostin and I. Tikhonov (1992), A. Prilepko and A. Kostin (1993a). The proofs of the basic assertions were carried out on the basis of Birkhoff–Tarsky principle for isotone operators. With regard to the problem concerned the reader may also refer to A. Prilepko and V. Soloviev (1987a,b,c), W. Rundell (1987), V. Isakov (1990c), K.-H. Hoffman and M. Yamamoto (1993) and others.

In Section 1.5 the inverse problem of recovering a source function in a modeling heat conduction equation was of special investigations. In contrast to (1.2.1) the source admits an alternative form and the unknown coefficient depends solely on time. Additional information is connected with a final result of measuring the temperature by a a perfect sensor located at a fixed point in the domain of space variables. Each such sensor is of finite size and, obviously, should make some averaging over the domain of action. In this view, it is reasonable to absorb from this sensor the subsidiary information in the form (1.5.4), where the function $\omega(x)$ is some sensor characteristic. In the case of a perfect sensor located at a point x_0 its characteristic is equal to

$$\omega(x) = \delta(x - x_0)$$

and the measurement of the temperature is well-characterized by (1.5.42). The initial stage of research was stipulated by the one-dimensional problem arising from the paper of B. Jones (1963). Later this problem was examined in various aspects by A. Prilepko (1973a), N. Beznoshchenko and A. Prilepko (1977), A. Prilepko and V. Soloviev (1987b), A. Prilepko and D. Orlovsky (1984). Further development of such theory in some functional spaces is connected with the results obtained by J. Cannon and L. Yanping (1986, 1988a,b, 1990), A. Prilepko and D. Orlovsky (1985a,b, 1987a,b, 1988, 1989, 1991), A. Prilepko and V. Soloviev (1987b), D. Orlovsky (1990, 1991a,b,c,d). An exhaustive survey on various statements of inverse problems for parabolic equations and well-established methods for solving them in some functional spaces is due to O. Alifanov (1979), G. Anger (1990), J. Beck et al. (1985), N. Beznoshchenko (1974, 1975a,b, 1983a,b,c), B. Bubnov (1988), B. Budak and A. Iskenderov (1967a,b), J. Cannon (1963, 1964, 1967, 1968, 1984), J. Cannon and P. DuChateau (1970, 1973a,b, 1978, 1979), J. Cannon and S. Perez-Esteva (1994), J. Cannon and L. Yanping (1986), J. Cannon et al. (1963), G. Chavent and P. Lemonnier (1974),

M. Choulli (1994), P. DuChateau (1981), P. DuChateau and W. Rundell (1985), Yu. Eidelman (1993a,b), A. Elcrat et al. (1995), H. Engl et al. (1994), V. Isakov (1976, 1982a,b, 1986, 1989, 1990a,b,c, 1991a,b, 1993, 1998), A. Iskenderov (1974, 1975, 1976), A. Friedman and M. Vogelius (1989), A. Friedman et al. (1989), B. Jones (1963), V. Kamynin (1992), M. Klibanov (1985, 1986), R. Kohn and M. Vogelius (1984, 1985), J.-L. Lions (1970), A. Lorenzi (1982, 1983a,b, 1985, 1987, 1992), A. Lorenzi and A. Prilepko (1993), L. Payne (1975), M. Pilant and W. Rundell (1986a,b, 1987b, 1988, 1990), A. Prilepko (1973a), C. Roach (1991), W. Rundell (1980, 1983, 1987), T. Suzuki (1983, 1986), T. Suzuki and R. Myrayma (1980), I. Tikhonov (1992), M. Yamamoto (1993, 1994a,b, 1995). Various topics and issues from the theory and practice for direct and inverse problems associated with parabolic equations are covered by the monographs by O. Alifanov (1979), O. Alifanov et al. (1988), Yu. Anikonov (1976, 1995), G. Anger (1990), A. Babin and M. Vishik (1989), J. Baumeister (1987), J. Beck et al. (1985), O. Besov et al. (1975), J. Cannon (1984), J. Cannon and U. Hornung (1986), J. Cannon et al. (1990), D. Colton et al. (1990), A. Denisov (1994), A. Dezin (1980), A. Fedotov (1982), A. Friedman (1964), A. Glasko (1984), C. Groetsch (1993), L. Hörmander (1965), V. Isakov (1998), V. Ivanov et al. (1978), L. Kantorovich and G. Akilov (1977), R. Lattes and J.-L. Lions (1967), M. Lavrentiev (1967, 1973), M. Lavrentiev et al. (1968, 1986), J.-L. Lions (1970), V. Mikhailov (1976), V. Morozov (1984), L. Payne (1975), V. Romanov (1973, 1984), A. Tikhonov and V. Arsenin (1977), G. Vainikko and A. Veretennikov (1986), V. Vladimirov (1971). A powerful support for the rapid development of this branch of science was provided by the original works of A. Akhundov (1988), A. Alessandrini (1988), A. Alessandrini and G. Vessella (1985), N. Beznoshchenko and A. Prilepko (1977), M. Choulli (1994), A. Calderón (1980), J. Canon and R. Ewing (1976), J. Canon et al. (1994), J. Canon and D. Zachmann (1982), W. Chewing and T. Seidman (1977), A. Denisov (1982), A. Goncharsky et al. (1973), A. Friedman and V. Isakov (1989), A. Friedman and M. Vogelius (1989), A. Friedman et al. (1989), B. Jones (1962, 1963), A. Il'in et al. (1962), V. Isakov (1988), R. Kohn and M. Vogelius (1984, 1985), A. Kostin and A. Prilepko (1996a,b), R. Langer (1933), A. Lorenzi (1992), A. Lorenzi and A. Lunardi (1990), A. Lorenzi and C. Pagani (1987), A. Lorenzi and E. Paparoni (1985, 1988), A. Lorenzi and A. Prilepko (1994), A. Lorenzi and E. Sinestrari (1987, 1988), V. Maksimov (1988), G. Marchuk (1964), V. Mikhailov (1963a,b), T. Moser (1964), N. Muzylev (1980), M. Nached (1970), D. Orlovsky (1994), I. Petrovsky (1934), A. Prilepko (1992), G. Richter (1981), T. Seidman (1981a,b), J. Sylvester and G. Uhlmann (1986, 1988), C. Talenti (1978), N.-S. Trudinger (1968) and A. Uzlov (1978).

Chapter 2 provides a unified approach and deep systematic study of first order linear hyperbolic systems and second order linear hyperbolic equations of two independent variables. If we confine ourselves to the case of two variables only, there is a relatively simple line of reasoning, making it possible to avoid cumbersome elements of the theory of partial differential equations and functional analysis. The main idea behind solution of direct and inverse problems here is to employ the method of characteristics. This simple method finds a wide range of applications. For more detail we recommend to see R. Courant and D. Hilbert (1953, 1962), S. Godunov (1971), B. Rozhdestvensky and N. Yanenko (1978), E. Sanchez-Palencia (1980).

Section 2.1 deals with x-hyperbolic systems of the canonical form (2.1.2). The basic part of this section is devoted to the problem of finding a nonhomogeneous term in the form (2.1.8) for the sought function with the boundary conditions (2.1.9). The results obtained are formulated in Theorems 2.1.2–2.1.3, where the existence and uniqueness of a solution of this inverse problem are proved in the class of functions of the exponential type if we make L small enough. The smallness of the value L is essential for the solvability of the inverse problem which interests us. To understand nature a little better, this obstacle is illustrated by one possible example from the paper of A. Kostin and I. Tikhonov (1991) in which the authors have shown that the value of L decreases with increasing the exponential type of the function v. However, in the particular case of the system (2.1.2), where all of the eigenvalues of the matrix K have constant sign, Theorem 2.1.4 asserts the existence and uniqueness of a solution of the inverse problem concerned. As an application of the above theorems we raise the question of the solution uniqueness in the problem of recovering a matrix D from the canonical equation (2.1.2) in the case when the matrix depends only on the variable x. It is worth emphasizing here that in the classical statement of the related direct problem the boundary conditions are somewhat different from those involved in (2.1.9). In the case of a direct problem the subscript of the functions $v_i(L,t) = \psi_i(t)$ and $v_i(0,t) = \psi(t)$ runs over $s < i \leq n$ and $1 \leq i \leq s$, respectively. The statement of problem (2.1.9) includes the boundary conditions with the subsidiary information enabling to recover not only the function $v(x,t)$, but also the coefficient $p(x)$. Deeper study of the inverse problem (2.1.2), (2.1.8) of finding a function $p(x)$ was initiated by V. Romanov (1977). However, the boundary conditions imposed in his paper happen to be overdeterminated even from the standpoint of inverse problems. Since the function $v(x,t)$ is known everywhere on the boundary of the half-strip, only the uniqueness of a solution of this inverse problem is ensured in such a setting. The general statement (2.1.9) in which the components of the function v are defined on a part of the boundary of the half-strip was suggested by D. Orlovsky (1983). He has proved therein the basic results set forth in Section 2.1.

Section 2.2 is devoted with the problem of finding a nonhomogeneous term of linear hyperbolic systems when its solution is given everywhere on the boundary of the half-strip $\{0 \leq x \leq L,\ t \geq 0\}$. As indicated above, for the structure of the right-hand side specified in Section 2.1 such a statement of the problem would be certainly overdeterminated. The decomposition $G(x,t) = H(x,t)\,p(t)$, where the vector function $p(t)$ is unknown, will be approved in mastering the difficulties involved. This amounts to investigating t-hyperbolic systems of the canonical form (2.2.1). The existence and uniqueness of their solutions are proved and appear to be of global character. In the body of this section we follow the paper of D. Orlovsky (1983).

Section 2.3 gives an illustration of applying the hyperbolic systems under consideration to investigating linear hyperbolic equations of second order within the framework of inverse problems.

Other statements of inverse problems for hyperbolic systems of first order and hyperbolic equations of second order as well as a survey of relevant results are available in many monographs, textbooks and papers. In this regard, it is worth mentioning Yu. Anikonov (1978), A. Baev (1985), H. Baltes (1987), S. Belinsky (1976), P. Berard (1986), I. Gelfand and B. Levitan (1951), S. Kabanikhin (1979, 1987), A. Khaidarov (1986), P. Lax and R. Phillips (1967), A. Louis (1989), V. Marchenko (1986), F. Natterer (1986), D. Orlovsky (1984a,b), J. Pöschel and E. Trubowitz (1986), A. Ramm (1992), V. Romanov (1968, 1972, 1973, 1977, 1978a,b,c,d, 1984), V. Romanov and E. Volkova (1982), V. Romanov and V. Yakhno (1980), V. Romanov et al. (1984), I. Tikhonov and Yu. Eidelman (1994), T. Tobias and Yu. Engelbrecht (1985), V. Yakhno (1977, 1990).

Plenty of inverse problems associated with hyperbolic equations is involved in the monographs by G. Alekseev (1991), Yu. Anikonov (1995), D. Colton and R. Kress (1992), D. Colton et al. (1990), R. Corenflo and S. Vessela (1991), K. Chadan and P. Sabatier (1989), I. Gelfand and S. Gindikin (1990), M. Gerver (1974), C. Groetsch (1993), G. Herglotz (1914), M. Imanaliev (1977), V. Isakov (1998), V. Kirejtov (1983), R. Lattes and J.-L. Lions (1967), P. Lax and R. Phillips (1967), B. Levitan (1984), J.-L. Lions (1970), L. Nizhnik (1991) and V. Romanov (1984).

The original works of A. Alekseev (1962, 1967), A. Amirov (1986a,b, 1987), D. Anikonov (1975, 1979, 1984), Yu. Anikonov (1976, 1978, 1992), Yu. Anikonov and B. Bubnov (1988), Yu. Antokhin (1966), M. Belishev (1989), A. Blagoveshensky (1966), A. Bukhgeim (1984, 1988), A. Bukhgeim and M. Klibanov (1981), J. Cannon and D. Dunninger (1970), J. Cannon et al. (1990), C. Cavaterra and M. Grasselli (1994), V. Dmitriev et al. (1976),

L. Faddeev (1976), A. Friedman (1987), I. Gelfand and B. Levitan (1951), M. Grasselli (1992, 1994), M. Grasselli et al. (1990, 1992), A. Ivankov (1983), M. Lavrentiev (1964), A. Lorenzi (1992), A. Lorenzi and A. Prilepko (1993, 1994), V. Maslov (1968), A. Nachman (1988), A. Prilepko and A. Ivankov (1984, 1985), A. Prilepko and A. Kostin (1993b), A. Prilepko and I. Tikhonov (1992, 1993, 1994), A. Prilepko and N. Volkov (1987, 1988), A. Prilepko, D. Orlovsky and I. Vasin (1992), V. Romanov and V. Yakhno (1980), P. Sabatier (1977a,b), V. Shelukhin (1993), J. Sylvester and G. Uhlmann (1987), T. Tobias and Yu. Engelbrecht (1985), V. Yakhno (1977, 1990) made significant contributions in the basic trends of such theory.

Chapter 3 is devoted to inverse problems in potential theory. The foundations of the theory of partial differential equations of the elliptic type may be found in many textbooks and monographs and papers by Yu. Berezanskij (1968), O. Besov et al. (1975), A. Bitsadze (1966, 1977), D. Colton and R. Kress (1992), R. Courant and D. Hilbert (1953, 1962), D. Gilbarg and N. Trudinger (1983), S. Godunov (1971), L. Hörmander (1965), N. Idel'son (1936), O. Ladyzhenskaya (1973), O. Ladyzhenskaya and N. Uraltseva (1968), E. Landis (1971), L. Likhtenshtein (1966), J.-L. Lions (1970), A. Lyapunov (1959), S. Michlin (1977), V. Mikhailov (1976), K. Miranda (1957), S. Sobolev (1954, 1988), A. Tikhonov and A. Samarskii (1963), V. Vladimirov (1971) and J. Wermer (1980). Much progress in potential theory has been achieved by serious developments due to N. Hunter (1953), V. Il'in et al. (1958), N. Landkhof (1966) and L. Sretensky (1946). The results presented in the third chapter have been obtained by A. Prilepko (1961, 1964, 1965a,b, 1966a,b,c, 1967, 1968a,b,c, 1969, 1970a,b,c, 1972, 1973a,b, 1974a,b). Many scientists are much interested in various aspects of uniqueness for the inverse problem I in the case of a classical potential (see Section 3.3). The first result regarding the solution uniqueness of the exterior inverse problem for the Newtonian potential in the category of "star-shaped" bodies with a constant density was proved by P. Novikov (1938). Later this problem was extensively investigated by L. Sretensky (1938, 1954), I. Rapoport (1940, 1941, 1950), A. Gelmins (1957), V. Ivanov (1955, 1956a,b, 1958a,b, 1962), L. Kazakova (1963, 1965), M. Lavrentiev (1955, 1956, 1963), Yu. Shashkin (1957, 1958, 1964), V. Simonov (1958), R. Smith (1961), I. Todorov and D. Zidorov (1958), and many others for the case of a constant density. The question of the solution uniqueness for the inverse problem I with a variable density for the logarithmic potential was studied by Yu. Shashkin (1957, 1958, 1964) and V. Simonov (1958) with the aid of conform mappings.

The aspect of the solution uniqueness in the study of various inverse problems is intimately connected with stability of their solutions. The general topological stability criterion ascribed to A. Tikhonov (1943) will imply certain qualitative conditions of stability if the corresponding uniqueness theorems hold true. For example, for the inverse problem related to the exterior potential with a constant density in the case of "star-shaped" bodies a few stability conditions were deduced from the uniqueness theorems established by P. Novikov (1938), V. Ivanov (1958a) and L. Kazakova (1963, 1965). Numerical estimates of deviation for two bodies are obtained in the papers of I. Rapoport (1940, 1941, 1950). They are expressed in terms of the difference between exterior potentials under certain restrictions on the properties of the potentials involved. There are also integral stability estimates which can be proved for "star-shaped" bodies of constant density. M. Lavrentiev (1967) derived for them the general numerical estimates depending on the deviation of potentials on a piece of the sphere, the interior of which contains the attracting bodies. In the plane-parallel problem for the logarithmic potential with a variable positive density stability estimates have been established by Yu. Shashkin (1964) for several classes of the generalized "star-shaped" bodies by means of relevant elements of the theory of functions of one complex variable and conform mappings. The extensive literature on this subject is also reviewed in the monographs by G. Anger (1990), V. Cherednichenko (1996), V. Isakov (1990c, 1998), V. Ivanov, V. Vasin and V. Tanana (1978), M. Lavrentiev (1967, 1973), A. Tikhonov and V. Arsenin (1977) and the original works of T. Angell et al. (1987), J. Cannon (1967), J. Cannon and W. Rundell (1987), V. Cherednichenko (1978), A. Friedman and B. Gustafsson (1987), V. Isakov (1990a), A. Khaidarov (1986, 1987), J.-L. Lions (1970), A. Lorenzi and C. Pagani (1981), D. Orlovsky (1989), G. Pavlov (1978), L. Payne (1975), M. Pilant and W. Rundell (1986b, 1987a), A. Prilepko (1973a,b, 1974b), A. Ramm (1992), C. Roach (1991), N. Weck (1972) and many others. We cite below for deeper study two collections of additional monographs and papers of Russian and foreign scientists whose achievements in potential theory and inverse problems for elliptic equations were recognized. The first one includes M. Aleksidze (1987), V. Antonov et al. (1988), P. Appel (1936), M. Brodsky (1990), S. Chandrasékhar (1973), V. Cherednichenko (1996), D. Colton et al. (1990), A. Friedman (1982), N. Idel'son (1936), V. Isakov (1990c, 1998), D. Kinderlehrer and G. Stampacchia (1980), M. Lavrentiev and B. Shabat (1973), L. Likhtenshtein (1966), P. Pitsetti (1933), M. Sacai (1982, 1987), V. Starostenko (1978), G. Talenti (1978), A. Tsirul'skii (1990), P. Vabishevich (1987), J. Wermer (1980), M. Zhdanov (1984) and D. Zidarov (1984). The second one contains D. Aharov and H. Shapiro (1976), D. Aharov et al. (1981), A. Alekseev and A. Chebotarev (1985), A. Alekseev and V. Cherednichenko (1982), M. Atakhodzhaev (1966), P. Balk (1977), H. Bellot and A. Friedman (1988),

E. Beretta and S. Vessella (1988), A. Bitsadze (1953), M. Brodsky and N. Nadirashvili (1982a,b), M. Brodsky and E. Panakhov (1990), M. Brodsky and V. Strakhov (1982, 1983, 1987), A. Cálderon (1980), V. Cherednichenko and G. Verjovkina (1992), A. Chudinova (1965a,b), Colli Franzone et al. (1984), P. Dive (1931, 1932), V. Filatov (1969), A. Friedman (1987), A. Gelmins (1957), C. Golizdra (1966), E. Hölder (1932), L. Hörmander (1976), V. Ivanov (1955, 1956a,b, 1958a,b, 1962), V. Ivanov and A. Chudinova (1966), V. Ivanov and L. Kazakova (1963), A. Iskenderov (1968), D. Kapanadze (1986), I. Kartisivadze (1963), M. Lavrentiev (1965), A. Loginov (1988), A. Lorenzi and C. Pagani (1981, 1987), O. Oleinik (1949, 1952), Yu. Osipov and A. Korotkii (1992), C. Pagani (1982), G. Pavlov (1975, 1976, 1982), D. Pinchon (1987), A. Prilepko (1985, 1992, 1996), A. Prilepko and V. Cherednichenko (1981), A. Prilepko and V. Sadovnichii (1996), V. Cherednichenko and S. Vessella (1993), C. Rempel (1973), M. Sacai (1978, 1987), J. Serrin (1971), V. Starostenko et al. (1988), V. Strakhov (1972, 1973, 1974a,b, 1977), V. Strakhov and M. Brodsky (1983a,b, 1984), P. Sulyandziga (1977), V. Tsapov (1989), V. Vinokurov and V. Novak (1983), B. Voronin and V. Cherednichenko (1981, 1983), G. Voskoboinikov and N. Nagankin (1969), L. Zalcman (1987) and A. Zamorev (1941a,b).

Chapter 4 treats the system of Navier–Stokes equations capable of describing the motion of a viscous incompressible fluid. As we have mentioned above, related direct problems for these equations have been examined by many scientists. For more a detailed information on this subject the reader can refer, in particular, to the monographs by O. Ladyzhenskaya (1970), R. Temam (1979), P. Constantin and C. Foias (1988). In Section 4.1 we expound certain exploratory devices which may be of help in investigating many inverse problems. It should be noted that the first statement of the inverse problem for the system of Navier–Stokes equations was formulated by A. Prilepko and D. Orlovsky (1987a). Additional information, that is, the overdetermination condition in setting up the inverse problem is a result of the pointwise measurement, making it possible to assign the value of the flux velocity at a fixed interior point in the domain of space variables within a certain interval of time. This inverse problem was investigated by means of methods of semigroup theory.

Section 4.2 deals with the inverse problem with the final overdetermination for the linearized systems. The results we outlined there were obtained by A. Prilepko and I. Vasin (1989a, 1990c). In the next section the same linearized Navier–Stokes system arises in the inverse problem of recovering a structure of time dependence for the external force function having representation (4.3.1) with the subsidiary information in the form of integral overdetermination. The material of this section is based on the works of A. Prilepko and I. Vasin (1989a), I. Vasin (1992b). We are exploring in Sections 4.4–4.5 the inverse problem for the nonlinear Navier–Stokes system with the final overdetermination both in the two-dimensional and three-dimensional domains of space variables. The existence theorems given in Section 4.4 have been proved by A. Prilepko and I. Vasin (1990b,c, 1991). In Section 4.5 we rely on the results obtained by A. Prilepko and I. Vasin (1989b, 1991). Section 4.6 includes the inverse problem in which the nonlinear Navier–Stokes system is accompanied by the integral overdetermination, whose use permits us to recognize how the external force function depends on the time. The theorems of the solution existence and uniqueness were established for this type of overdetermination. The paper of I. Vasin (1993) is recommended for further development and deeper study in this area. The nonlinear inverse problem of recovering the evolution of a coefficient at the unknown function of velocity was completely posed and examined in Section 4.7 in the light of the preceding results of A. Prilepko and I. Vasin (1993). Section 4.8 reflects the contemporary stage of research in connection with the inverse problem of the combined recovery of two coefficients of the linearized Navier–Stokes system with the integral overdetermination prescribed in the papers of I. Vasin (1995, 1996). Some other aspects of setting up and investigating inverse problems of hydrodynamics are covered by the developments of A. Chebotarev (1995), A. Prilepko and I. Vasin (1990a, 1992), A. Prilepko, D. Orlovsky and I. Vasin (1992), I. Vasin (1992a), Yu. Anikonov (1992), V. Kamynin and I. Vasin (1992). Similar inverse problems associated with the Navier–Stokes system emerged in several works on exact controllability (see, for example, A. Fursikov and O. Imanuvilov (1994)).

Relevant results from functional analysis, operator theory and the theory of differential equations in Banach spaces are given in Chapter 5. All proofs of the main statements of Section 5.1 as well as a detailed exposition of the foundations of functional analysis and operator theory are outlined in textbooks and monographs by N. Akhiezer and I. Glazman (1966), A. Balakrishnan (1976), G. Birkhoff (1967), N. Dunford and J. Schwartz (1971a,b,c), R. Edwards (1965), E. Hille and R. Phillips (1957), L. Kantorovich and G. Akilov (1977), T. Kato (1966), A. Kolmogorov and S. Fomin (1968), M. Krasnoselskii et al. (1966), L. Lyusternik and V. Sobolev (1982), A. Plesner (1965), F. Riesz and B. Sz.-Nagy (1972), W. Rudin (1975), S. Sobolev (1988, 1989), H. Schaefer (1974), L. Schwartz (1950, 1951), V. Trenogin (1980), B. Vulikh (1967) and K. Yosida (1965). The theory of differential equations of the first order in a Banach space is discussed in Section 5.2. The preliminaries of such theory have been appeared quite long time ago. A rapid development in this area over recent years is due to A. Babin and M. Vishik (1989), C. Batty and

D. Robinson (1984), A. Balakrishnan (1976), R. Beals and V. Protopescu (1987), A. Belleni-Morante (1989), Ph. Clément et al. (1987), E. Davies (1980), H. Fattorini (1983), H. Gajewski et al. (1974), J. Goldstein (1985), V. Gorbachuk and M. Gorbachuk (1984), D. Henry (1981), S. Krein (1967), S. Krein and M. Khazan (1983), G. Ladas and V. Lakshmihanthan (1972), J.-L. Lions (1961), S. Mizohata (1977), A. Pazy (1983), V. Vasiliev et al. (1990), M. Vishik and O. Ladyzhenskaya (1956), K. Yosida (1965).

Section 5.3 is devoted to differential equations of the second order in a Banach space. At the initial stage the theory of equations of the second order was less advanced as compared with differential equations of the first order. We are unaware of any textbook or monograph on this subject in the modern literature. Much progress in this area has been achieved later and reflected in the papers of H. Fattorini (1969a,b, 1985), J. Goldstein (1969), J. Kisynski (1972), S. Kurepa (1982), D. Lutz (1982), H. Serizawa and M. Watanabe (1986), P. Sobolevsky and V. Pogorelenko (1967), M. Sova (1966, 1975, 1977), C. Travis and G. Webb (1978), V. Vasiliev et al. (1990) and S. Yakubov (1985).

Section 5.4 is connected with the theory of differential equations with variable coefficients. The subject of investigation is relatively broad and the reader may confine yourself to the results of A. Fisher and J. Marsden (1972), H. Fattorini (1983), H. Gajewski et al. (1974), M. Gil (1987), D. Henry (1981), T. Kato (1953, 1956, 1961, 1970, 1973, 1975a,b, 1982), J.-L. Lions (1961), F. Lomovtsev and M. Yurchuk (1976), S. Mizohata (1977), P. Sobolevsky (1961), P. Sobolevsky and V. Pogorelenko (1967), H. Tanabe (1960), S. Yakubov (1970) and K. Yosida (1956, 1963).

In this book we outline only those abstract problems for equations with variable operator coefficients which are essential for subsequent applications. Section 5.5 includes boundary value problems for abstract elliptic equations. A detailed overview of relevant results obtained for elliptic equations is available in A. Balakrishnan (1960), S. Krein and G. Laptev (1962, 1966a,b), G. Laptev (1968), P. Sobolevsky (1968) and V. Trenogin (1966).

In Section 6.1 some tools of applying the theory of abstract differential equations are aimed at solving direct and inverse problems in mathematical physics. For the first time this technique was used by A. Iskenderov and R. Tagiev (1979) with regard to the inverse problem for the heat conduction equation (6.1.1), (6.1.11), (6.1.12) involving the function $\Phi(x,t) \equiv 1$. Advanced theory of abstract inverse problems was mainly connected with this problem due to Yu. Eidelman (1983, 1987), D. Orlovsky (1988), A. Prilepko and D. Orlovsky (1987), W. Rundell (1980). Historically, the abstract inverse problem supplied by conditions (6.1.3)–(6.1.4) came next. However, this problem was initially considered not for the heat conduction equation, but for symmetric hyperbolic systems of the first order (see D. Orlovsky

(1984)). Later the theory of abstract inverse problems was well-developed by Yu. Eidelman (1990, 1991, 1993a,b), A. Egorov (1978), M. Grasselli (1992), M. Grasselli et al. (1990, 1992), A. Lorenzi (1988), A. Lorenzi and A. Prilepko (1993), A. Lorenzi and E. Sinestrari (1986, 1987), D. Orlovsky (1989, 1990, 1991a,b,c,d, 1992a,b, 1994), A. Prilepko and D. Orlovsky (1984, 1985a,b,c, 1987, 1988, 1989, 1991), A. Prilepko and I. Tikhonov (1992, 1993, 1994), A. Prilepko et al. (1992a,b), I. Tikhonov (1992, 1995), I. Tikhonov and Yu. Eidelman (1994).

The linear inverse problem with smoothing overdetermination was studied in Section 6.2. The concept of its weak solution introduced by D. Orlovsky (1991a) permits one to solve this problem under the minimum restrictions on the input data. Under such an approach the existence of smooth solutions can be established by means of weak solutions of related problems "in variations". Later the results of Section 6.2 are being used in generalized form in Sections 6.3 and 6.4 to cover on the same footing the case of a semilinear equation. The papers of D. Orlovsky (1991a), A. Prilepko and D. Orlovsky (1984, 1985a,b,c, 1987, 1988, 1989, 1991), A. Prilepko and I. Tikhonov (1992) are devoted to the mathematical apparatus adopted in this area. In Section 6.5 we study quasilinear parabolic equations by means of the method of solving direct problems ascribed to P. Sobolevsky (1961). The same approach to inverse problems is adopted in the work of D. Orlovsky (1991b). Section 6.6 is devoted to semilinear equations with variable operator coefficients and singular overdetermination and is based on the results of A. Prilepko and D. Orlovsky (1991). The remaining part of Chapter 6 deals with equations of hyperbolic type. In Sections 6.7 and 6.8 we consider one class of abstract equations which are associated in applications with hyperbolic systems of the first order. Section 6.7 involves a smoothing overdetermination, while Section 6.8 – a singular one. In Section 6.9 we are concerned with an abstract equation corresponding in applications to those systems of the first order to which hyperbolic equations of the second order can amount.

Chapter 7 deals with two-point inverse problem (7.1.1)–(7.1.4) for a linear differential equation in a Banach space X, where an element $p \in X$ is sought. Just this statement of an inverse problem provides sufficient background for applying the methods of the theory of abstract differential equations to the exploration of inverse problems in mathematical physics. This problem was first considered by A. Iskenderov and R. Tagiev (1979) by appeal to the operator function $\Phi(t) \equiv I$. In this paper the explicit formula (7.1.33) was established under the condition that the operator $V(T) - I$ is invertible. As a matter of fact, the same result was obtained by W. Rundell (1980), where the semigroup $V(t)$ was supposed to be exponentially decreasing. As the outcome of restrictions arising from these two papers one should take for granted that the operator A is invertible. In the paper of Yu. Eidelman (1983) it was supposed with regard to problem (7.1.1)–(7.1.4) that $\Phi(t) \equiv I$ and the semigroup is analytic, that is, the case of a parabolic equation occurred. It is worth mentioning here that problem (7.1.1)–(7.1.4) may be well-posed even if the operator A is not invertible. We refer the reader to the work of A. Prilepko and D. Orlovsky (1987) where the unique solvability of problem (7.1.1)–(7.1.4) was proved for $\Phi(t) \equiv I$ and the operator A for which $\lambda = 0$ is one of the eigenvalues of finite multiplicity. A final answer concerning a well-posedness of problem (7.1.1)–(7.1.4) with $\Phi(t) \equiv I$ is known from the paper of Yu. Eidelman (1990) in which necessary and sufficient conditions of both uniqueness and existence of the solution to an equation with an arbitrary strongly continuous semigroup. In particular, a solution is unique if and only if the spectrum of the operator A contains no points of the type $2\pi i k/T$, where k is integer, $k \neq 0$, and i is the imaginary unit. The solvability condition is expressed in terms of Cesaro summability of series composed by solutions of resolvent equations. The results outlined in Section 7.2 for problem (7.2.1)–(7.2.4) with the scalar function Φ and the self-adjoint operator were obtained by D. Orlovsky (1990). The reference to the paper of Yu. Eidelman (1991) is needed in this context. In this paper the function Φ is scalar and the operator A generates a semigroup $V(t)$, which is continuous for $t > 0$ in a uniform operator topology of the space $\mathcal{L}(X)$ as well as necessary and sufficient conditions are established for the inverse problem to be uniquely solvable under any admissible input data. These conditions include the constraint $\Phi(T) \neq 0$ and the requirement that function (7.2.8) has no zeroes on the spectrum of the operator A. The theorem on Fredholm-type solvability of the inverse problem (7.1.1)–(7.1.4) was proved by D. Orlovsky (1988, 1990).

The inverse problem (7.3.3)–(7.3.5) in a Banach lattice was investigated by D. Orlovsky (1994), A. Prilepko and I. Tikhonov (1993, 1994), A. Prilepko et al. (1992), I. Tikhonov (1995). The proof of Theorem 7.3.1 was carried out following the paper of A. Prilepko and I. Tikhonov (1994) with minor changes and involving a new, more general than (7.1.4), type of overdetermination:

$$\int_0^T u(t)\, d\mu(t) = u_1 ,$$

where μ is a function of bounded variation.

In Chapter 8 we touch upon abstract inverse problems for equations of the second order in a Banach space. As stated above, the theory of abstract inverse problems began by equations of the first order. The study of equations of the second order was initiated by D. Orlovsky (1989), A. Prilepko and D. Orlovsky (1989) and was continued by D. Orlovsky (1990, 1991a,d, 1992a) and G. Pavlov (1988). As in the case of first order equations we approve here two basic settings when the subsidiary information is provided either by condition (8.1.3) or conditions of the type (8.2.3), (8.3.2). In Section 8.1 we consider the overdetermination of the first type with regard to a semilinear hyperbolic equation. The basic tool here is connected with the theory of abstract cosine functions whose role for second order equations is identical to that played by semigroup theory for first order equations. Section 8.2 presents two-point inverse problems for equations of hyperbolic type. In Section 8.3 two-point problems are viewed within the framework of elliptic equations. In conclusion we note that all of the main results of Chapter 8 were obtained by D. Orlovsky (1989, 1990, 1991a,d, 1992a), A. Prilepko and D. Orlovsky (1989).

Chapter 9 offers one possible way of applying the theory of abstract inverse problems to equations of mathematical physics. Section 9.1 deals with symmetric hyperbolic systems. General information about hyperbolic systems and methods of solving them is available in textbooks and monographs by R. Courant and D. Hilbert (1953), S. Godunov (1971), S. Mizohata (1977), B. Rozhdestvensky and N. Yanenko (1978) as well as in research papers by A. Fischer and J. Marsden (1972), K. Friedrichs (1944, 1954, 1958), T. Kato (1970, 1973, 1975a,b, 1982), P. Lax and R. Phillips (1960), D. Ludwig (1960), E. Massey (1972), S. Mizohata (1959a,b) and R. Phillips (1957). We refer the reader for deeper study of hyperbolic systems by the method of abstract differential equations to T. Kato (1970, 1973, 1975a,b, 1982), P. Lax and R. Phillips (1960), E. Massey (1972), S. Mizohata (1959a,b, 1977) and R. Phillips (1957).

Section 9.1 continues to develop the approach of T. Kato (1970, 1973) and E. Massey (1972). The study of inverse problems for hyperbolic systems was initiated by V. Romanov and L. Slinyucheva (1972). Various statements of inverse problems associated with hyperbolic systems and an extensive literature on this subject are reviewed in the papers of S. Belinsky (1976), M. Lavrentiev et al. (1969), D. Orlovsky (1984), A. Prilepko (1973), A. Prilepko and D. Orlovsky (1985a), A. Prilepko et al. (1992b) and V. Romanov (1978a,b,c,d). Some overview of the semigroup approach to inverse problems can be found in D. Orlovsky (1984), A. Prilepko and D. Orlovsky (1984, 1985a, 1989).

Section 9.2 deals with a linear second order equation of hyperbolic type. In preparation for this, our approach amounts to reducing a second order equation to a first order system and adopting the tool developed by M. Ikawa (1968). An alternative scheme is based on the theory of abstract cosine functions. This scheme is not applicable in Section 9.2. It is worth noting here that the paper of J. Kisynski (1972) establishes an equivalence between the method of cosine functions and the method of reduction to a first order system in applications to abstract differential equations of the second order with constant operator coefficients. An extensive literature on inverse problems for hyperbolic systems of the second order is reviewed in A. Bukhgeim (1988), M. Lavrentiev et al. (1969), D. Orlovsky (1989, 1991a), A. Prilepko (1973), A. Prilepko and D. Orlovsky (1985a,b,c), V. Romanov (1978b). The applications of the abstract method to the inverse problems concerned are discussed in A. Amirov (1986), D. Orlovsky (1989, 1990, 1991a,d, 1992a,b), A. Prilepko and I.Tikhonov (1993).

Section 9.3 is devoted to a system of differential equations from elasticity theory where the coefficients satisfy only the conditions of symmetry, positive definiteness and smoothness. A good look at the system of elasticity theory as an abstract differential equation in a Banach space is due to G. Duvaut and J.-L. Lions (1972), E. Sanchez-Palencia (1980). For the purposes of the current study it was expedient to use some technique of abstract cosine functions. To make our exposition more complete, it is necessary to point out several other works where the inverse problems of interest were investigated by the methods of the theory of abstract differential equations: A. Prilepko and D. Orlovsky (1985c, 1989), A. Prilepko et al. (1992b).

Section 9.4 deals with a second order equation of parabolic type. The theory of parabolic equations as a part of the general theory of partial differential equations is the most broad and advanced. For an overview of this subject we refer the reader to H. Amann (1986, 1987), R. Courant and D. Hilbert (1953, 1962), S. Eidelman (1964), H. Fattorini (1983), A. Friedman (1964), D. Henry (1981), T. Kato (1961), S. Krein (1967), O. Ladyzhenskaya et al. (1968), S. Mizohata (1977), P. Sobolevsky (1961), M. Solomyak (1960) and V. Vladimirov (1971).

The theory of inverse problems for parabolic equations is also well-developed. For more detail the reader can see J. Cannon (1968), M. Lavrentiev et al. (1969), D. Orlovsky (1991b, 1992b, 1994), A. Prilepko (1973), A. Prilepko and D. Orlovsky (1985b,c, 1991), A. Prilepko and I. Tikhonov (1993, 1994), A. Prilepko et al. (1992a), W. Rundell (1980). It is worth mentioning here the works where inverse problems for parabolic equations are purely treated on the basis of abstract methods: Yu. Eidelman (1983, 1987, 1990, 1991, 1993a,b), A. Iskenderov and R. Tagiev (1979), D. Orlovsky (1988, 1990, 1991b,c, 1992b, 1994), A. Prilepko and D. Orlovsky (1985b,c, 1989, 1991), A. Prilepko and I. Tikhonov (1993, 1994), A. Prilepko et al. (1992a,b), I. Tikhonov (1995) and W. Rundell (1980).

Section 9.5 deals with the equation of neutron transport. A classical theory for this equation is available in K. Case and P. Zweifel (1963, 1972),

C. Cercignani (1975), T. Germogenova (1986), V. Vladimirov (1961).

A new semigroup approach to this equation was developed by S. Albertoni and B. Montagnini (1966), R. Beals and V. Protopescu (1987), H. Hejtmanek (1970), K. Jorgens (1968), J. Lehner and G. Wing (1955), C. Lekkerkerker and H. Kaper (1986), M. Ribaric (1973), R. Richtmyer (1978), S. Shikhov (1967, 1973), I. Vidav (1968, 1970), J. Voight (1984), G. Wing (1962), P. Zweifel and E. Larsen (1975). Inverse problems for the transport equation under this approach were considered by A. Prilepko and D. Orlovsky (1985b), A. Prilepko and I. Tikhonov (1992), A. Prilepko et al. (1992b), I. Tikhonov (1995).

Section 9.6 is devoted to the linearized Bolzman equation. A solution to this equation describes a distribution of rarefied gas particles with respect to coordinates and velocities. Some general information about the Bolzman equation is available in A. Arseniev (1965), C. Cercignani (1975), O. Lanford et al. (1983), N. Maslova (1978, 1985). Both works of J.-P.Guirand (1970, 1978) continue to develop this semigroup approach in this area. The main results of Section 9.6 are taken from the papers of A. Prilepko and D. Orlovsky (1987, 1988).

In Section 9.7 we consider the system of Navier–Stokes equations describing the motion of a viscous incompressible fluid. The basic properties of its solutions are described in O. Ladyzhenskaya (1970) and R. Temam (1979). A new semigroup approach to solving direct problems for this system began by investigations of H. Fujita and T. Kato (1964), T. Kato and H. Fujita (1962). For inverse problems for Navier–Stokes equations we refer the reader to V. Kamynin and I. Vasin (1992), A. Prilepko and D. Orlovsky (1985c), A. Prilepko and I. Vasin (1989a,b, 1990a,b,c, 1991, 1992, 1993), A. Prilepko et al. (1992b), I. Vasin (1992, 1993, 1995, 1996).

Under a semigroup approach inverse problems for Navier–Stokes equations were treated in the paper of A. Prilepko and D. Orlovsky (1985c).

Section 9.8 deals with the system of Maxwell equations which has been under consideration within the semigroup framework in E. Bykhovsky (1957), G. Duvaut and J.-L. Lions (1972), R. Richtmyer (1978). In this section one inverse problem is resolved following the approach of A. Prilepko and D. Orlovsky (1989, 1991).

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Index